Physics 5153 Classical Mechanics
Velocity Dependent Potentials

1 Introduction

We have so far only considered forces and therefore potentials, that are dependent only on the spatial coordinates. In this lecture, we will consider velocity dependent potentials.

1.1 Rayleigh's Dissipation Function

The standard form of Lagrange's equations of motion, ignoring the Lagrange multiplier term, are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \tag{1}$$

where $L = T - V$ and the gradient of the potential V is assumes to be a generalized force

$$Q'_i = \frac{\partial V}{\partial q_i} \tag{2}$$

which could in principle be remove from the Lagrangian and added to the right hand side. But what if the we have a velocity dependent damping force of the form

$$F \propto -\dot{q} \tag{3}$$

Examples of this type of force are atmospheric drag, resistive motion through a viscous fluid, and damping force in a spring. In these cases, we can define a new function, the Rayleigh dissipation function

$$\mathcal{F} = \frac{1}{2}c\dot{q}^2 \quad \text{or in general} \quad \mathcal{F} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\dot{q}_i\dot{q}_j \tag{4}$$

and write the equations of motion in the following form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial \mathcal{F}}{\partial \dot{q}_i} = 0 \tag{5}$$

where we are assuming that the only generalized forces are those that can be written as potentials and in terms of $\mathcal{F}$. The last term, therefore, represents a generalized force

$$Q = -\frac{\partial \mathcal{F}}{\partial \dot{q}} \tag{6}$$

To interpret $\mathcal{F}$, we recall that the rate of energy lost or gained from an applied force in rectangular coordinates is given by

$$\frac{dW}{dt} = \sum_i \vec{\mathbf{F}}_i \cdot \vec{\mathbf{v}}_i \tag{7}$$

This equation can be transformed to generalized coordinates

$$\frac{dW}{dt} = -\sum_i F_i \frac{\partial x_i}{\partial q_i}\dot{q}_i = -\sum_i Q_i\dot{q}_i = -2\mathcal{F} \tag{8}$$

where the minus sign is due to the force and velocity acting in opposite directions, and we use

$$Q_i = F_i \frac{\partial x_i}{\partial q_i} = \frac{\partial \mathcal{F}}{\partial \dot{q}_i} \tag{9}$$

in the last step. Hence, $\mathcal{F}$ is half the energy lost by the system, and is independent of the set of coordinates used as would be expected.

1.2 Velocity Dependent Potentials

Another example of a velocity dependent force that can be incorporated into the Lagrangian, is the Lorentz force. Recall that the basic form of Lagrange's equation is

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i \tag{10}$$

If the force is derivable from a position dependent scalar function (potential), then the Lagrange equations of motion are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{11}$$

where $L = T - V$ and $V \equiv V(q, t)$. Let's now consider a simple generalization of the definition of a conservative potential

$$Q_i = -\frac{\partial V}{\partial q_i} \quad \Rightarrow \quad Q_i = -\frac{\partial U}{\partial q_i} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_i} \quad \text{with} \quad U \equiv U(\dot{q}, q, t) \tag{12}$$

A potential that satisfies this condition, also allows the Lagrange equations to be put into the same form as Eq. 11

1.2.1 Example—The Lorentz Force

As an example of a velocity dependent potential, we will consider the Lorentz force

$$\vec{\mathbf{F}} = e(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \mathbf{B}) \tag{13}$$

To use it in the Lagrangian, it has to be written in terms of the scalar and vector potentials

$$\left.\begin{aligned} \vec{\mathbf{E}} &= -\vec{\boldsymbol{\nabla}}\phi - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \vec{\boldsymbol{\nabla}} \times \mathbf{A} \end{aligned}\right\} \quad \Rightarrow \quad \vec{\mathbf{F}} = e\left(-\vec{\boldsymbol{\nabla}}\phi - \frac{\partial \mathbf{A}}{\partial t} + \vec{\mathbf{v}} \times \vec{\boldsymbol{\nabla}} \times \mathbf{A}\right)$$

