

Physics 5153 Classical Mechanics

Properties of the Lagrangian

1 Introduction

From the previous lectures, we have deduced from D'Alembert's principle the Lagrangian $L = T - V$ for a system where the constraints are workless and given as

$$\sum_{i=1}^n a_{ji} dq_i + a_{jt} dt = 0 \quad (1)$$

where j corresponds to the j th constraint. Notice that this form for the constraint is valid for both holonomic and nonholonomic cases, where $a_{ji} = \frac{\partial f_j}{\partial q_i}$ with $f_j(q_i) = 0$ for the holonomic case. The general form of the equations of motion for the Lagrangian, the equivalent of Newton's second law, are given by

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = Q_i + \sum_{j=1}^m \lambda_j a_{ji} \quad (2)$$

where Q_i are those components of the generalized force that can not be written as a gradient of a potential and the sum over the Lagrange multipliers corresponds to the constraints, which we will see later correspond to the forces of constraint.

We will now concern ourselves with determining the integrals of the motion. We start with n second order differential equations, one for each degree of freedom, along with $2n$ initial conditions, and aim to find solutions for the differential equations. One method of expressing the general solution is as a set of $2n$ functions

$$f(q, \dot{q}, t) = \alpha_j \quad \text{for } (j = 1, \dots, 2n) \quad (3)$$

where the α_j are arbitrary constants. These $2n$ functions are called the integrals of the motion. These functions maintain a constant value as the motion of the system evolves. In principle, these equations can be solved for the q_i 's and \dot{q}_i 's as functions of the constants α 's and the time t .

Usually it is not possible to obtain all the f 's by any direct method. None-the-less, we will search for methods that allow us to transform the coordinates so as to simplify the finding of the integrals of motion. We will in fact look for transformations where the coordinates and momenta are constants, thereby forming the $2n$ integrals of motion.

We will start by considering some elementary characteristics of the dynamical systems that allow us to obtain the integrals of motion by quadratures, that is, in terms of known elementary functions or indefinite integrals of such functions.

1.1 Cyclic Variables

Consider a holonomic system that can be described by the standard form of Lagrange's equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \quad (4)$$

Suppose that $L(q, \dot{q}, t)$ depends on n \dot{q} 's but the first k q 's are missing. The k missing coordinates are called cyclic or ignorable coordinates. The equations of motion for these coordinates are given by

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \quad \text{for } (i = 1, \dots, k) \quad (5)$$

which implies

$$\frac{\partial L}{\partial \dot{q}_i} = \beta_i \quad \text{for } (i = 1, \dots, k) \quad (6)$$

which is an integral of the motion. The velocity derivative of the Lagrangian can be interpreted by considering the case of a free particle in rectangular coordinates. Assume that the particle propagates along the x -axis, then $L = \frac{1}{2}m\dot{x}^2$. The velocity derivative of this Lagrangian is

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad (7)$$

which is the momentum. Therefore, as we did for the coordinates and the force, we define a generalized momentum

$$p_q \equiv \frac{\partial L}{\partial \dot{q}} \quad (8)$$

where as for the other generalized quantities, the generalized momentum does not necessarily have units of momentum. Therefore, we conclude that if the a coordinate is cyclic, then the associated generalized momentum is conserved. The generalized momenta that are associated with the cyclic variables are integrals of the motion.

As a simple example, consider the Kepler problem, that is problem of the motion of a point particle of mass m under the influence of a $1/r$ potential. The Lagrangian in rectangular coordinates for this system is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{\mu}{\sqrt{x^2 + y^2}} \quad (9)$$

where we use the fact that the motion is in a plane. Notice that there are no cyclic variables in this Lagrangian. Therefore no conserved momenta. But if we convert to polar coordinates, we find that one of the variables (θ) is cyclic

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{\mu}{r} \quad (10)$$

This shows that a judicious choice of coordinates can simplify the problem. The conserved momentum is

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad (11)$$

which is an integral of the motion.

