Physics 5153 Classical Mechanics
Solution by Quadrature

1 Introduction

In the previous lectures, we have reduced the number of effective degrees of freedom that are needed to solve the equations of motion. This was done by finding integrals (constants) of the motion. The first step was to determine which variables are cyclic, since these lead to conserved momenta. Once the initial conditions are specified for the problem, the momenta are known for all time.

If the system is conservative, we found that the Jacobi integral ($T_2 - T_0 + V = h$) is also an integral (constant) of the motion. In cases where all the variables are cyclic except for one, the Jacobi integral can be directly integrated

$$\dot{q} = f(q) \quad \Rightarrow \quad dt = \frac{dq}{f(q)} \tag{1}$$

and the time dependence of the remaining generalized coordinate determined. In this case we say that the solution is found by quadratures. That is, the solution is given as an indefinite integral of a single variable. Our goal in this lecture is to determine when an equation of motion is separable, and can be put into a number of equations of the form of Eq. 1, that is we would like to know when an equation of motion can be solved by quadratures.

1.1 Liouville' System

First recall that a system having n degrees of freedom requires $2n$ integrals (constants) of the motion for the complete solution of the equations of motion. These $2n$ integrals of motion are effectively the initial conditions for the second order differential equation of motion. As stated in the introduction, is $n-1$ variables are cyclic, then the variables can be removed by use of the Routhian method. This leaves us with an equation of motion in one variable. Furthermore, is the system is conservative, then the equation is solved directly by quadratures.

Let's assume that we have a conservative holonomic system. In addition, assume that there are not enough cyclic variables to guarantee separability as given above. It may still be separable, if it is an orthogonal system, that is, it is a natural system and there are no cross terms ($\dot{q}_i \dot{q}_j \; i \neq j$) in the kinetic energy.

Let's consider the following general example, which we will show is separable, that is, solvable by quadratures. Assume that the kinetic and potential energies are given in the following forms

$$T = \frac{1}{2} f \sum_{i=1}^{n} \dot{q}_i^2 \qquad\qquad V = \frac{1}{f} \sum_{i=1}^{n} v_i(q_i) \tag{2}$$

where

$$f = \sum_{i=1}^{n} f_i(q_i) > 0 \tag{3}$$

Let's consider the Lagrange equations in the following form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0 \tag{4}$$

Substituting the kinetic and potential energies from Eq. 2 into Lagrange's equation gives the following equation

$$\frac{d}{dt}(f\dot{q}_i) - \frac{1}{2}\frac{\partial f_i}{\partial q_i}\sum_{j=1}^{n}\dot{q}_j^2 + \frac{1}{f}\frac{\partial v_i}{\partial q_i} - \frac{V}{f}\frac{\partial f_i}{\partial q_i} = 0 \tag{5}$$

Since this is a natural system, the Jacobi (energy) integral is given by

$$T + V = \frac{1}{2}f\sum_{j=1}^{n}\dot{q}_j^2 + V = h \quad \Rightarrow \quad \frac{1}{2}\sum_{j=1}^{n}\dot{q}_j^2 = \frac{1}{f}(h - V) \tag{6}$$

Next substitute Eq 6 into Eq. 5 and simplify

$$\frac{d}{dt}(f\dot{q}_i) - \frac{h}{f}\frac{\partial f_i}{\partial q_i} + \frac{1}{f}\frac{\partial v_i}{\partial q_i} = 0 \tag{7}$$

Now multiply this equation by $2f\dot{q}_i$ and obtain

$$\frac{d}{dt}(f^2\dot{q}_i^2) - 2h\frac{\partial f_i}{\partial q_i}\dot{q}_i + 2\frac{\partial v_i}{\partial q_i}\dot{q}_i = 0 \quad \Rightarrow \quad \frac{d}{dt}(f^2\dot{q}_i^2) = 2\frac{d}{dt}(hf_i - v_i) \tag{8}$$

where the right hand side of the second term is a total derivative since each term only depends on q_i. Integrating, the result is

$$f^2\dot{q}_i^2 = 2\left[hf_i - v_i + c_i\right] \tag{9}$$

Next sum over i, where we find the following

$$\sum_{i=1}^{n} c_i = 0 \tag{10}$$

which can be shown using Eqs. 2, 6 and 9. Hence the c_i and h together form n independent constants of the motion. (Note that there are only $n-1$ independent c_i, otherwise all the c_i would have to be zero.)

