Physics 4183 Electricity and Magnetism II
Waveguides

1 Introduction

An important application of electrodynamics in bounded regions is that of guiding signals from location to location through the use of wave guides. Waveguides can be as simple as a wire or trace on a printed circuit board, or more complex, such as coaxial cables, rectangular waveguides, and fiber optic. All of these rely on the same principles requiring that the fields match boundary condition on the various surface.

The discussion in this set of lectures will concentrate on rectangular waveguides. We will start the discussion by giving a physical description of how wave travels through the guide by using the case a pair of conducting parallel plates. From this point we will move on to formally solve the problem by applying the boundary conditions required by the Maxwell equations.

1.1 Propagation Between Parallel Conducting Plates

As a simple physical problem, let's consider an electromagnetic wave confined between two parallel conducting planes separated by a distance b (see Fig. 1). As for any other case where we have waves incident on an interface, the problem can be broken up into two case, the TE case where the electric field is perpendicular to the plane of the wave vectors, and the TM case where the magnetic field is perpendicular to the plane of the wave vectors.

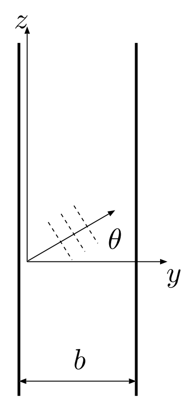

Figure 1: The figure shows the wave along with its direction confined between two parallel conducting plates. The parallel dashed lines correspond to the plane of constant phase, which is also the plane the fields are in.

To determine the conditions that the conducting plates impose on the wave, let's take a wave that is incident on the plates making an angle θ with respect to the y-axis and propagates in the y-z plane. From our previous discussion, and assuming the plates are perfect conductors ($\sigma = \infty$)

the amplitude of the reflected waves is

$$E_r^s = \frac{n_1\cos\theta_i - \sqrt{\mathcal{N}^2 - n_1\sin\theta_i}}{n_1\cos\theta_i + \sqrt{\mathcal{N}^2 - n_1\sin\theta_i}} = -E_i^s \qquad E_r^p = \frac{-\mathcal{N}^2\cos\theta_i + \sqrt{\mathcal{N}^2 - n_1\sin\theta_i}}{\mathcal{N}^2\cos\theta_i + \sqrt{\mathcal{N}^2 - n_1\sin\theta_i}} = -E_i^p \quad (1)$$

where $\mathcal{N}$ is the complex index of refraction of media 2 (the conducting walls), and we have used the approximation for a good conductor

$$\mathcal{N} \approx (1+i)\sqrt{\frac{\mu\sigma}{2\omega\mu_0\epsilon_0}} \quad (2)$$

Notice that in both cases the amplitude (E_0) is inverted, this allows for the boundary conditions to be satisfied at the surface. The component of the electric field parallel to the surface cancel (incident + reflected). This is required since the parallel component of the field must be continuous across the interface and it must be zero inside a conductor. The components of the electric field normal to the surface for the incident and reflected waves add, since a charge is expected to be induced on the surface. The remaining discussion will focus only on the TE case, the TM case is similar. The wave-vector for the incident wave is

$$\vec{\mathbf{k}}_i = k(\hat{\mathbf{y}}\cos\theta + \hat{\mathbf{z}}\sin\theta) \quad (3)$$

and for the reflected wave it is

$$\vec{\mathbf{k}}_r = k(-\hat{\mathbf{y}}\cos\theta + \hat{\mathbf{z}}\sin\theta) \quad (4)$$

The electric field associated with the TE wave between the plates is

$$\vec{\mathbf{E}} = \hat{\mathbf{x}}E_0\left[e^{ik(y\cos\theta + z\sin\theta - \omega t)} - e^{ik(-y\cos\theta + z\sin\theta-\omega t)}\right] \quad (5)$$