1.2 The Routhian Function

In this section, we introduce a procedure that eliminates the cyclic variables from consideration as separate degrees of freedom from the Lagrangian. Consider a holonomic system with the first k

variables being cyclic. Now define a Routhian function $R(q_{k+1}, \dots, q_n, \dot{q}_{k+1}, \dots, \dot{q}_n, \beta_1, \dots, \beta_k, t)$ as follows

$$R = L - \sum_{i=1}^k \beta_i \dot{q}_i \quad (12)$$

where we eliminate the \dot{q} 's for the cyclic coordinates by solving the k equations

$$\frac{\partial L}{\partial \dot{q}_i} = \beta_i \quad \text{for } (i = 1, \dots, k) \quad (13)$$

for \dot{q}_i as functions of q_i , β_i and t .

To determine the equations of motion for the Routhian, we take variations on it and its definition given in Eq. 12. We start with the left hand side

$$\delta R = \sum_{i=k+1}^n \frac{\partial R}{\partial q_i} \delta q_i + \sum_{i=k+1}^n \frac{\partial R}{\partial \dot{q}_i} \delta \dot{q}_i + \sum_{i=1}^k \frac{\partial R}{\partial \beta_i} \delta \beta_i + \frac{\partial R}{\partial t} \delta t \quad (14)$$

Next we calculate the right hand side of Eq. 12

$$\left. \begin{aligned} \delta L &= \sum_{i=k+1}^n \frac{\partial L}{\partial q_i} \delta q_i + \sum_{i=1}^k \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \sum_{i=k+1}^n \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{\partial L}{\partial t} \delta t \\ \delta \sum_{i=1}^k \beta_i \dot{q}_i &= \sum_{i=1}^k \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \sum_{i=1}^k \dot{q}_i \delta \beta_i \end{aligned} \right\} \Rightarrow \delta R = \sum_{i=k+1}^n \frac{\partial L}{\partial q_i} \delta q_i + \sum_{i=k+1}^n \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i - \sum_{i=1}^k \dot{q}_i \delta \beta_i + \frac{\partial L}{\partial t} \delta t \quad (15)$$

Since the variables are independent, we arrive at the following relations

$$\frac{\partial L}{\partial q_i} = \frac{\partial R}{\partial q_i} \quad \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial R}{\partial \dot{q}_i} \quad (i = k + 1, \dots, n) \quad (16)$$

$$\frac{\partial L}{\partial t} = \frac{\partial R}{\partial t} \quad \dot{q}_i = -\frac{\partial R}{\partial \beta_i} \quad (i = 1, \dots, k) \quad (17)$$

We can now substitute into the Lagrange equations of motion

$$\frac{d}{dt} \left(\frac{\partial R}{\partial \dot{q}_i} \right) - \frac{\partial R}{\partial q_i} = 0 \quad \text{for } (i = k + 1, \dots, n) \quad (18)$$

These equations have the same form as those for the Lagrangian, but there are only $n - k$ equations. Therefore, we have succeeded in eliminating the cyclic variables, and only have second order differential equations in the non-cyclic variables. If the cyclic variables need to be calculated, they can be soled for as follows

$$\dot{q}_i = -\frac{\partial R}{\partial \beta_i} \Rightarrow q_i = -\int \frac{\partial R}{\partial \beta_i} dt \quad \text{for } (i = 1, \dots, k) \quad (19)$$

Let's again consider the Kepler problem. The Lagrangian, as we saw before, is

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{\mu}{r} \quad (20)$$

where, as we noted before, is independent of θ , therefore θ is a cyclic variable. Next we calculate the Routhian is given by

$$R = L - p_\theta\dot{\theta} = \frac{1}{2}m\dot{r}^2 - \frac{\beta^2}{2mr^2} + \frac{\mu}{r} \quad (21)$$

where we calculated the generalized momentum as follows

$$\beta = p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad (22)$$

Therefore, we have reduced the Lagrangian to one degree of freedom, and we therefore have one differential equation to solve

$$m\ddot{r} - \frac{\beta^2}{mr^3} + \frac{\mu}{r^2} = 0 \quad (23)$$

which is the same equation we arrived at using the Lagrangian method.