The remaining n constants of the motion, can be derived using Eq. 9 in the following form

$$\frac{dq_i}{dt} = \frac{\sqrt{2(hf_i - v_i + c_i)}}{f} \tag{11}$$

which implies

$$\frac{dq_1}{\sqrt{2(hf_1 - v_1 + c_1)}} = \frac{dq_2}{\sqrt{2(hf_2 - v_2 + c_2)}} = \cdots = \frac{dq_n}{\sqrt{2(hf_n - v_n + c_n)}} = \frac{dt}{f} \equiv d\tau \tag{12}$$

where τ is a time-like parameter. Each differential expression is a function of a single q_i, so the problem is reduced to quadratures. Integrating these equations produces the remaining constants of the motion.

This system can be generalized to become a Liouville system, replacing dq_i with $M_i(q_i)dq_i$ to obtain

$$T = \frac{1}{2}f\sum_{i=1}^{n} M_i(q_i)\dot{q}_i^2 \tag{13}$$

where we assume that $M_i(q_i) > 0$. The potential energy (V) remains as before. A natural system having the kinetic and potential energies in this form is called a Liouville system.

By making the transformation $q_i \to \sqrt{M_i(q_i)}\, q_i$, the solutions for a Liouville system are as in Eq. 12 given by

$$\frac{dq_1}{\sqrt{\phi_1(q_1)}} = \frac{dq_2}{\sqrt{\phi_2(q_2)}} = \cdots = \frac{dq_n}{\sqrt{\phi_n(q_n)}} = \frac{dt}{f} = d\tau \tag{14}$$

where

$$\phi_i(q_i) = \frac{2}{M_i}(hf_i - v_i + c_i) \qquad (i = 1, 2, \ldots, n) \tag{15}$$

Using Eq. 3, multiplying Eq. 14 by f_i, and summing over i, we obtain

$$\sum_{i=1}^{n} \frac{f_i dq_i}{\sqrt{\phi_i(q_i)}} = dt \quad \Rightarrow \quad \sum_{i=1}^{n} \int \frac{f_i dq_i}{\sqrt{\phi_i(q_i)}} = t - \beta_1 \tag{16}$$

Similarly, taking differences of the indefinite integrals of Eq. 14, we have

$$\int \frac{dq_1}{\sqrt{\phi_1(q_1)}} - \int \frac{dq_j}{\sqrt{\phi_j(q_j)}} = \beta_j \qquad (j = 1, 2, \ldots, n) \tag{17}$$

where the first integral is chosen arbitrarily as a reference. Thus, these two sets of equations provide n independent constant the $\beta_1, \beta_2, \ldots, \beta_n$, which, with the previous $n - 1$ c_i and h, constitute the required $2n$ independent constants of the motion.

In evaluating the integrals of Eqs. 16 and 17, a question arises concerning the sign of $\sqrt{\phi_i(q_i)}$. Since f is positive, we from Eq. 14 find that $\sqrt{\phi_i(q_i)}$ has the same sign as dq_i. This is important in the study of *libration* motions, that is, motions in which one or more q's oscillate between fixed limiting values.

1.2 Example

As an example, let's consider the spherical pendulum shown in Fig. 1. Reduce the problem to quadratures and obtain the integrals of the motion.

