Physics 5153 Classical Mechanics
Hamilton-Jacobi Equation

1 Introduction

We have shown that a transformation can be carried out on the Hamiltonian that leaves Hamilton's equations of motion in canonical form. The equations of motion after the transformation become

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i} \qquad K = H + \frac{\partial F}{\partial t} \tag{1}$$

where Q_i and P_i are the new variables, K is the new Hamiltonian, and F is the generating function, which is a function of the new and old variables where half are select as the old variables and half new to provide a bridge between the old and the new[1]. The goal of a canonical transformation is to simplify the the equations of motion. For instance, a transformation that leaves all the variables cyclic leads to simple equations of motion for the momentum. An alternate approach is to transform the canonical coordinates at time t to time $t = 0$. In this case the transformation equations are the desired solution to the problem

$$q = q(q_0, p_0, t) \quad \text{and} \quad p = p(q_0, p_0, t) \tag{2}$$

This last procedure applies to both time dependent and independent Hamiltonian, and corresponds to the Hamilton-Jacobi method. This is the method that will be discussed in this set of lectures. This will also provide a method to determine the generating function.

1.1 Hamilton-Jacobi Equation

As stated above, if the new variables are selected to be the initial conditions associated with the old variables

$$q_i = q_i(q_{0i}, p_{0i}, t) \qquad p_i = p_i(q_{0i}, p_{0i}, t) \tag{3}$$

then we have found the desired solution to our problem; note that we select the constant at a fixed time, therefore the transformed Hamiltonian is independent of time also $K(t_0)$. Therefore, the canonical equations of motion for the transformed coordinates are

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0 \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0 \tag{4}$$

which implies that K is a constant as would be expected since it is now a function of constants.

Let's select the generating function to be $F_2(q_i, P_i, t)$, where the relation between the variables is

$$p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i} \tag{5}$$

Using these expressions, we can write the transformation equation for the Hamiltonian as

$$H\left(q_i, \frac{\partial F_2}{\partial q_i}, t\right) + \frac{\partial F_2}{\partial t} = K \tag{6}$$

[1]Keep in mind that only $2n$ of the variables are linearly independent.

but since K is a constant, we can select it to be zero since it enters the equations of motion as a partial derivative, therefore changing the Hamiltonian by a constant does not effect the motion

$$H\left(q_i, \frac{\partial F_2}{\partial q_i}, t\right) + \frac{\partial F_2}{\partial t} = 0 \tag{7}$$

This last equation is the Hamilton-Jacobi equation, which depends on q_i and t only; the momentum has been remove by the use of the partial derivatives of the generating function. Note, that we could also have selected the generating function F_1 since p_i is given by the same partial derivative of F_1 as of F_2. The generating function is defined in terms of Hamilton's principle function[2] as follows

$$F_2 \equiv S(q_i, P_i, t) = S(q_i, \alpha_i, t) \tag{8}$$

where we use the fact that the P_i are constants. If we have n q_i, then we have $n+1$ α_i, since we have partial derivatives with respect to a total of $n+1$ variables; the n q_i and t. But since S only appears as a derivative $S+\alpha$ could be a solution without altering the equations, also in the transformation equations S appears in the derivatives, therefore this additive constant has no meaning and can be ignored. Therefore there are only n α_i and the solution is given by

$$S = S(q_i, \alpha_i, t) \qquad i = 1, 2, \ldots, n \tag{9}$$

The generating function S is exactly of the form that we were looking for. It is composed of the time t, n coordinates q_i and n constant α_i that we are at liberty to identify with the new constant momenta

$$P_i \equiv \alpha_i \tag{10}$$

This identification does not contradict our original desire that the new momenta be connected with the initial values of q and p at time t_0. The n equations relating the new momenta to the old coordinates and momenta are

$$p_i = \frac{\partial S(q, \alpha, t)}{\partial q_i} \tag{11}$$

which can be solved for α in terms of q and p at $t = t_0$. The remaining n equations, which give the new constant coordinate in terms of the generating function are