As an aside, the Routhian contains two terms that depend only on the variable r . If we look at the Routhian from the viewpoint of an observer rotating with the line drawn from the attracting center to the particle, we have a fictitious force, and write the Routhian as follows

$$R = T' - V' \quad \Rightarrow \quad \begin{aligned} T' &= \frac{1}{2}m\dot{r}^2 \\ V' &= \frac{\beta^2}{2mr^2} - \frac{\mu}{r} \end{aligned} \quad (24)$$

1.3 Conservative Systems

Earlier we found that a conservative force field has the properties that:

1. The components of the generalized force can be written in terms of the first derivative with respect to the generalized coordinates of a scalar that depends only on the configuration

$$Q_i = -\frac{\partial V(q)}{\partial q_i} \quad (25)$$

2. The path integral of the generalized force is independent of path taken between the end points

$$W = \int_a^b \vec{Q} \cdot d\vec{q} = \sum_{i=1}^n \int_{a_i}^{b_i} Q_i dq_i \quad (26)$$

Assuming that no other forces act on the system, the total mechanical energy $E = T + V$ is an integral of the motion.

It is possible to find an energy-like integral of the motion that is of greater generality, which will give a more suitable definition of a conservative system. The definition is as follows:

1. The standard form of the Lagrange's equation applies (the system can be either holonomic or nonholonomic);

2. The Lagrangian function L is not an explicit function of time;
3. Any constraint equations can be expressed in differential form

$$\sum_{i=1}^n a_{ji} dq_i = 0 \quad (27)$$

where all the $a_{jt} = 0$.

To show that these conditions are sufficient to ensure the existence of an energy integral, we will consider a system in nonholonomic form with a Lagrangian L independent of time

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^m \lambda_j a_{ji} \quad (28)$$

Notice that for any holonomic system, the constraint functions $f_j(q)$ must be independent of time for the condition $a_{jt} = 0$ to hold. Next take the total time derivative of the Lagrangian

$$\frac{dL}{dt} = \sum_{i=1}^n \left[\frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i + \frac{\partial L}{\partial q_i} \dot{q}_i \right] \quad (29)$$

But Eq. 28 gives an expression for

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \sum_{j=1}^m \lambda_j a_{ji} \quad (30)$$

which can be substituted into Eq. 29

$$\frac{dL}{dt} = \sum_{i=1}^n \left[\frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \dot{q}_i \right] + \sum_{i=1}^n \sum_{j=1}^m \lambda_j a_{ji} \quad (31)$$

where the last term is zero from Eq. 27. Therefore, the total time derivative of the Lagrangian is

$$\frac{dL}{dt} = \sum_{i=1}^n \left[\frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \dot{q}_i \right] = \frac{d}{dt} \left[\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right] \quad (32)$$

which can be directly integrated since both sides of the equation are total derivatives with respect to time

$$\sum_{i=1}^n \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = h \quad (33)$$

where h is a constant.

Next recall that the kinetic energy can be written as the sum of the a term quadratic in the generalized velocity, one that is linear, and one that is independent

$$T = \frac{1}{2} \sum_{k=1}^{3N} m_k \left(\sum_{i=1}^n \frac{\partial x_k}{\partial q_i} \dot{q}_i + \frac{\partial x_k}{\partial t} \right)^2 = T_2 + T_1 + T_0 \quad \Rightarrow \quad L = T_2 + T_1 + T_0 - V \quad (34)$$

Since the potential energy is independent of the generalized velocity, the summation in Eq. 33 can be written as follows

$$\sum_{i=1}^n \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i = 2T_2 + T_1 \quad (35)$$

which leads to

$$T_2 - T_0 + V = h \quad (36)$$

which has units of energy. This integral of the motion is called the Jacobi integral and sometimes the energy integral.

Notice that Eq. 36 can be expressed as the sum of a term quadratic in the generalized velocity $T' = T_2$ and a second term that depends only on the coordinates $V' = V - T_0$, therefore the Jacobi integral can be expressed as

$$T' + V' = h \quad (37)$$

The quantity V' represents the real potential plus a term that comes from the inertial terms (fictitious force). These arise because some of the generalized coordinates are measured relative to a moving frame. T' is the kinetic energy assuming that any moving constraints or reference frames are held fixed. Therefore, the quantity $T' + V'$ is a constant for any conservative system, but it is not always the total energy of the system measured relative to an inertial frame.

