April 16, 2010

PHYS 5583 (E & M II)

Exam 2

1. 33%

A time dependent electric dipole, located at the origin has 4-current

$$J^{\alpha}(t,\mathbf{r}) = \left[-c\,\mathbf{p}(\mathbf{t})\cdot\vec{\nabla}\delta^3(\mathbf{r}),\ \dot{\mathbf{p}}(\mathbf{t})\delta^3(\mathbf{r})\right],$$

and a retarded 4-potential (in Gaussian units)

$$A^{\alpha}(t,\mathbf{r}) = \left[-\nabla\cdot\left\{\frac{\mathbf{p}(t-r/c)}{r}\right\},\ \frac{\dot{\mathbf{p}}(t-r/c)}{cr}\right].$$

And all parts

(a) Compute the radiation part (i.e., the $\sim 1/r$ part) of the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$ for the above dipole. {Hints: The radiation parts depend on second derivatives of the dipole moment and you should find that $\mathbf{B}_{rad} = \hat{r}\times\mathbf{E}_{rad}$.}

(b) What type of polarization, plane or circular, will be seen by a distant observer coming from a dipole with

$$\mathbf{p}(t) = p_0\cos(\omega t)\hat{k}?$$

(c) Use the Poynting vector to compute

$$\left\langle\frac{dP(\theta)}{d\Omega}\right\rangle,$$

the time averaged power radiated into a unit solid angle, as a function of the spherical polar angle θ for a dipole with

$$\mathbf{p}(t) = p_0[\cos(\omega t)\hat{\imath} + \sin(\omega t)\hat{\jmath}].$$

d) Derive the A's

e) what is the polarization for the dipole in part c

Prob 1 (Gaussian)

A time dependent electric dipole located at origin has 4-current

$$J^{\alpha}(t,\bar{r}) = \left[-c\bar{p}(t)\cdot\bar{\nabla}\delta^3(\bar{r}),\ \dot{\bar{p}}(t)\delta^3(\bar{r})\right]$$

and a retarded 4-current

$$A^{\alpha}(t,\bar{r}) = \left[-\nabla\cdot\left\{\frac{\bar{p}(t-r/c)}{r}\right\},\ \frac{\dot{\bar{p}}(t-r/c)}{cr}\right]$$

a) Find the retarded potentials

$$\Box A^{\alpha}(x^{\alpha}) = \frac{4\pi}{c} J^{\alpha}(x^{\alpha})$$

$$\Rightarrow \quad A^{\alpha}(x) = \int D_{ret}(x-x')\, J^{\alpha}(x')\, d^4x'$$

Thus,

$$\Phi^{ret}(t,\bar{r}) = \int_{\text{allspace}} \frac{\delta^4(x^0 - x^{0\prime} - |\bar{r}-\bar{r}'|)}{4\pi\,|\bar{r}-\bar{r}'|}\,\frac{4\pi}{c}\, cP(t',\bar{r}')\, dx^{0\prime}\, d^3x'$$

$$= \int_{(\infty)^4} \frac{\delta^4\left(c\left(t-t'-\frac{|\bar{r}-\bar{r}'|}{c}\right)\right)}{c\,|\bar{r}-\bar{r}'|}\left(-c\bar{p}(t')\cdot\nabla'\delta^3(\bar{r}')\right) c\, dt'\, d^3r'$$

$$= -\frac{c}{c}\int_{(\infty)^4} \frac{\delta^4\left(-t' + \left(t - \frac{|\bar{r}-\bar{r}'|}{c}\right)\right)}{|\bar{r}-\bar{r}'|} \bar{p}(t') \cdot \nabla' \delta^3(\bar{r}') \, dt' d^3\bar{r}'$$

$$= -\int_{(\infty)^3} \bar{p}\left(t - \frac{|\bar{r}-\bar{r}'|}{c}\right) \cdot \nabla' \frac{\delta^3(\bar{r}')}{|\bar{r}-\bar{r}'|} \, d^3\bar{r}'$$

NOW, $\bar{\nabla}\cdot(U\bar{A}) = (\bar{\nabla}U)\cdot\bar{A} + U(\bar{\nabla}\cdot\bar{A})$

$$\Rightarrow \bar{A}\cdot(\bar{\nabla}U) = \bar{\nabla}\cdot(U\bar{A}) - U(\bar{\nabla}\cdot\bar{A})$$

$$= -\int_{(\infty)^3} \bar{\nabla}' \cdot \left(\frac{\bar{p}\left(t - \frac{|\bar{r}-\bar{r}'|}{c}\right)}{|\bar{r}-\bar{r}'|} \delta^3(\bar{r}') \right) d^3\bar{r}'$$

$$+ \int_{(\infty)^3} \delta^3(\bar{r}') \, \bar{\nabla}' \cdot \frac{\bar{p}\left(t - \frac{|\bar{r}-\bar{r}'|}{c}\right)}{|\bar{r}-\bar{r}'|} \, d^3\bar{r}'$$

$$= \underbrace{-\int_{\partial V_3} \frac{\bar{p}\left(t - \frac{|\bar{r}-\bar{r}'|}{c}\right)}{|\bar{r}-\bar{r}'|} \delta^3(\bar{r}') \, d^2\bar{r}'}_{\to 0 \text{ as the fun}^c \text{ vanishes at the boundary}} + \int_{(\infty)^3} \delta^3(\bar{r}') \, \bar{\nabla}' \cdot \frac{\bar{p}\left(t - \frac{|\bar{r}-\bar{r}'|}{c}\right)}{|\bar{r}-\bar{r}'|} \, d^3\bar{r}'$$