At $y = 0$ the electric field has to vanish since it is parallel to the surface, inside a conductor there is no field, and the boundary conditions require it to be continuous (as stated above). This condition is automatically satisfied, since we used the calculated reflection amplitude, which already had this condition imposed. Next, the field is required to be zero at $y = b$ for the same reason as at $y = 0$. First we rewrite Eq. 5 as follows

$$\vec{\mathbf{E}} = \hat{\mathbf{x}}E_0\left[e^{iky\cos\theta} - e^{-iky\cos\theta}\right]e^{i(kz\sin\theta-\omega t)} = \hat{\mathbf{x}}E_0\left[2i\sin(ky\cos\theta)\right]e^{i(kz\sin\theta-\omega t)} \quad (6)$$

From this expression, the condition that the field be zero at $y = b$, is easily seen to be

$$\sin(kb\cos\theta) = 0 \quad\Rightarrow\quad kb\cos\theta = n\pi \quad \text{with} \quad n = 1,\,2,\,3,\,\ldots \quad (7)$$

where we ignore $n = 0$, since it leads to a trivial solution; $\vec{\mathbf{E}} = 0$ for all times and all positions. Notice that we have set up a standing wave in the y direction, but the wave continues to propagate along the z-axis.

To understand what the boundary condition imposes on the wave as it propagates between the plates, we write the wave-vector as a two wavelengths: One associated with the standing wave, and one with the propagating wave. The wavelength associated with the y-direction (standing wave) is

$$k_c = k\cos\theta \quad\Rightarrow\quad \frac{2\pi}{\lambda_c} = \frac{2\pi}{\lambda_0}\cos\theta \quad\Rightarrow\quad \lambda_c = \frac{2\pi}{k\cos\theta} = \frac{\lambda_0}{\cos\theta} \quad \text{where} \quad \lambda_0 = \frac{2\pi}{k} \quad (8)$$

and λ_0 is the wavelength in vacuum. The wavelength associated with the z-direction (propagation) is

$$k_g = k\sin\theta \quad \Rightarrow \quad \lambda_g = \frac{2\pi}{k\sin\theta} = \frac{\lambda_0}{\sin\theta} \tag{9}$$

Notice that the three wavelengths are related by

$$k^2 = k_c^2 + k_g^2 \quad \Rightarrow \quad \frac{1}{\lambda_c^2} + \frac{1}{\lambda_g^2} = \frac{1}{\lambda_0^2} \quad \Rightarrow \quad \frac{1}{\lambda_0^2} - \frac{1}{\lambda_c^2} = \frac{1}{\lambda_g^2} \tag{10}$$

The value of λ_c is fixed by the boundary conditions, and the value of the incident angle. If we take $n = 1$, this gives $\lambda_c = 2b/n = 2b$, next increase λ_0 such that λ_g is negative, the wave no longer propagates, but is extinguished since λ_g is imaginary

$$\vec{\mathbf{E}} = \hat{\mathbf{x}}E_0\left[\sin(2\pi y/\lambda_c)\right]e^{-2\pi z/\lambda_g}e^{-i\omega t} \quad \text{where} \quad E_0 \to 2iE_0 \quad \text{and} \quad \lambda_0 > \lambda_c \tag{11}$$

Another interesting phenomena occurs in the waveguide. The phase velocity is greater than c. This can be seen by noticing that the phase in the direction z direction has to advance in the same time a longer distance than in the $\hat{\mathbf{k}}$ direction. The phase velocity along the waveguide is

$$v_p = \frac{\lambda_0}{\sin\theta}\frac{1}{\Delta t} = \frac{c}{\sin\theta} \tag{12}$$

The group velocity is given by $d\omega/dk_g$. Using Eq. 10, the group velocity is found to be

$$ck_0 = \omega = c\sqrt{k_g^2 - k_c^2} \quad \Rightarrow \quad v_g = \frac{d\omega}{dk_g} = c\frac{k_g}{k} = c\sin\theta \tag{13}$$

notice that the $v_g v_p = c^2$. The group velocity is the velocity at which energy (signals) propagate, therefore being consistent with special relativity.