1.4 Natural Systems

A natural system is a conservative system that has the following additional properties:

1. It is described by the standard holonomic form of the Lagrange equations;
2. The kinetic energy is expressed as a homogeneous quadratic function of the \dot{q}_i

$$T = T_2 = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m m_{ij} \dot{q}_i \dot{q}_j \quad (38)$$

where the m_{ij} can be functions of the q 's but are independent of time.

The Jacobi integral in such a system requires $T_0 = 0$. Note, that this means that the transformation equations from rectangular coordinates to generalized coordinates must be independent of time for a natural system since $T_0 = 0$, this is not the case for a conservative system in general.

1.5 Example Jacobi Integral

Consider the system shown in Fig. 1. Calculate the Jacobi integral for it. We start with the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgy \quad (39)$$

Notice that the system can be defined in terms of a single variable θ , therefore we transform the Lagrangian to this single coordinate incorporating the constraint

$$\begin{aligned} x &= r \sin \theta \cos \omega t \\ y &= r \sin \theta \sin \omega t \\ z &= r \cos \theta \end{aligned} \quad (40)$$

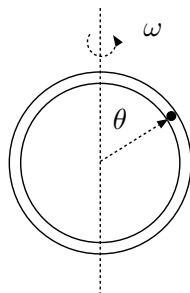


Figure 1: Rotating hollow ring with particle of mass m free to slide within hollow region. The contact between the ring and particle is massless.

which leads to the Lagrangian

$$L = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\omega^2 \sin^2 \theta) - mgr \cos \theta \quad (41)$$

where the constraint is Incorporated into the generalized coordinates, so it is not given independently at this point. Since the Lagrangian is independent of time, the system is conservative. The Jacobi integral is

$$h = T_2 - T_0 + V = \frac{1}{2}m(r^2\dot{\theta}^2 - r^2\omega^2 \sin^2 \theta) + mgr \cos \theta \quad (42)$$

Notice that the term T_0 does not depend on any velocities, therefore it could be considered as part of the potential. In fact if we take a frame of reference that moves with the ring, then one can consider the T_0 term as a fictitious force. In this case the Jacobi integral is given in the form

$$\left. \begin{aligned} T' &= \frac{1}{2}m\dot{r}^2 \\ V' &= mgr \cos \theta - \frac{1}{2}mr^2\omega^2 \sin^2 \theta \end{aligned} \right\} \Rightarrow T' + V' = h \quad (43)$$

where this is a natural system, since the kinetic energy depends only on the square of the velocity, and V' is a function only of the coordinates.

An interesting aspect of the potential V' is that for $\omega^2 > g/r$ there are two minimum in the effective potential, while for $0 \geq \omega^2 \leq g/r$ there is a single symmetric minimum. This is an example of dynamical symmetry breaking.

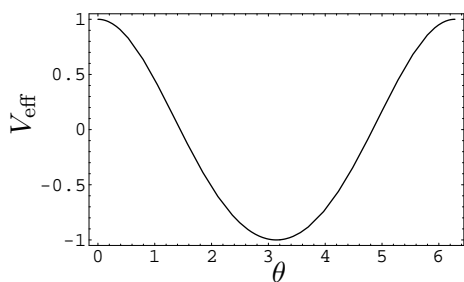


Figure 2: This plot shows the effective potential when $0 \leq \omega^2 \leq g/r$

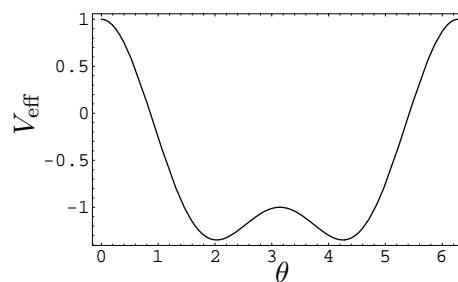


Figure 3: This plot shows the effective potential when $\omega^2 > g/r$