Before attempting the solution, we determine the kinetic and potential energies using an appropriate set of generalized coordinates. We start by writing the Lagrangian in Cartesian coordinates

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \tag{18}$$

Before specifying the point transformation to an appropriate set of generalized coordinates, we must determine the number of degrees of freedom. The Lagrangian as given above has 3 coordinates, but there is a constraint among the coordinates ($x^2 + y^2 + z^2 - \ell = 0$). Hence, there are 2 degrees of freedom. The point transformations that are required to express the Lagrangian in the generalized coordinates (θ, ϕ) given in the Fig. 1 are

$$\left.\begin{aligned} x &= \ell \sin\theta \cos\phi \\ y &= \ell \sin\theta \sin\phi \\ z &= \ell \cos\theta \end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned} \dot{x} &= \dot{\theta}\ell \cos\theta \cos\phi - \dot{\phi}\ell \sin\theta \sin\phi \\ \dot{y} &= \dot{\theta}\ell \cos\theta \sin\phi + \dot{\phi}\ell \sin\theta \cos\phi \\ \dot{z} &= -\dot{\theta}\ell \sin\theta \end{aligned}\right. \tag{19}$$

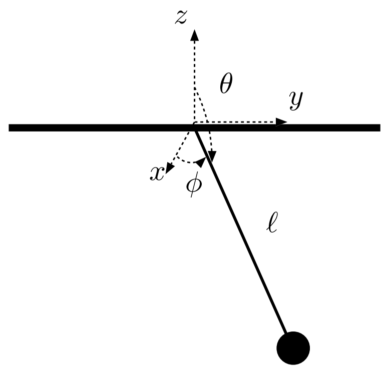

Figure 1: Spherical pendulum along with coordinate system.

where the constraint has been taken into account. Hence, the Lagrangian is transformed to the following form

$$L = \frac{1}{2}m\ell^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) - mg\ell\cos \tag{20}$$

We can now solve the problem using two different methods.

1.2.1 Method 1

In this method, the problem will be solved by removing the cyclic variables from the Lagrangian by using the Routhian method. We use the Lagrangian given in Eq. 20

$$L = \frac{1}{2}m\ell^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) - mg\ell\cos\theta \tag{21}$$

Note that the Lagrangian has a single cyclic coordinate ϕ, with the associated generalized momentum given by

$$\frac{\partial L}{\partial \dot{\phi}} = m\ell^2\dot{\phi}\sin^2\theta = \alpha_\phi \tag{22}$$

where α_ϕ is the constant momentum in the ϕ direction.

Next we calculate the Routhian

$$R = L - \alpha_\phi\phi = \frac{1}{2}m\ell^2\dot{\theta}^2 - \frac{\alpha_\phi^2}{2m\ell^2\sin^2\theta} - mg\ell\cos\theta \tag{23}$$

where $\dot{\phi}$ has been eliminated as expected. Notice, that the Routhian can be written as a natural system

$$\left.\begin{aligned} T' &= \frac{1}{2}m\ell^2\dot{\theta}^2 \\ V' &= \frac{\alpha_\phi^2}{2m\ell^2\sin^2\theta} + mg\ell\cos\theta \end{aligned}\right\} \quad \Rightarrow \quad R = T' - V' \tag{24}$$

Hence, we can immediately write down the (Jacobi) energy integral

$$h = T' + V' = \frac{1}{2}m\ell^2\dot{\theta}^2 + \frac{\alpha_\phi^2}{2m\ell^2\sin^2\theta} + mg\ell\cos\theta \tag{25}$$

This can be solved for the velocity $\dot{\theta}$

$$\dot{\theta} = \sqrt{\frac{2}{m\ell^2}(h - mg\ell\cos\theta - \alpha_\phi^2/(2m\ell^2)\sin^2\theta)} \tag{26}$$

which can be written as follows

$$dt = \frac{m\ell^2 \sin\theta d\theta}{2m\ell^2\sin^2\theta(h - mg\ell\cos\theta) - \alpha_\phi^2} \tag{27}$$

This equation can be integrated to give

$$t - t_0 = \int_{\theta_0}^{\theta} \frac{m\ell^2 \sin\theta d\theta}{2m\ell^2\sin^2\theta(h - mg\ell\cos\theta) - \alpha_\phi^2} \tag{28}$$

The sign of the square root should be the same as $d\theta$ stated earlier, since any change in $d\theta$ causes a positive change in dt.