To show that the Lorentz force satisfies Eq. 12, we will work in Cartesian coordinates. First, we note that the term $-e\vec{\boldsymbol{\nabla}}\phi$ is equivalent to the standard potential that we use. Therefore, we only need to consider the term

$$e\left(-\frac{\partial \mathbf{A}}{\partial t} + \vec{\mathbf{v}} \times \vec{\boldsymbol{\nabla}} \times \mathbf{A}\right) \tag{14}$$

To simplify the algebra significantly, we will consider only the x coordinate and generalize in the end. The triple cross product can be expanded out to give

$$\begin{aligned}(\vec{\mathbf{v}} \times \vec{\boldsymbol{\nabla}} \times \mathbf{A})_x &= v_y \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - v_z \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \\ &= v_y \frac{\partial A_y}{\partial x} + v_z \frac{\partial A_z}{\partial x} + v_x \frac{\partial A_x}{\partial x} - v_x \frac{\partial A_x}{\partial x} - v_y \frac{\partial A_x}{\partial y} - v_z \frac{\partial A_x}{\partial z}\end{aligned} \tag{15}$$

where we add and subtract the third and forth terms in the last equation. Since $\mathbf{A}$ is a function of position and time, the total derivative is given by

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial x} v_x + \frac{\partial A_y}{\partial y} v_y + \frac{\partial A_z}{\partial z} v_z + \frac{\partial A_x}{\partial t} \tag{16}$$

Using this expression, the triple cross product can be written in a more compact form

$$(\vec{\mathbf{v}} \times \vec{\boldsymbol{\nabla}} \times \mathbf{A})_x = \frac{\partial}{\partial x} (\vec{\mathbf{v}} \cdot \mathbf{A}) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t} \tag{17}$$

where we use the fact that the velocity is not an explicit function of the coordinates. Next we observe that

$$\frac{dA_x}{dt} = \frac{d}{dt} \left[\frac{\partial}{\partial \dot{x}} (\vec{\mathbf{v}} \cdot \mathbf{A}) \right] \tag{18}$$

since $\mathbf{A}$ is not an explicit function of the velocity. Hence

$$F_x = e \left\{ -\frac{\partial \phi}{\partial x} + \frac{\partial \vec{\mathbf{v}} \cdot \mathbf{A}}{\partial x} - \frac{d}{dt} \left[\frac{\partial}{\partial \dot{x}} (\vec{\mathbf{v}} \cdot \mathbf{A}) \right] \right\} \tag{19}$$

which can be generalized to the following expression

$$\vec{\mathbf{F}} = e \left[-\vec{\boldsymbol{\nabla}} \left(\phi - \vec{\mathbf{v}} \cdot \mathbf{A} \right) + \frac{d}{dt} \left(\vec{\boldsymbol{\nabla}}_v \left(\phi - \vec{\mathbf{v}} \cdot \mathbf{A} \right) \right) \right] \tag{20}$$

This is in the required form

$$Q_i = \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{q}_i} \right) - \frac{\partial U}{\partial q_i} \quad \text{with} \quad U = e \left(\phi - \vec{\mathbf{v}} \cdot \mathbf{A} \right) \tag{21}$$

The Lagrangian for a charged particle of mass m traversing an electromagnetic field is given by

$$L = \frac{1}{2} m v^2 - e \left(\phi - \vec{\mathbf{v}} \cdot \mathbf{A} \right) \tag{22}$$

From the Lagrangian, we can calculate the generalized momentum, which is

$$p = \frac{\partial L}{\partial \dot{q}_i} = m \vec{\mathbf{v}} + e \mathbf{A} \tag{23}$$

so if any of the coordinates are cyclic, it is this momentum that is conserved. That is a portion of the momentum of the particle is associated with the electromagnetic field. The energy of the particle is

$$E = T + e\phi \tag{24}$$

since the magnetic field does not work on the system, because the force acts perpendicular to the velocity.