Physics 5153 Classical Mechanics
Introductory Remarks

1 Introduction

Since Newton laid down the foundations of classical mechanics in his three laws of motion and various definitions, there have been two fundamental approaches to mechanics. The vectorial approach that has it origins with Newton, and the analytical approach that originated with Leibniz[1]. In the vector approach, the motion of a particle is completely specified by the known forces acting on it at every point along its path. The forces determine the change in momentum, this being the quantity that defines the motion, both the force, and momentum being vector quantities. The analytic approach replaces the momentum by the kinetic energy, and the forces by the work function[1]. These two scalar functions along with the principle of energy conservation lead to a single scalar equation. This at first sight might appear puzzling, since particles move in three dimensions and the analytical mechanics has a single scalar equation.

During this semester, we will concentrate on the analytic approach to mechanics. In particular, we will concentrate on the methods developed by Lagrange and the later discoveries of Hamilton and Jacobi. These approaches to mechanics allow one to discuss systems that could not have been imagined by Newton or for that matter, Lagrange, Hamilton and Jacobi. For instance, the Lagrangian approach allows one to discuss classical field theories such as electromagnetism. The Lagrangian is the starting point for relativistic quantum field theory. The Hamiltonian approach to mechanics comes into play in the discussion of non-relativistic quantum mechanics, and statistical mechanics. All these topics are beyond what Newton had intended with his original formulation.

One might ask why develop alternate formulations of mechanics? The answer is simple, they provide new insights into nature that the original formulation might not have provided. In addition, they allow one to expand beyond the boundaries of what the original formulation allowed as is pointed out above.

In this lecture, we begin with a brief discussion of kinematics, that is, what do we mean by motion. This is floowed by a critique of Newton's laws along with a reformulation that is more self-consistent. We will then discuss two integrals of the equations of motion and what we learn from each.

1.1 Elementary Kinematics

Classical mechanics deals with the motion of particles. To describe motion, we need a method of describing the change in position of the particle under study. We do this by defining a reference frame and measure position relative to its origin. We assume that the space we deal with is Euclidean. In a Cartesian coordinate system, the position is given by three numbers that are projections onto the three orthogonal axis

$$\vec{\mathbf{x}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} = \sum_i x_i\hat{\mathbf{x}}_i \equiv x_i\hat{\mathbf{x}}_i \tag{1}$$

[1]Most modern text call this the work function the work.

As the particle moves, the position vector changes, and we need a parameter t that increases monotonically as $\vec{\mathbf{x}}(t)$ runs through successively later positions. This is an intuitive definition that assumes we can distinguish what is meant by *before* and *after*. Therefore, if $t_2 > t_1$, then the particle is at the position $\vec{\mathbf{x}}(t_2)$ later than at the position $\vec{\mathbf{x}}(t_1)$. In addition we require that the first and second derivatives with respect to t exist.

Note that we have not said how we define the reference frame or how we measure time. These will come later.

1.2 Newton's Laws

The standard starting point for mechanics is Newton's three laws along with some associated definitions[3]. The definitions that are necessary for understanding the three laws are:

1. The quantity of matter in a body will be measured by its density and volume conjointly. This quantity of matter will be known by the name of body or mass; it will also be known by the weight of the body in question. That mass is proportional to weight has been found by very careful experiments on pendulums.

2. Quantity of motion is measured by the velocity and the quantity of matter (mass) conjointly.

3. The innate force of matter is a power of resisting by which every body, as much as in it lies, endeavors to preserve in its present state, whether it be of rest or of moving uniformly forward in a straight line.

4. An impressed force is an action exerted upon a body in order to change its state either of rest or of moving uniformly forward in a straight line.

and the three laws are:

1. Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it is compelled by forces to change that state.

2. Change in motion is proportional to the force and takes place in the direction of the straight line in which the force acts
$$\vec{\mathbf{F}} = m\vec{\mathbf{a}}$$

3. To every action there is always an equal and contrary reaction; or, the mutual actions of any two bodies are always equal and oppositely directed along the same straight line.

There have been many discussion throughout history on the problems with Newton's formulation of his laws. In this brief discussion we will attempt to outline the basic criticism. The first law has the basic problem of not defining a particular frame of reference in which it holds. In addition, to define a velocity, one needs to specify how one measures time, that is define a clock. As an example, consider a frame that is in free-fall relative to the Earth. An object at rest relative to this frame clearly experiences a force. On the other hand, the object relative to a frame tied to the Earth is not in uniform motion (*this of course means we have an appropriate clock to measure time*). Therefore, before the first law can be tested experimentally, an appropriate reference frame and clock need to be defined. Once we define an appropriate frame of reference and a clock, the first law defines what we mean by force.

Having discussed the basic problems with the first law, we move onto the second law. The lack of a definition of a reference frame and a clock hold also for the second law. In addition, the second law has two undefined quantities, force and mass.