$$\Phi^{ret}(t,\bar{r}) = \int \delta^3(\bar{r}') \; \bar{\nabla}' \cdot \frac{\bar{P}\left(t - \frac{|\bar{r}-\bar{r}'|}{c}\right)}{|\bar{r}-\bar{r}'|} \, d^3\bar{r}'$$

$$\frac{\partial}{\partial x'^i}\left(|\bar{r}-\bar{r}'|\right) = \frac{1}{2}\,\frac{2(x-x')^i(-1)}{|\bar{r}-\bar{r}'|} = -\frac{(x-x')^i}{|\bar{r}-\bar{r}'|}$$

$$\& \quad \frac{\partial}{\partial x^i}\left(|\bar{r}-\bar{r}'|\right) = \frac{(x-x')^i}{|\bar{r}-\bar{r}'|}$$

$$= \int -\delta^3(\bar{r}') \; \bar{\nabla} \cdot \frac{\bar{P}\left(t - \frac{|\bar{r}-\bar{r}'|}{c}\right)}{|\bar{r}-\bar{r}'|} \, d^3\bar{r}'$$

$$= -\frac{\bar{\nabla}\cdot\bar{P}(t-r/c)}{r}$$

$$A^{ret}(t,\bar{r}) = \int_{(\infty)^4} \frac{\delta^4(x^0 - x'^0 - |\bar{r}-\bar{r}'|)}{4\pi\,|\bar{r}-\bar{r}'|} \, \frac{4\pi}{c} J(\bar{r}') \, d^4x'$$

$$= \frac{1}{c}\int \frac{\delta^4\left(t - t' - \frac{|\bar{r}-\bar{r}'|}{c}\right)}{|\bar{r}-\bar{r}'|} \, \dot{\bar{P}}(t')\,\delta^3(\bar{r}') \, dt' \, d^3\bar{r}'$$

$$= \frac{1}{c} \int_{(\infty)^3} \frac{\dot{\bar{p}}\left(t - \frac{|\bar{r}-\bar{r}'|}{c}\right)}{|\bar{r}-\bar{r}'|} \delta^3(\bar{r}')\, d^3\bar{r}'$$

$$= \frac{\dot{\bar{p}}(t-r/c)}{cr}$$

NOW, $\bar{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \bar{A}}{\partial t}$

$$\sim \quad -\nabla\Phi = \nabla\left\{ \bar{\nabla}\cdot \frac{\bar{p}(t-r/c)}{r} \right\}$$

$$= \nabla\left\{ \frac{\bar{\nabla}\cdot\bar{p}()}{r} + \bar{\nabla}\left(\frac{1}{r}\right)\cdot\bar{p}() \right\}$$

$$\Rightarrow \bar{\nabla}\left(\frac{1}{r}\right) = -\frac{1}{2}(\ )^{-3/4} \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = -\frac{\bar{r}}{r^3}$$

$$\Rightarrow \bar{\nabla}\cdot\bar{p}(t-r/c) \Rightarrow \frac{\partial}{\partial x^i} p^i\left(t - \frac{(x^2+y^2+z^2)^{1/2}}{c}\right)$$

$$= \frac{\partial p^i}{\partial t} \frac{\partial}{\partial x^i}\left(t - \frac{(x^2+y^2+z^2)^{1/2}}{c}\right)$$

$$= \dot{p}^i\left(-\frac{\frac{1}{2}(\ )^{-\frac{1}{2}} 2x^i}{c}\right)$$

$$= -\dot{p}^i \frac{x^i}{cr}$$

$$\Rightarrow \quad \bar{\nabla}\cdot\bar{p}(t-r/c) = -\frac{\dot{\bar{p}}\cdot\bar{r}}{cr}$$

$$\Rightarrow \quad -\nabla\Phi = -\nabla\left\{ \frac{\dot{\bar{p}}()\cdot\bar{r}}{cr^2} + \frac{\bar{r}\cdot\bar{p}()}{r^3} \right\}$$

$$\rightsquigarrow \quad -\bar{\nabla}\left(\frac{\bar{r}\cdot\bar{p}()}{r^3}\right)$$

$$= -\frac{\partial}{\partial x^i}\left(\frac{x^j p^j()}{r^3}\right)$$

$$= -\frac{\partial}{\partial x^i}\left(\frac{1}{r^3}\right)x^j p^j - \frac{\partial}{\partial x^i}(x^j)\frac{p^j()}{r^3} - \frac{\partial}{\partial x^i}(p^j)\frac{x^j}{r^3}$$

$$= +\frac{3\,(2x^i)}{2r^5}x^j p^j - \frac{\delta^{ij}p^j()}{r^3} - \dot{p}^j\frac{\partial}{\partial x^i}\left(t-\frac{r}{c}\right)\frac{x^j}{r}$$

$$= \frac{3x^i x^j p^j}{r^5} - \frac{p^j()}{r^3} - \dot{p}^j\left(-\frac{1}{2}\frac{2x^i}{cr}\right)\frac{x^j}{r^3}$$