1.2 Waves in Hollow Conductors

We will now consider the general case of an electromagnetic wave bound by conducting surfaces on all sides; the surfaces are assumed to be perfect conductors and the bounded media vacuum. The only condition that we will impose is that the bounding surface have a uniform cross section. We start with the boundary conditions on the surface of the conductor

$$\vec{\mathbf{E}}_\parallel = 0 \qquad\qquad \vec{\mathbf{B}}_\perp = 0 \tag{14}$$

$$\vec{\mathbf{E}}_\perp = \frac{\sigma}{\epsilon_0} \qquad\qquad \vec{\mathbf{B}}_\parallel = \mu_0\vec{\mathbf{K}} \tag{15}$$

The boundary conditions on the first line determine the shape of the field. The second set determine the induced currents and charges of the boundaries once the fields are known.

To determine the wave solutions that propagate through the guide, we align the z-axis of our coordinate system along the axis of the guide. Since the wave is assumed to propagate in the z-direction, we assume that the fields associated with the wave have the form

$$\vec{\mathbf{E}} = \vec{\boldsymbol{\mathcal{E}}}(x,y)\,e^{i(k_g z - \omega t)} \qquad\qquad \vec{\mathbf{B}} = \vec{\boldsymbol{\mathcal{B}}}(x,y)\,e^{i(k_g z - \omega t)} \tag{16}$$

where I have used the notation k_g from the last section. With this assumed form of the field, the Maxwell equations between the conductors in Cartesian coordinates take on the following forms: The divergence of the electric field

$$\vec{\nabla} \cdot \vec{\mathbf{E}} = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{E}_x}{\partial x} + \frac{\partial \mathcal{E}_y}{\partial y} + ik_g \mathcal{E}_z = 0 \tag{17}$$

The divergence of the magnetic field

$$\vec{\nabla} \cdot \vec{\mathbf{B}} = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{B}_x}{\partial x} + \frac{\partial \mathcal{B}_y}{\partial y} + ik_g \mathcal{B}_z = 0 \tag{18}$$

The curl of the electric field

$$\vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \quad \Rightarrow \quad \begin{cases} \dfrac{\partial \mathcal{E}_z}{\partial y} - ik_g \mathcal{E}_y = i\omega \mathcal{B}_x \\ ik_g \mathcal{E}_x - \dfrac{\partial \mathcal{E}_z}{\partial x} = i\omega \mathcal{B}_y \\ \dfrac{\partial \mathcal{E}_y}{\partial x} - \dfrac{\partial \mathcal{E}_x}{\partial y} = i\omega \mathcal{B}_z \end{cases} \tag{19}$$

The curl of the magnetic field

$$\vec{\nabla} \times \vec{\mathbf{B}} = \frac{1}{c^2}\frac{\partial \vec{\mathbf{E}}}{\partial t} \quad \Rightarrow \quad \begin{cases} \dfrac{\partial \mathcal{B}_z}{\partial y} - ik_g \mathcal{B}_y = -\dfrac{i\omega}{c^2} \mathcal{E}_x \\ ik_g \mathcal{B}_x - \dfrac{\partial \mathcal{B}_z}{\partial x} = -\dfrac{i\omega}{c^2} \mathcal{E}_y \\ \dfrac{\partial \mathcal{B}_y}{\partial x} - \dfrac{\partial \mathcal{B}_x}{\partial y} = -\dfrac{i\omega}{c^2} \mathcal{E}_z \end{cases} \tag{20}$$

To determine the fields in the waveguide, take the second of the curl of $\vec{\mathbf{E}}$ equations and the first of the curl of $\vec{\mathbf{B}}$ equations, and solve for $\mathcal{E}_x$ and $\mathcal{B}_y$ in terms of the z component of the fields

$$\mathcal{E}_x = \frac{i}{k_c^2}\left(k_g \frac{\partial \mathcal{E}_z}{\partial x} + \omega \frac{\partial \mathcal{B}_z}{\partial y}\right) \tag{21}$$

$$\mathcal{B}_y = \frac{i}{k_c^2}\left(k_g \frac{\partial \mathcal{B}_z}{\partial y} + \frac{\omega}{c^2} \frac{\partial \mathcal{E}_z}{\partial x}\right) \tag{22}$$