$$Q_i = \beta_i = \frac{\partial S(q, \alpha, t)}{\partial \alpha_i} \tag{12}$$

The constants β_i can be calculated in terms of the initial conditions by evaluating the partial derivative at $t = t_0$. The equations for the β_i can be inverted to give

$$q_i = q_i(\beta, \alpha, t) \tag{13}$$

After performing the differentiation in Eq 11, we can substitute q_i and we have the momenta

$$p_i = p_i(\beta, \alpha, t) \tag{14}$$

The generating function S is related to the action, which was defined earlier. To see this relation, take the total derivative of S

$$\frac{dS}{dt} = \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial t} \tag{15}$$

[2]This will be shown to be the action later.

Using Eqs. 7, 11, and 12, the total derivative can be written as follows

$$\frac{dS}{dt} = p_i \dot{q}_i - H = L \quad \Rightarrow \quad S = \int L dt + \text{constant} \tag{16}$$

which is the indefinite integral of the Lagrangian; recall the action is the definite integral of the Lagrangian.

Let's consider the case where the Hamiltonian is independent of time. In this case the Hamiltonian is conserved and in most cases is the total energy. Starting with Eq. 7

$$H(p_i, q_i) + \frac{\partial S}{\partial t} = 0 \quad \Rightarrow \quad H\left(q_i, \frac{\partial S}{\partial q_i}\right) + \frac{\partial S}{\partial t} = 0 \tag{17}$$

we can write the generating function as

$$S(q_i, \alpha_i, t) = W(q_i, \alpha_i) - at \quad \Rightarrow \quad a = H(p_i, q_i) \tag{18}$$

where we will take the constant a to be α_1, which is the Jacobi integral.

As in the general case, we can arrive at a partial differential equation for the generating function, which in this case we take as $W(q_i, P_i)$ (this is referred to as Hamilton's characteristic function). As before, we are lead to the partial differential equation by using the transformation properties of a generating function that depends on q_i and P_i

$$p_i = \frac{\partial W}{\partial q_i} \qquad Q_i = \beta_i = \frac{\partial W}{\partial P_i} = \frac{\partial W}{\partial \alpha_i} \qquad H(q_i, p_i) = H\left(q_i, \frac{\partial W}{\partial P_i}\right) = \alpha_1 \tag{19}$$

Notice that if we use W as the generating function, the transformed Hamiltonian is not equal to zero

$$H + \frac{\partial W}{\partial t} = \alpha_1 = K \tag{20}$$

In this case, the transformed momenta are still constants of the motion

$$\dot{P}_i = \frac{\partial K}{\partial Q_i} = 0 \quad \Rightarrow \quad P_i = \alpha_i \tag{21}$$

but the transformed coordinates are not all constants

$$\dot{Q}_i = \frac{\partial K}{\partial \alpha_i} = \begin{cases} 1 & i = 1 \\ 0 & i \neq 1 \end{cases} \quad \Rightarrow \quad \begin{cases} Q_1 = t + \beta_1 \equiv \frac{\partial W}{\partial \alpha_1} & i = 1 \\ Q_i = \beta_i \equiv \frac{\partial W}{\partial \alpha_i} & i \neq 1 \end{cases} \tag{22}$$

Notice that in this case not all the Q_i are constants, the transformed coordinate conjugate to the energy is dependent on the time. We now have all the elements to solve for q_i and p_i.