$$= \frac{3x^i x^j p^j}{r^5} - \frac{p^j()}{r^3} + \frac{\dot{p}^j x^j\, x^i}{cr^4}$$

$$= \frac{3\bar{r}\;\bar{r}\cdot\bar{p}()}{r^5} - \frac{\bar{p}()}{r^3} + \frac{(\dot{\bar{p}}()\cdot\hat{r})\,\hat{r}}{cr^2}$$

$$\rightsquigarrow \; -\bar{\nabla}\left(\frac{\dot{\bar{p}}()\cdot\bar{r}}{cr^2}\right)$$

$$= -\frac{\partial}{\partial x^i}\left(\frac{\dot{p}^j()\,x^j}{cr^2}\right)$$

$$= -\frac{\partial}{\partial x^i}\left(\frac{1}{r^2}\right)\frac{\dot{p}^j()\,x^j}{c} - \frac{\partial}{\partial x^i}\left(\dot{p}^j()\right)\frac{x^j}{cr^2} - \frac{\partial}{\partial x^i}(x^j)\,\frac{\dot{p}^j()}{cr^2}$$

$$= +\frac{2x^i}{r^4}\,\frac{\dot{p}^j()\,x^j}{c} - \ddot{p}^j\,\frac{\partial}{\partial x^i}(t-r/c)\,\frac{x^j}{cr^2} - \frac{\delta^{ij}\,\dot{p}^j()}{cr^2}$$

$$= \frac{2\bar{r}\;\dot{\bar{p}}()\cdot\bar{r}}{cr^4} - \ddot{p}^j\left(-\frac{1}{2cr}(2x^i)\right)\frac{x^j}{cr^2} - \frac{\dot{\bar{p}}()}{cr^2}$$

$$= \frac{2\hat{r}(\dot{\bar{p}}\cdot\hat{r})}{cr^2} + \frac{\hat{r}(\ddot{\bar{p}}\cdot\hat{r})}{c^2 r} - \frac{\dot{\bar{p}}()}{cr^2}$$

$$\rightsquigarrow \; -\frac{1}{c}\frac{\partial \bar{A}(t,r)}{\partial t} = -\frac{1}{c}\frac{\partial}{\partial t}\left\{\frac{\dot{\bar{p}}(t-r/c)}{cr}\right\}$$

$$= -\frac{\ddot{\bar{p}}(t-r/c)}{c^2 r}$$

$$\bar{E} = \frac{3\hat{r}(\bar{p}\cdot\hat{r}) - \bar{p}}{r^3} + \frac{(\dot{\bar{p}}()\cdot\hat{r})\hat{r}}{c r^2}$$

$$+ \frac{2\hat{r}(\dot{\bar{p}}\cdot\hat{r})}{c r^2} + \frac{\hat{r}(\ddot{\bar{p}}\cdot\hat{r})}{c^2 r} - \frac{\dot{\bar{p}}()}{c r^2}$$

$$- \frac{\ddot{\bar{p}}(t-r/c)}{c^2 r}$$

$$\Rightarrow \bar{E} = \underbrace{\frac{3\hat{r}(\bar{p}\cdot\hat{r}) - \bar{p}}{r^3}}_{\text{near}} + \underbrace{\frac{3\hat{r}(\dot{\bar{p}}()\cdot\hat{r}) - \dot{\bar{p}}}{c r^2}}_{\text{intermediate}}$$

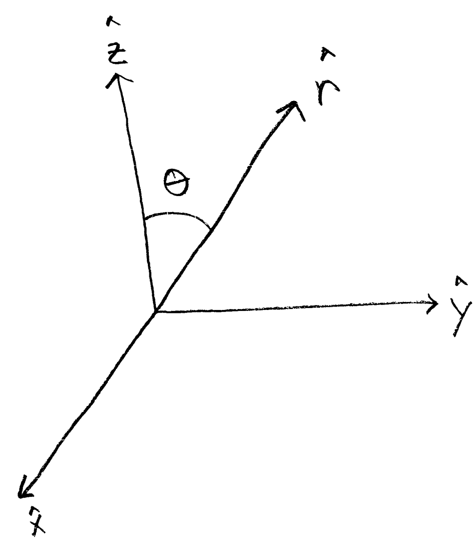
$$+ \underbrace{\frac{\hat{r}(\ddot{\bar{p}}()\cdot\hat{r}) - \ddot{\bar{p}}()}{c^2 r}}_{\text{radiation field / far}}$$

$$\bar{E}_{rad} = \frac{\hat{r}(\ddot{\bar{P}}(\,)\cdot\hat{r}) - \ddot{\bar{P}}(\,)}{c^2 r}$$

$$\bar{B}_{rad} = \hat{r} \times \bar{E}_{rad}$$

$$= \frac{-\hat{r} \times \ddot{\bar{P}}(\,)}{c^2 r}$$

b) $\bar{p}(t) = P_0 \cos(\omega t)\,\hat{z}$



$\Rightarrow \ddot{\bar{p}}(t) = -\omega^2 P_0 \cos(\omega t)\,\hat{z}$

$$\bar{E}_{rad} = \frac{-\omega^2 P_0 \cos(\omega t)\{\hat{r}(\hat{z}\cdot\hat{r}) - \hat{z}\}}{c^2 r}$$

$$= -\underbrace{\frac{\omega^2 P_0 \cos(\omega t)}{c^2 r}}_{=\alpha}(\cos\theta\,\hat{r} - \hat{z})$$