Likewise, we can solve for $\mathcal{E}_y$ and $\mathcal{B}_x$ in terms of the z component of the fields using the first of the curl of $\vec{\mathbf{E}}$ equations and the second of the curl of $\vec{\mathbf{B}}$ equations

$$\mathcal{E}_y = \frac{i}{k_c^2}\left(k_g \frac{\partial \mathcal{E}_z}{\partial y} - \omega \frac{\partial \mathcal{B}_z}{\partial x}\right) \tag{23}$$

$$\mathcal{B}_x = \frac{i}{k_c^2}\left(k_g \frac{\partial \mathcal{B}_z}{\partial x} - \frac{\omega}{c^2} \frac{\partial \mathcal{E}_z}{\partial y}\right) \tag{24}$$

where $k_c^2 = \omega^2/c^2 - k_g^2 = k_0^2 - k_g^2$. This set of equations give the transverse fields in terms of the longitudinal fields. We would still like a set of equations that contain only the longitudinal fields. These can be found by substituting $\mathcal{E}_x$ and $\mathcal{E}_y$ into Eq 17, and $\mathcal{B}_x$ and $\mathcal{B}_y$ into Eq. 18

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) \mathcal{E}_z = 0 \qquad\qquad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) \mathcal{B}_z = 0 \tag{25}$$

or in a more compact notion, which is more general

$$\left(\nabla_T^2 + k_c^2\right)\mathcal{E}_z = 0 \qquad\qquad \left(\nabla_T^2 + k_c^2\right)\mathcal{B}_z = 0 \tag{26}$$

where ∇_T^2 is the transverse Laplacian operator, therefore we can work in any coordinate system we choose.

Let's now ask what type of waves can propagate down the guide. First we consider the case where the fields are transverse (TEM). In this case $\mathcal{E}_z = \mathcal{B}_z = 0$. The equations for the divergence of the fields in this case become

$$\frac{\partial \mathcal{E}_x}{\partial x} + \frac{\partial \mathcal{E}_y}{\partial y} = 0 \qquad\qquad \frac{\partial \mathcal{B}_x}{\partial x} + \frac{\partial \mathcal{B}_y}{\partial y} = 0 \tag{27}$$

and the third of the two curl equations become

$$\frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y} = 0 \qquad\qquad \frac{\partial \mathcal{B}_y}{\partial x} - \frac{\partial \mathcal{B}_x}{\partial y} = 0 \tag{28}$$

The curl equations imply that each of the transverse fields can be written as potentials

$$\mathcal{E}_x = -\frac{\partial \Phi_E}{\partial x} \qquad \mathcal{E}_y = -\frac{\partial \Phi_E}{\partial y} \qquad\qquad \mathcal{B}_x = -\frac{\partial \Phi_B}{\partial x} \qquad \mathcal{B}_y = -\frac{\partial \Phi_B}{\partial y} \tag{29}$$

Since the bounding surface is a conductor, it is at a constant electric potential. Therefore, the electric field along the surface is zero. Equation 27 states that the electric field is constant inside the region bounded by the conductors. Therefore, the electric field must be zero everywhere. Using the curl equations for the electric and magnetic fields, the magnetic field is likewise zero. Therefore, we must conclude that no TEM wave can propagate through a hollow waveguide; we have assumed that there is only one continuous conducting surface, so only in this case does this hold.