1.2 Example—The Simple Harmonic Oscillator

As an example, let's consider the simple harmonic oscillator. The Hamiltonian for this system is

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2 = E \tag{23}$$

where q is the displacement from the natural length of the spring, and the Hamiltonian is a constant of the motion, which in this case is the energy. Since the Hamiltonian is independent of time, we can use the second form of the Hamilton-Jacobi equation

$$\left.\begin{array}{l} \frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m\omega^2 q^2\right] + \frac{\partial S}{\partial t} = 0 \\ S(q,\alpha,t) = W(q,\alpha) - \alpha t \end{array}\right\} \quad \Rightarrow \quad \frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2 + m\omega^2 q^2\right] = \alpha = E \tag{24}$$

where in the last step we identify α with E the total energy. Next we need to find β, which is given by

$$\beta + t = \frac{\partial W}{\partial E} \tag{25}$$

Hence, in order to determine β, we have to calculate the generating function using Eq. 24

$$W = \sqrt{2mE}\int dq\sqrt{1 - \frac{m\omega^2 q^2}{2E}} \quad \Rightarrow \quad \frac{\partial W}{\partial E} = \sqrt{\frac{m}{2E}}\int \frac{dq}{\sqrt{1 - \frac{m\omega^2 q^2}{2E}}} \tag{26}$$

where we carry out the derivative before performing the integration. Next we perform the integration

$$\beta + t = \frac{1}{\omega}\sin^{-1}\left(q\sqrt{\frac{m\omega^2}{2E}}\right) \quad \Rightarrow \quad q = \sqrt{\frac{2E}{m\omega^2}}\sin(\omega t + \phi) \tag{27}$$

where we replace $\omega\beta$ with ϕ and used the general integration formula

$$\int \frac{dq}{\sqrt{1 - a^2 q^2}} = \frac{1}{a}\sin^{-1}(aq) \tag{28}$$

To calculate the momentum, we differentiate the Eq 26 with respect to the coordinate q

$$p = \frac{\partial W}{\partial q} = \sqrt{2mE - m^2\omega^2 q^2} = \sqrt{2mE\left[1 - \sin(\omega t + \phi)\right]} = \sqrt{2mE}\cos(\omega t + \phi) \tag{29}$$

which can be seen to be equivalent to not performing the integration. The only quantity left to find is ϕ. Since the transformed coordinates were selected to be constants, they are related to the original coordinates at $t = 0$. The phase is found by solving the equations at $t = 0$

$$\frac{q_0}{p_0} = \frac{1}{m\omega}\frac{\sin\phi}{\cos\phi} \quad \Rightarrow \quad \tan\phi = m\omega\frac{q_0}{p_0} \tag{30}$$

1.3 Separation of Variables

We have already see that if the Hamiltonian is independent of time, the Hamilton-Jacobi equation can be simplified by separating out the time variable. Let's consider a further simplification of the Hamilton-Jacobi equation where there are no cross terms in the coordinates in the time independent Hamiltonian. That is, the Hamiltonian can be written as a sum of terms

$$E = H(q,p) = \sum_{i=1}^{n} H(q_i, p_i) \tag{31}$$

where we have assume that the Hamiltonian is the total energy[3] In this case, the generating function can also be written as a sum of terms

$$S(q,\alpha,t) = \sum_{i=1}^{n} W_i(q_i,\alpha_i) + Et \tag{32}$$

Therefore, the Hamilton-Jacobi equation becomes

$$H\left(q, \frac{dW}{dq}\right) = \sum_{i=1}^{n} H_i\left(q_i, \frac{dW_i}{dq_i}\right) = E \tag{33}$$

note that we have gone from a partial derivative to a total derivative since each W_i depends on only one coordinate. Since each term depends on only one coordinate and the sum must equal a constant, each term must be a constant that we can take as follows

$$H_i(q_i,\alpha_i) = \alpha_i \quad \Rightarrow \quad \sum_{i=1}^{n} \alpha_i = E \tag{34}$$

As a simple example of a system that can be separated, consider a two-dimensional anisotropic harmonic oscillator. The Hamiltonian is given by

$$H = \frac{p_1^2}{2m} + m\omega_1^2 q_1 + \frac{p_2^2}{2m} + m\omega_1^2 q_2 \tag{35}$$