Now, $\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

$$= \alpha \sin\theta\left\{\cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}\right\}$$

$$= \alpha\sin\theta\,\hat{\theta}$$

Since $\bar{E}_{rad}$ points only in $\hat{\theta}$ direction, its plane polarized

$$\rightsquigarrow \ddot{\vec{p}}(t) = P_0\left[-\omega^2\cos(\omega t)\,\hat{x} - \omega^2\sin(\omega t)\,\hat{y}\right]$$

$$\rightsquigarrow \ddot{\vec{p}}\cdot\hat{r} = P_0[\quad]\cdot\left(\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}\right)$$

$$= -\omega^2 P_0 \sin\theta\left[\cos\phi\cos\omega t + \sin\phi\sin\omega t\right]$$

$$\Rightarrow |\ddot{\vec{P}}|^2 = \omega^4 P_0^2$$

$$\Rightarrow (\ddot{\vec{P}}\cdot\hat{r})^2 = \omega^4 P_0^2 \sin^2\theta\left\{\cos^2\phi\cos^2\omega t + \sin^2\phi\sin^2\omega t + 2\sin\phi\cos\phi\sin\omega t\cos\omega t\right\}$$

$$\left\langle \frac{dP(\theta)}{d\Omega}\right\rangle = \frac{1}{4\pi c^3}\left\{\omega^4 P_0^2 - \omega_0^4 P^2 \sin^2\theta\left(\langle\cos^2\phi\rangle\cos^2\omega t + \langle\sin^2\phi\rangle\sin^2\omega t\right.\right.$$

$$\Rightarrow \langle\sin\phi\cos\phi\rangle = 0$$

$$\Rightarrow \langle\sin^2\phi\rangle = \langle\cos^2\phi\rangle = \frac{1}{2}$$

$$= \frac{1}{4\pi c^3}\,\omega^4 P_0^2\left(1 - \frac{1}{2}\sin^2\theta\right)$$

$$= \frac{\omega^4 P_0^2}{8\pi c^3}\left(1 + \cos^2\theta\right)$$

c) $\bar{p}(t) = P_0 \left[\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y} \right]$

$$\vec{S} = \frac{c}{4\pi}\left(\bar{E}_{rad} \times \bar{B}_{rad}\right)$$

$$= \frac{c}{4\pi}\left(\bar{E}_{rad} \times \hat{r} \times E_{rad}\right)$$

$$= \frac{c}{4\pi}\left\{ \hat{r}.(\bar{E}.\bar{E})_{rad} - \bar{E}(\hat{r}.\bar{E}) \right\}$$

$\rightarrow \hat{r}.\bar{E}_{rad} = 0$

$$= \frac{c}{4\pi}|E_{rad}|^2 \hat{r}$$

$$\frac{dP}{d\Omega} = \vec{S}.\hat{r}\, r^2 (1 - \hat{r}.\bar{\beta}) = \frac{c}{4\pi}|E_{rad}|^2 r^2$$

for dipole $\bar{\beta} = 0$

$$|E_{rad}|^2 = \frac{1}{c^4 r^2}\left\{ \hat{r}(\ddot{\bar{p}}.\hat{r}) - \ddot{\bar{p}} \right\}^2$$

$$= \frac{1}{c^4 r^2}\left\{ (\ddot{\bar{p}}.\hat{r})^2 + |\ddot{\bar{p}}|^2 - 2(\ddot{\bar{p}}.\hat{r})^2 \right\}$$

$$= \frac{1}{c^4 r^2}\left\{ |\ddot{\bar{p}}|^2 - (\ddot{\bar{p}}.\hat{r})^2 \right\}$$

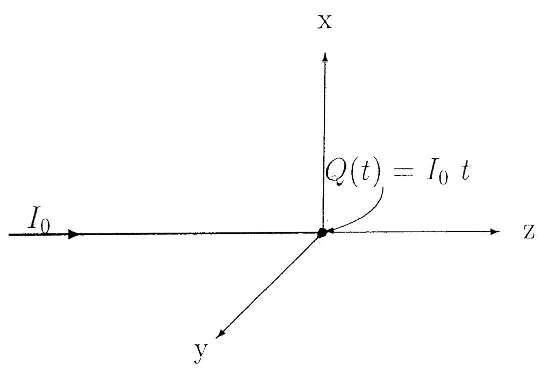
$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3}\left\{ |\ddot{\bar{p}}|^2 - (\ddot{\bar{p}}.\hat{r})^2 \right\}$$

/ 3. 33%

In the absence of polarizable and/or magnetizable material (i.e., only free charges and currents present) Maxwell's equations, in the Lorentz gauge, reduce to the inhomogeneous wave equation:

$$\Box \begin{Bmatrix} \Phi \\ A^x \\ A^y \\ A^z \end{Bmatrix} = \frac{4\pi}{c} \begin{Bmatrix} c\rho \\ J^x \\ J^y \\ J^z \end{Bmatrix}, \quad \text{where } \Box \equiv \left(\frac{\partial}{c\partial t}\right)^2 - \nabla^2 .$$

A time dependent charge $Q(t) = I_0\, t,\ t \geq 0$ is fixed at the origin



of a cylindrical polar coordinate system (ρ, ϕ, z) The charge increases linearly with time because a constant current I_0 flows in along a thin wire attached to the charge on its left, see the figure. Assume the wire carries no current for $t < 0$, however, at $t = 0$ a current I_0 abruptly starts flowing in the +z direction and remains constant for $t \geq 0$. Assume the wire remains neutral as the charge at the origin grows. Find the following quantities at time t for points (ρ, ϕ, z):

(a) The charge density $\rho(t, \rho, \phi, z)$,

(b) The current density $\mathbf{J}(t, \rho, \phi, z)$,

(c) The retarded scalar potential $\Phi(t, \rho, \phi, z)$,

(d) The retarded vector potential $\mathbf{A}(t, \rho, \phi, z)$.