Let's now consider the other two possible modes, TE where $\mathcal{E}_z = 0$, and TM where $\mathcal{B}_z = 0$. For the TE mode, Eqs. 21-24 become

$$\begin{aligned} \mathcal{E}_x &= c\frac{ik_0}{k_c^2}\frac{\partial \mathcal{B}_z}{\partial y} & \qquad\qquad \mathcal{E}_y &= -c\frac{ik_0}{k_c^2}\frac{\partial \mathcal{B}_z}{\partial x} \\ \mathcal{B}_x &= \frac{ik_g}{k_c^2}\frac{\partial \mathcal{B}_z}{\partial x} & \qquad\qquad \mathcal{B}_y &= \frac{ik_g}{k_c^2}\frac{\partial \mathcal{B}_z}{\partial y} \end{aligned} \tag{30}$$

The lower set of equations lead to the following relation between the transverse gradient of the longitudinal magnetic field and the transverse magnetic field components

$$\vec{\boldsymbol{\nabla}}_T \mathcal{B}_z = -\frac{ik_c^2}{k_g}\vec{\boldsymbol{\mathcal{B}}}_t \tag{31}$$

Combining this equation with the upper equations of Eq. 30, leads to

$$\vec{\boldsymbol{\mathcal{E}}}_t = -\frac{ck_0}{k_g}(\hat{\mathbf{z}} \times \vec{\boldsymbol{\mathcal{B}}}_t) \tag{32}$$

A similar set of relations can be derived for the TM case, where $\mathcal{B}_z = 0$

$$\vec{\boldsymbol{\mathcal{B}}}_t = \frac{k_0}{k_g c}(\hat{\mathbf{z}} \times \vec{\boldsymbol{\mathcal{E}}}_t) \qquad\qquad \vec{\boldsymbol{\nabla}}_T \mathcal{E}_z = -\frac{ik_c^2}{k_g}\vec{\boldsymbol{\mathcal{E}}}_t \tag{33}$$

These two sets of equations (Eqs. 31-33), along with the differential equations

$$(\nabla_T^2 + k_c^2)\mathcal{B}_z = 0 \quad \text{TE} \tag{34}$$
$$(\nabla_T^2 + k_c^2)\mathcal{E}_z = 0 \quad \text{TM} \tag{35}$$

and the boundary conditions

$$\begin{aligned} \vec{\mathbf{E}}_{\parallel}|_S &= \hat{\mathbf{n}} \times \vec{\mathbf{E}}|_S = 0 \\ \vec{\mathbf{B}}_{\perp}|_S &= \hat{\mathbf{n}} \cdot \vec{\mathbf{B}}|_S = 0 \end{aligned} \tag{36}$$

give the fields in the waveguide. It should be clear from these equations that the TE fields are completely defined in terms of $\mathcal{B}_z$, while the TM fields are completely defined in terms of $\mathcal{E}_z$. To see what these boundary conditions mean, let's define a local coordinate system on the inner wall of the waveguide as follows

- z is parallel to the axis of the waveguide;
- y is normal to the surface;
- x is parallel to the surface, and normal to the y-z plane ($\hat{\mathbf{y}} \times \hat{\mathbf{z}}$).

For the TE mode ($\mathcal{E}_z = 0$, Eq. 36 imply that $\mathcal{E}_x = 0$ and $\mathcal{B}_y = 0$. But Eq. 30 says that these two components are proportional to $\frac{\partial \mathcal{B}_z}{\partial y}$, which leads to

$$\left.\frac{\partial \mathcal{B}_z}{\partial n}\right|_S \equiv \left.\frac{\partial \mathcal{B}_z}{\partial y}\right|_S = 0 \tag{37}$$

For the TM case ($\mathcal{B}_z = 0$), the boundary conditions (Eq. 36) require that $\mathcal{E}_x|_S = 0$, $\mathcal{E}_z|_S = 0$, and $\mathcal{B}_y = 0$. We find from Eq. 21 that

$$\frac{\partial \mathcal{E}_z}{\partial x} = 0 \tag{38}$$

which is automatically implied, since $\mathcal{E}_z = 0$ and the derivative is along the surface. The only condition for the TM mode is

$$\mathcal{E}_z|_S = 0 \tag{39}$$

1.3 Rectangular Waveguide

The most common shape of a hollow waveguide is rectangular, which we will consider in this section. We assume that the waveguide has dimensions $a \times b$ with $a > b$, and that the walls are perfect conductors forming an equipotential surface. The most common waveguides are used for transporting TE waves, therefore we start with the TE case. The longitudinal (z) component of the magnetic field, from which all the other components can be calculated, is found by solving the differential equation

$$(\nabla_T^2 + k_c^2)\mathcal{B}_z = 0 \tag{40}$$

Since the there are no cross terms involving the coordinates, the equation can be solved by separation of variables where we assume a solution of the form $\mathcal{B}_z(x, y) = X(x)Y(y)$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) X(x)Y(y) = 0 \quad \Rightarrow \quad \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = -k_c^2 \tag{41}$$