It is clear that this Hamiltonian can be split into the sum of two terms. One with coordinates q_1 and the second with coordinates q_2. The Hamiltonian for this system is independent of time, and it represents the total energy. Therefore, the function S can be written as follows

$$S(q,\alpha,t) = W_1(q_1,\alpha_1) + W_2(q_2,\alpha_2) - Et \tag{36}$$

The Hamilton-Jacobi equation for this system is therefore

$$\frac{1}{2m}\left[\left(\frac{\partial W_1}{\partial q_1}\right)^2 + m^2\omega_1^2 q_1^2 + \left(\frac{\partial W_2}{\partial q_2}\right)^2 + m^2\omega_2^2 q_2^2\right] = E \tag{37}$$

Since this equation must be true for any arbitrary value of the coordinates, each term must be a constant

$$\begin{aligned} \frac{1}{2m}\left(\frac{\partial W_1}{\partial q_1}\right)^2 + \frac{1}{2} m\omega_1^2 q_1^2 &= \alpha_1 \\ \frac{1}{2m}\left(\frac{\partial W_2}{\partial q_2}\right)^2 + \frac{1}{2} m\omega_2^2 q_2^2 &= \alpha_2 \end{aligned} \tag{38}$$

where $E = \alpha_1 + \alpha_2$. From this point forward, the equation is solved in the same manner as the one-dimensional simple harmonic oscillator.

[3] If the Hamiltonian is not the total energy, then we can replace E with h, the Jacobi integral since we are assuming that the Hamiltonian is independent of time.

1.4 Central Potential

One of the problems that is greatly simplified by the use of the Hamilton-Jacobi equation is that of the central potential. This problem has a time independent Hamiltonian, is seperable, and has cyclic variables. Let's consider the case of a central body of infinite mass[4] that generates an attractive inverse square force. A second body of mass m is free to move under the influence of the potential. The Lagrangian for the orbiting mass is

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{k}{r} \tag{39}$$

where we use the fact that conservation of momentum dictates that the orbit is in a plane. Next we convert from the Lagrangian to the Hamiltonian. The first step is to calculate the canonical (generalized) momenta

$$p_r = m\dot{r} \qquad p_\theta = mr^2\dot{\theta} \tag{40}$$

Notice that the Lagrangian is independent of θ, therefore p_θ is conserved. Using the just calculated generalized momenta, the Hamiltonian is found to be

$$H = p_r\dot{r} + p_\theta\dot{\theta} - L = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{k}{r} \tag{41}$$

which is equal to the total energy and is a conserved quantity.

Before proceeding to solve for the motion, notice that the second term in the Hamiltonian has the form of a repulsive potential. Therefore, the last two terms can be thought of as an effective potential. One piece that is repulsive, centrifugal barrier, the second piece is attractive, gravitational potential. Notice that there are several different orbits possible. If the total energy is equal to the minimum of the effective potential, the orbit will be circular since their is only one allowed radius. If the total energy is greater than the minimum and less than zero ($0 > E > E_{\min}$), then the orbit will be elliptic since the turning points are at two different radii. If the total energy is zero, then one of the turning points is at infinity. In this case the path is parabolic. Finally, if the energy is greater than zero, the path is hyperbolic. This is a transition between the conic sections depending on the value of the total energy.

We will now proceed to solve the problem. Hamilton's characteristic function for this system is given by

$$W = W_1(r) + \alpha_\theta\theta \tag{42}$$

where $\alpha_\theta \equiv p_\theta$ the conserved momentum. The Hamilton-Jacobi equations is

$$\left(\frac{\partial W_1}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{r^2} - \frac{2mk}{r} = 2m\alpha_1 \quad \Rightarrow \quad \left(\frac{\partial W_1}{\partial r}\right)^2 + \frac{p_\theta^2}{r^2} - \frac{2mk}{r} = 2mE \tag{43}$$

where we have identified α_1 with the total energy as stated in the earlier discussion. The next step is to solve for the characteristic function, which is

$$\frac{\partial W_1}{\partial r} = \sqrt{2m\left(E + \frac{k}{r}\right) - \frac{p_\theta^2}{r^2}} \quad \Rightarrow \quad W = \left\{\int d\vec{\mathbf{r}}\sqrt{2m\left(E + \frac{k}{r}\right) - \frac{p_\theta^2}{r^2}}\right\} + p_\theta\theta \tag{44}$$