Hints: The retarded Green's function for the $\Box$ operator is:

$$G^{ret}(\mathbf{r}, t; \mathbf{r}', t') = \frac{\delta\left(t - t' - |\mathbf{r} - \mathbf{r}'|/c\right)}{4\pi\, |\mathbf{r} - \mathbf{r}'|}.$$

You might need the integral

$$\int \frac{dX}{\sqrt{X^2 + a^2}} = \ln(\sqrt{X^2 + a^2} + X).$$

* compute E & B fields

Prob 3 (Gaussian)

$$\Box \begin{Bmatrix} \Phi \\ A^x \\ A^y \\ A^z \end{Bmatrix} = \frac{4\pi}{c} \begin{Bmatrix} c\rho \\ J^x \\ J^y \\ J^z \end{Bmatrix} \quad ; \quad \Box \equiv \left(\frac{\partial}{c\,\partial t}\right)^2 - \nabla^2$$

(a) $$\rho(t, x, y, z) = Q(t)\,\delta(x)\,\delta(y)\,\delta(z)\,\Theta(t)$$

$$\Rightarrow \rho(t, \rho, \phi, z) = I_0 t \,\frac{\delta(\rho)\,\delta(z)}{2\pi\rho}\,\Theta(t)$$

(b) $$\bar{\nabla}\cdot\bar{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}\left(I_0 t\,\Theta(t)\,\delta(x)\,\delta(y)\,\delta(z)\right)$$

$$= -I_0\,\Theta(t)\,\delta(x)\,\delta(y)\,\delta(z)$$

$$- I_0\,\delta(t)\,\delta(x)\,\delta(y)\,\delta(z)$$

but $I_0\,\delta(t) = 0$

$$\Rightarrow \frac{\partial J^z}{\partial z} = -I_0\,\Theta(t)\,\delta(x)\,\delta(y)\,\delta(z)$$

$$\Rightarrow J^z = -\int I_0\,\Theta(t)\,\delta(x)\,\delta(y)\,\delta(z)\,dz$$

$$\Rightarrow \bar{J} = I_0\,\Theta(t)\,\delta(x)\,\delta(y)\,\Theta(-z)\,\hat{z}$$

$$= I_0\,\Theta(t)\,\frac{\delta(\rho)}{2\pi\rho}\,\Theta(-z)\,\hat{z}$$

c)

$$A^{\alpha}(x) = \int D_{ret}(x,x') \frac{4\pi}{c} J^{\alpha}(x') d^4x'$$

$$\Phi^{ret}(t,r) = \int \frac{\delta^4 (x^0 - x'^0 - |\bar{r}-\bar{r}'|)}{4\pi |\bar{r}-\bar{r}'|} c\rho(x') d^4x'$$

$$= \int_{(\infty)^3} \frac{\rho\left(t - \frac{|\bar{r}-\bar{r}'|}{c}\right)}{|\bar{r}-\bar{r}'|} d^3\bar{r}'$$

$$= \int I_0\left(t - \frac{|\bar{r}-\bar{r}'|}{c}\right) \Theta\left(t - \frac{|\bar{r}-\bar{r}'|}{c}\right) \frac{\delta^3(\bar{r}')}{|\bar{r}-\bar{r}'|} d^3\bar{r}'$$

$$= \frac{I_0}{|\bar{r}|}\left(t - \frac{r}{c}\right) \Theta\left(t - \frac{r}{c}\right)$$

d)

$$\bar{A}^{ret}(x) = \frac{1}{c}\int_{(\infty)^3} \frac{\bar{J}\left(t-\frac{|\bar{r}-\bar{r}'|}{c}\right)}{|\bar{r}-\bar{r}'|} d^3r'$$

$$|\bar{r}-\bar{r}'| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$

$$= \frac{\hat{z}}{c}\int \frac{I_0 \Theta\left(t-\frac{|\bar{r}-\bar{r}'|}{c}\right)\Theta(-z')\,\delta(x')\,\delta(y')}{\left\{(x-x')^2+(y-y')^2+(z-z')^2\right\}^{1/2}} d^3r'$$

$$= \frac{\hat{z}}{c}\int \frac{I_0 \,\Theta\left(t-\frac{\sqrt{x^2+y^2+(z-z')^2}}{c}\right)\Theta(-z')}{\sqrt{x^2+y^2+(z-z')^2}} dz'$$