Next we calculate $\phi(\theta)$. Start with the generalized momentum

$$d\phi = \frac{dt}{m\ell^2\sin^2\theta} \tag{29}$$

Next substitute for dt

$$d\phi = \frac{\alpha_\phi^2\, d\theta}{\sin\theta\sqrt{2m\ell^2\sin^2\theta(h - mg\ell\cos\theta) - \alpha_\phi^2}} \tag{30}$$

integrating this equation gives

$$\phi - \phi_0 = \int_{\theta_0}^{\theta} \frac{\alpha_\phi^2\, d\theta}{\sin\theta\sqrt{2m\ell^2\sin^2\theta(h - mg\ell\cos\theta) - \alpha_\phi^2}} \tag{31}$$

Thus, we have obtained the 4 constants of the motion α_ϕ, h, t_0, and ϕ_0.

1.2.2 Method 2

We will express the system as a Liouville system. The kinetic and potential energies in the generalized coordinates shown in Fig. 1 are

$$T = \frac{1}{2}m\ell^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) \qquad\qquad V = mg\ell\cos\theta \tag{32}$$

Notice that the system is orthogonal, since the kinetic energy depends only on the square of the generalized velocities and there are no cross terms. The standard form of the Liouville problem,

$$T = \frac{1}{2}f\sum_{i=1}^{n} M_i(q_i)\dot{q}_i^2 \qquad V = \frac{1}{f}\sum_{i=1}^{n} v_i(q_i) \quad \text{with} \quad f = \sum_{i=1}^{n} f_i(q_i) > 0 \tag{33}$$

Writing the kinetic and potential energies in this form we get

$$T = \frac{1}{2}\left(m\ell^2\sin^2\theta\right)\left\{\frac{1}{\sin^2\theta}\dot{\theta}^2 + \dot{\phi}^2\right\} \tag{34}$$

$$V = \frac{1}{m\ell^2\sin^2\theta}\left\{m^2g\ell^3\sin^2\theta\cos\theta\right\}$$

Therefore, we find the coefficients to be

$$\begin{aligned} f_\theta(\theta) &= m\ell^2 \sin^2\theta & M_\theta(\theta) &= \frac{1}{\sin^2\theta} & v_\theta(\theta) &= m^2 q\ell^3 \sin^2\theta\cos\theta \\ f_\phi(\phi) &= 0 & M_\phi(\phi) &= 1 & v_\phi(\phi) &= 0 \end{aligned} \tag{35}$$

Next, we give the $\phi_i(q_i)$ for this problem

$$\phi_i(q_i) = \frac{2}{M_i}(hf_i - v_i + c_i) \quad \Rightarrow \quad \begin{cases} \phi_\theta = 2\sin^2\theta[m\ell^2\sin^2\theta(h - mg\ell\cos\theta) + c_\theta] \\ \phi_\phi = 2c_\phi \end{cases}$$

and using Eq. 10, we find the c_i to be

$$2c_\phi = -2c_\theta = \left[m\ell^2\dot{\phi}\sin^2\theta\right]^2 = \alpha_\phi^2 \tag{36}$$

Finally, we arrive at the result

$$\int_{\theta_0}^{\theta} \frac{m\ell^2\sin^2\theta d\theta}{\sqrt{\phi_\theta(\theta)}} = t - t_0 \quad \text{and} \quad \int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\phi_\theta(\theta)}} = \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{2c_\phi}} \quad \Rightarrow \quad \int_{\theta_0}^{\theta} \frac{\alpha_\phi d\theta}{\sqrt{\phi_\theta(\theta)}} = \phi - \phi_0 \tag{37}$$

which after substituting for $\phi_i(q_i)$, we find the same result as using the other method. Notice that we have arrived at the 4 constant need, c_θ, c_ϕ, ϕ_0, and t_0, which all given by the initial conditions.