Since the two terms on the right hand side depend on a single variable, the only way that the equality can be satisfied is if the two terms are independently equal to a constant

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = -\alpha^2 \qquad \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = -\beta^2 \quad \text{where} \quad k_c^2 = \alpha^2 + \beta^2 \tag{42}$$

The two differential equations are trivial to solve, the solutions being

$$X(x) = A\sin\alpha x + B\cos\alpha x \qquad Y(y) = C\sin\beta y + D\cos\beta y \tag{43}$$

We now need to apply the boundary conditions. These are given by Eq. 36. Along the surfaces at $x = 0$ and $x = a$, the boundary conditions are

$$\mathcal{E}_y = 0 \qquad \mathcal{B}_x = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{B}_z}{\partial x} = 0 \tag{44}$$

where the equality comes from Eqs. 30. Along the surfaces at $y = 0$ and $y = b$, the boundary conditions are

$$\mathcal{E}_x = 0 \qquad B_y = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{B}_z}{\partial y} = 0 \tag{45}$$

Starting with the surface at $x = 0$, we get

$$\frac{\partial \mathcal{B}_z}{\partial x} = A\alpha\cos\alpha x - B\beta\sin\alpha x \quad \Rightarrow \quad \left.\frac{\partial \mathcal{B}_z}{\partial x}\right|_{x=0} = 0 = A\alpha \quad \Rightarrow \quad A = 0 \tag{46}$$

where we impose the condition on A not α to avoid getting a trivial solution. The boundary conditions at $x = b$ are

$$\frac{\partial \mathcal{B}_z}{\partial x} = -B\beta\sin\alpha x \quad \Rightarrow \quad \left.\frac{\partial \mathcal{B}_z}{\partial x}\right|_{x=0} = 0 = -B\beta\sin\alpha a \quad \Rightarrow \quad \alpha = \frac{n\pi}{a} \tag{47}$$

again we select the condition such that we don't get a trivial solution. Therefore, the solution for $X(x)$ is

$$X(x) \propto \cos\frac{n\pi}{a}x \tag{48}$$

We can carry through the same procedure on the surfaces at $y = 0$ and $y = b$ to get

$$Y(y) \propto \cos\frac{m\pi}{b}y \tag{49}$$

Therefore the z component of the magnetic field is

$$\mathcal{B}_z = B_0\cos\frac{n\pi}{a}x\,\cos\frac{m\pi}{b}y \quad \text{with} \quad k_c^2 = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right) \tag{50}$$

with the longest allowed wavelength being $\lambda = 2a$, which corresponds to $n = 1$, $m = 0$ (lowest frequency $\omega = 2\pi c/a$).

To calculate the remaining components of the fields, we use Eqs. 30. The components for the TE_{10} mode are

$$\begin{aligned}
\mathcal{E}_x &= 0 & \mathcal{B}_x &= -i\left[\frac{k_g a}{\pi}B_0\sin\frac{\pi}{a}x\right]\, e^{i(k_g z-\omega t)} \\
\mathcal{E}_y &= i\left[\frac{k_0 a}{\pi}B_0\sin\frac{\pi}{a}x\right]\, e^{i(k_g z-\omega t)} & \mathcal{B}_y &= 0 \\
\mathcal{E}_z &= 0 & \mathcal{B}_z &= B_0\left[\cos\frac{\pi}{a}x\right]\, e^{i(k_g z-\omega t)}
\end{aligned} \tag{51}$$

with the actual fields being the real part of these equations.