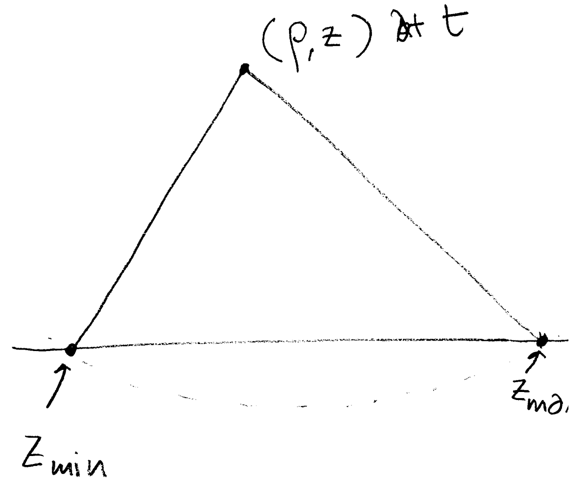
$$= \frac{\hat{z}}{c}\int_{-\infty}^{+\infty} I_0 \frac{\Theta\left(t-\frac{\sqrt{\rho^2+(z-z')^2}}{c}\right)}{\sqrt{\rho^2+(z-z')^2}}\Theta(-z')\,dz'$$

* The argument of the Heaviside func^c has to be positive

i.e. $t - \frac{1}{c}\sqrt{\rho^2+(z-z')^2} > 0$

$$\Rightarrow \quad ct > \sqrt{\rho^2+(z-z')^2}$$

$$\Rightarrow \quad (ct)^2-\rho^2 > (z-z')^2$$



for, $z - z' > 0$

$$\sqrt{(ct)^2 - \rho^2} > z - z'$$

$$\Rightarrow z' > z - \sqrt{(ct)^2 - \rho^2}$$

for, $z' - z > 0$

$$\sqrt{(ct)^2 - \rho^2} > z' - z$$

$$\Rightarrow z' < z + \sqrt{(ct)^2 - \rho^2}$$

$$\rightsquigarrow z_{min} = z - \sqrt{(ct)^2 - \rho^2}$$

$$\rightsquigarrow z_{max} = z + \sqrt{(ct)^2 + \rho^2}$$

- **Case 1:** $0 < z_{min} < z < z_{max}$

$$\bar{A} = (0) \quad \because \quad \Theta(-z) = 0$$

• Case 2: $z_{max} > 0$

$$\bar{A} = \frac{\hat{z}}{c}\int_{z_{min}}^{0} \frac{I_0 \, dz'}{\sqrt{\rho^2 + (z-z')^2}}$$

$z_{min} = z - \sqrt{(ct)^2 - \rho^2}$

$z - z' = x$

$\Rightarrow dz' = -dx$

$$= \frac{\hat{z}}{c}\int_{z}^{\sqrt{(ct)^2-\rho^2}} \frac{I_0 \, dx}{\sqrt{\rho^2 + x^2}}$$

$z' = 0, \; x = z$

$z' = z_{min},$

$x = \sqrt{(ct)^2 - \rho^2}$

$$= \frac{\hat{z} I_0}{c}\left[\ln\left(\sqrt{x^2+\rho^2} + x\right)\right]_{z}^{\sqrt{(ct)^2-\rho^2}}$$

$$= \hat{z} I_0 \ln\left\{\frac{ct + \sqrt{(ct)^2 - \rho^2}}{\sqrt{z^2+\rho^2} + z}\right\}$$

• Case 3 $\quad z_{max} < 0$

$$\bar{A}(t,r) = \frac{\hat{z}}{c}\int_{z_{min}}^{z_{max}} \frac{I_0 \, dz'}{\sqrt{\rho^2 + (z-z')^2}}$$

$z - z' = x$

$\Rightarrow dz' = -dx$

$$= \frac{\hat{z} I_0}{c}\int_{-\sqrt{(ct)^2-\rho^2}}^{\sqrt{(ct)^2-\rho^2}} \frac{dx}{\sqrt{\rho^2+x^2}}$$

$z' = z_{min}$

$x = \sqrt{(ct)^2-\rho^2}$

$z' = z_{max}$

$x = -\sqrt{(ct)^2-\rho^2}$

$$= \frac{\hat{z} I_0}{c}\left[\ln \sqrt{x^2+\rho^2} + x\right]$$

$$= \hat{z}\frac{I_0}{c}\ln\left\{\frac{ct + \sqrt{(ct)^2-\rho^2}}{ct - \sqrt{(ct)^2-\rho^2}}\right\}$$

e)

$$\bar{E} = -\nabla\bar{\Phi} - \frac{1}{c}\frac{\partial\bar{A}}{\partial t}$$

$$\rightsquigarrow \quad -\nabla\bar{\Phi} = -\nabla\left\{\frac{I_0(t-\frac{r}{c})}{|\bar{r}|}\,\Theta(t-\tfrac{r}{c})\right\}$$

$$= -\nabla\left(\frac{1}{r}\right) I_0(t-r/c)\,\Theta(t-r/c)$$

$$-\nabla(t-r/c)\frac{I_0}{r}\,\Theta(t-r/c)$$

$$-\frac{I_0}{r}(t-r/c)\,\bar{\nabla}\left(\Theta(t-r/c)\right)$$

$$= \frac{1}{2}\frac{1}{r^3}\,2\bar{r}\;I_0(t-r/c)\,\Theta(t-r/c)$$

$$+\frac{1}{2c}\frac{1}{r}\,2\bar{r}\;\frac{I_0}{r}\,\Theta(t-r/c)$$

$$+\frac{I_0}{r}(t-r/c)\,\delta(t-r/c)\,\frac{1}{2}\frac{1}{rc}\,2\bar{r}$$

$$= \frac{\bar{r}}{r^3}\,I_0\,(t-r/c)\,\Theta(t-r/c) + \frac{\bar{r}}{cr^2}\,I_0\,\cancel{\Theta}(t-r/c)$$