[4]There is no lose of generality by assuming an infinite mass, since we can work in the center of mass frame with the central mass being effectively infinite and the orbiting mass being the reduced mass $\mu = Mm/(M+m)$.

But we don't need to calculate this integral, since we are only interested in its partial derivatives with respect to the canonical momenta. The partial derivatives are

$$t+\beta_1 = \frac{\partial W}{\partial \alpha_1} = \frac{\partial W}{\partial E} = \int \frac{m dr}{\sqrt{2m(E+k/r) - p_\theta^2/r^2}} \tag{45}$$

$$\beta_2 = \frac{\partial W}{\partial \alpha_\theta} = \frac{\partial W}{\partial p_\theta} = -\int \frac{p_\theta dr}{r^2\sqrt{2m(E+k/r) - p_\theta^2/r^2}} + \theta \tag{46}$$

where Eq. 45 gives the time dependence of the radial coordinate ($r(t)$), and Eq. 46 gives the angular dependence of the radial coordinate ($r(\theta)$).

We will integrate Eq. 46 first, since it provides all the information necessary to describe the orbit. First we use the change of variables $u = 1/r$ with the differential being $du = -dr/r^2$. The integral becomes

$$\theta = \beta_2 - \int \frac{du}{\sqrt{2m/p_\theta(E+ku) - u^2}} \quad \Rightarrow \quad \theta = \theta_0 - \cos^{-1}\left[\frac{\frac{p_\theta^2 u}{mk} - 1}{\sqrt{1+\frac{2Ep_\theta^2}{mk^2}}}\right] \tag{47}$$

Next we replace u with $1/r$ and invert the equation to get the angular dependence of r

$$\frac{1}{r} = \frac{mk}{p_\theta^2}\left[1 + \sqrt{1+\frac{2Ep_\theta^2}{mk^2}}\cos(\theta-\theta_0)\right] \tag{48}$$

where θ_0 is identified as the of the turning angles of the orbit. This equation can be simplified by introducing the eccentricity parameter e, which is defined as

$$e = \sqrt{1+\frac{2Ep_\theta^2}{mk^2}} \quad \Rightarrow \quad r(\theta) = \frac{p_\theta^2/mk}{1+e\cos(\theta-\theta_0)} \tag{49}$$

This equation defines the conic sections. The different conic sections are defined by the energy through the eccentricity

$e > 1$	$E > 0$	hyperbola
$e = 1$	$E = 0$	parabola
$e < 1$	$E < 0$	ellipse
$e = 0$	$E = -mk^2/2p_\theta^2$	circle

where the value of the energy for the circular orbit corresponds to the minimum of the effective potential. In case of an elliptic orbit, the semimajor axis corresponds to half the distance between the turning points of the orbit. The turning points correspond to the case where $\dot{r} = 0$, therefore the total energy is given by

$$E = \frac{p_\theta^2}{2mr^2} - \frac{k}{r} \quad \Rightarrow \quad r^2 + \frac{k}{E}r - \frac{p_\theta^2}{2mE} = 0 \quad \Rightarrow \quad r = -\frac{k \pm \sqrt{k^2 + 2mEp_\theta^2}}{2E} \tag{50}$$

with the semimajor axis being

$$a = \frac{r_+ + r_-}{2} = -\frac{k}{2E} \tag{51}$$

Using the expression for the semimajor axis, the orbit equation becomes

$$r(\theta) = \frac{a(1 - e^2)}{1 + e\cos(\theta - \theta_0)} \tag{52}$$