$$+\frac{\bar{r}}{r^2c}\,I_0\,\delta(t-r/c)\,(t-r/c)$$

$$\rightsquigarrow \quad -\nabla\Phi = \frac{\bar{r}}{r^3} I_0 t \; H(t - r/c) + \frac{\bar{r}}{c r^2} I_0 \, (t - r/c) \, \delta(t - r/c)$$

$$= \frac{\hat{r}}{r^2} I_0 t \; H\left(t - \frac{r}{c}\right) + \frac{\hat{r}}{c r} I_0 t \, \delta(t - r/c)$$

$$- \hat{r} \frac{I_0}{c^2} \delta\left(t - \frac{r}{c}\right)$$

$$\bar{B} = \nabla \times \bar{A}$$

$$= \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ A^z \end{pmatrix}$$

$$= \frac{\partial A_z}{\partial y}\hat{x} - \frac{\partial A_z}{\partial x}\hat{y}$$

for $z_{max} < 0$

$$\bar{A}(t, \bar{r}) = \hat{z}\frac{I_0}{c} \ln\left\{ \frac{\left[ct + \sqrt{(ct)^2 - \rho^2}\right]}{\left(ct - \sqrt{(ct)^2 - \rho^2}\right)} \right\}$$

$$\frac{\partial A^z}{\partial y} = \frac{I_0}{c}\frac{1}{\{\ \}} \frac{(\ )\frac{1}{2}(\sqrt{\ })^{-1}(-2y) - [\ ](-\frac{1}{2}(\sqrt{\ })^{-1}(-2y)}{\left(ct - \sqrt{(ct)^2 - \rho^2}\right)^2}$$

$$= \frac{I_0}{c\{\ \}} \frac{-\left(ct - \sqrt{(ct)^2 - \rho^2}\right)\frac{y}{\sqrt{(ct)^2 - \rho^2}} - \left(ct + \sqrt{(ct)^2 - \rho^2}\right)\frac{y}{\sqrt{(ct)^2 - \rho^2}}}{(\ )^2}$$

$$= \frac{I_0}{c} \frac{ct - \sqrt{(ct)^2 - \rho^2}}{ct + \sqrt{(ct)^2 + \rho^2}} \frac{y}{\sqrt{(ct)^2 - \rho^2}} \frac{-2ct}{\left(ct - \sqrt{(ct)^2 - \rho^2}\right)^2}$$

$$= \frac{I_0}{c} \frac{y}{\rho^2} \frac{(-2ct)}{\sqrt{(ct)^2 - \rho^2}}$$

$$= -2I_0 t \frac{y}{(x^2 + y^2)\sqrt{c^2t^2 - x^2 - y^2}}\hat{x}$$

$$\frac{\partial A_z}{\partial x} = -2I_0 t \frac{x}{(x^2 + y^2)\sqrt{c^2t^2 - x^2 - y^2}}\hat{y}$$

$$\bar{B} = \frac{-2I_0 t}{\rho^2\sqrt{(ct)^2-\rho^2}}\underbrace{(y\hat{x} - x\hat{y})}_{\shortparallel}$$

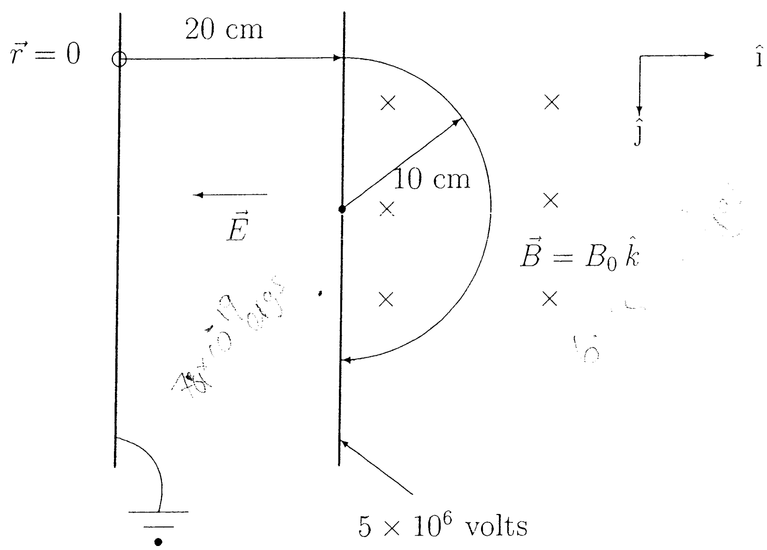
$$\left\{\rho\sin\phi\,(\cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi}) - \rho\cos\phi\,(\sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi})\right.$$

$$= -\rho(\sin^2\phi + \cos^2\phi)\,\hat{\phi}$$

$$\bar{B} = \frac{2I_0 t}{\rho\sqrt{(ct)^2-\rho^2}}\,\hat{\phi}$$

2. 33%

An electron starts from $\vec{r} = 0$ at rest on the negative plate of a capacitor and is accelerated 20 cm to the positive plate through a potential difference of 5×10^6 Volts (see the figure). The electron then (through a pin hole in the positive plate) enters a uniform magnetic field $\vec{B}_0 = B_0 \hat{k}$ which turns the electron in a semi-circle orbit of radius 10 cm.



(a) Find the (lab) time the electron takes to reach the positive plate and the (lab) time it takes to move around the 1/2 circle.

(b) How much total energy does the electron loose as radiation during its linear acceleration in the capacitor and during its 1/2-circle orbit in the magnetic field? What fraction of the electron's total energy is lost to radiation?

(c)

{Hints: The Larmor formula is

$$P(t) = -\frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}\right) = \frac{2}{3}\frac{q^2}{c}\gamma^6\,[(\dot{\vec{\beta}})^2 - (\vec{\beta} \times \dot{\vec{\beta}})^2]$$

and the acceleration is caused by the Lorentz force

$$\frac{dp^\mu}{d\tau} = \frac{q}{c}F^{\mu\lambda}u_\lambda,$$

$m_ec^2 = 0.5MeV$, 1 eV=1.6×10^{-12} ergs, e=4.8×10^{-10} statcoul. }

Prob 2 (Gaussian)

$V = 5\times10^{6}$ Volts

$d = 20$ cm

Initial energy $=$ rest mass energy $+ qV_0$

$= mc^2 + 0$

Final energy $= \gamma mc^2 + qV$

Thus,

$$\gamma mc^2 + qV = mc^2$$

$$\Rightarrow \gamma = 1 - \frac{qV}{mc^2} = 1 + \frac{eV}{mc^2}$$

$$\Rightarrow \gamma_{max} = 1 + \frac{eV}{mc^2} \quad \& \quad \beta_{max} = \sqrt{1-\gamma_{max}^{-2}} = \left\{1 - \frac{1}{\left(1+\frac{eV}{mc^2}\right)^2}\right\}^{1/2}$$

$$= \left\{\frac{\left(1+\frac{eV}{mc^2}\right)^2 - 1}{\left(1+\frac{eV}{mc^2}\right)^2}\right\}^{1/2}$$

If the potential difference is V

then, $E = \frac{V}{d}$

Now,

$$\frac{d\bar{P}}{d\tau} = e\bar{E}$$

$$\Rightarrow mc\frac{d}{dt}(\gamma\beta) = \frac{eV}{d}$$

$$\Rightarrow \frac{d}{dt}(\gamma\beta) = \frac{eV}{mcd}$$

$$\Rightarrow \int_0^{(\gamma\beta)_{max}} d(\gamma\beta) = \int_{t_i}^{t_f} \frac{eV}{mcd}\,dt$$

$$\Rightarrow \frac{eV}{mcd}(t_f - t_i) = (\gamma\beta)_{max} = \sqrt{\frac{eV}{mc^2}\left(2 + \frac{eV}{mc^2}\right)}$$

$$\Rightarrow \Delta t = \frac{mcd}{eV}\left\{\frac{eV}{mc^2}\left(2 + \frac{eV}{mc^2}\right)\right\}^{1/2}$$

The power radiated is

$$P = \frac{2}{3}\frac{e^2}{c}\gamma^6\left((\dot{\bar{\beta}})^2 - (\bar{\beta}\times\dot{\bar{\beta}})^2\right)$$

$$\rightsquigarrow \bar{\beta} \parallel \dot{\bar{\beta}} \Rightarrow \bar{\beta}\times\dot{\bar{\beta}} = 0$$

$$P(t) = \frac{2}{3}\frac{e^2}{c}\gamma^6(\dot{\bar{\beta}})^2$$

Again

$$\frac{dP}{d\tau} = eE$$

$$\dot{\gamma} = \gamma^3 \bar{\beta} \dot{\bar{\beta}}$$

$$\Rightarrow \frac{d}{dt}(\gamma \bar{\beta}) = \frac{eV}{mcd}$$

$$\Rightarrow \dot{\gamma} \bar{\beta} + \gamma \dot{\bar{\beta}} = \frac{eV}{mcd}$$

$$\gamma^2 = (1-\beta^2)^{-1}$$

$$\Rightarrow \gamma^3 \bar{\beta} \dot{\bar{\beta}} \bar{\beta} + \gamma \dot{\bar{\beta}} = (\ )$$

$$\beta^2 = 1 - \gamma^{-2}$$

$$\Rightarrow \gamma \dot{\bar{\beta}} \underbrace{(\gamma^2 \beta^2 + 1)}_{=\gamma^2} = \frac{eV}{mcd}$$

$$\gamma^2 \beta^2 + 1 = \gamma^2(1-\gamma^{-2}) + 1 = \gamma^2$$

$$\Rightarrow \gamma^3 \dot{\bar{\beta}} = \frac{eV}{mcd}$$

$$\Rightarrow \dot{\bar{\beta}} = \frac{eV}{\gamma^3 mcd}$$

$$P = \frac{2}{3} e^2 \gamma^6 (\dot{\bar{\beta}})^2$$

$$P = \frac{dW}{dt}$$

$$\Rightarrow dW = P\,dt$$

$$\Rightarrow \Delta W = \frac{2e^4 V^2}{3 m^2 c^2 d^2} \Delta t$$

$$\bar{E}_{rad} = \frac{e}{c} \frac{\hat{n} \times (\hat{n} - \bar{\beta}) \times \dot{\bar{\beta}}}{(1 - \bar{\beta} \cdot \hat{n})^3 R}$$