Finally, we look at the third law. We have the same problems with the lack of a reference frame and lack of a definition of force. On the other hand, we do arrive at a definition of mass. Assume an isolated system of two particles, of course we need a proper definition of a force to define isolated, then the third law allows us to define mass as follows

$$\frac{m_2}{m_1} = \frac{a_1}{a_2} \tag{2}$$

where one can define define the mass of all objects relative to the mass of a single object. But as we have already stated, this requires an isolated system, which requires a proper definition of force and reference frame.

Many people have attempted to give alternate formulations to Newton's laws that overcome the difficulties. A cleaner version that has in origins in the work of Mach [5] and given by Eisenbud [4] states them as two principles. Before stating them, we point out that they apply to point particles, even though they can be extended to objects of finite size by summing up their effect on differential elements of the object. Next we need to introduce the concept of an isolated particle. By this we mean the object is sufficiently small and sufficiently far from other matter, where sufficiently refers to the accuracy that we can measure the motion of the object. For example, assume that we can only measure the position of an object to 1 mm. Assume that the object has a velocity of 1 nm/s. It would take it 11.6 days to move 1 mm. Therefore, if the measurements we plan to carry out on the object takes a couple of hours, the object is at rest.

Given the statements in the preceding paragraph, the two principle are [2]

1. There exist certain frames of reference, called inertial, with the following two properties:

    (a) Every isolated particle moves in a straight line in such a frame. The unit of time can be defined by using the motion of any such isolated particle as a standard: equal lengths are marked off on its trajectory, and the time intervals in which the particle crosses successive marks are defined as equal intervals of time. It then follows by definition that the standard particle is moving at constant velocity.

    (b) When time is so defined, every other isolated particle also moves at constant velocity in this frame. In other words, the definition of time is independent of the particle by which it is defined.

2. There are two parts to this principle. The first deals momentum conservation, and the second with the existence of mass.

    (a) *Conservation of momentum.* Consider two particles 1 and 2 isolated from all matter, but not from each other, and observed from an inertial frame. In general they will not move at constant velocity: their interactions cause them to accelerate. Let $v_i(t)$ be their velocities. Then their exists a positive constant μ_{12} and a constant vector $\vec{\mathbf{K}}$ such that

$$\vec{\mathbf{v}}_1(t) + \mu_{12}\vec{\mathbf{v}}_2(t) = \vec{\mathbf{K}} \tag{3}$$

In addition, μ_{12} is independent of the inertial frame and of the motion. If the experiment is performed with particles 1 and 3, and 2 and 3, similar results are obtained

$$\vec{\mathbf{v}}_2(t) + \mu_{23}\vec{\mathbf{v}}_3(t) = \vec{\mathbf{L}} \tag{4}$$

$$\vec{\mathbf{v}}_3(t) + \mu_{31}\vec{\mathbf{v}}_1(t) = \vec{\mathbf{M}} \tag{5}$$

(b) *Existence of Mass.* The μ_{ij} are related according to

$$\mu_{12}\mu_{23}\mu_{31} = 1 \tag{6}$$

Notice that the first principle is a statement of Newton's first law, while the second principle leads to momentum conservation

$$\begin{aligned} m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 &= \vec{\mathbf{P}}_{12} \\ m_2\vec{\mathbf{v}}_2 + m_3\vec{\mathbf{v}}_3 &= \vec{\mathbf{P}}_{23} \\ m_3\vec{\mathbf{v}}_3 + m_1\vec{\mathbf{v}}_1 &= \vec{\mathbf{P}}_{31} \end{aligned} \tag{7}$$

Taking the derivative of these equations leads to

$$m_i\vec{\mathbf{v}}_i + m_j\vec{\mathbf{v}}_j = 0 \tag{8}$$

Defining the force as $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, we have a statement of Newton's third law and with the definition of the force all three laws are contained in the two principles given above.

1.3 Integral's of Motion

To calculate the motion[2] of a particle, one starts with Newton's second law

$$\vec{\mathbf{F}} = m\frac{d\vec{\mathbf{v}}}{dt} \tag{9}$$

where we assume that the mass, m, of the particle is constant. An obvious first integral can be taken over time variable

$$\vec{\mathbf{F}} = m\frac{d\vec{\mathbf{v}}}{dt} \quad \Rightarrow \quad m\vec{\mathbf{v}} - m\vec{\mathbf{v}}_0 = \int_{t_0}^{t} \vec{\mathbf{F}}dt \tag{10}$$

The integral leads to three cases that have straightforward applications: When the force is constant in space and has only a time dependence the integral is easily calculated analytically or numerically. The second case is when the force is of a very short duration. Here we can use the impulse approximation, $d\vec{\mathbf{p}} = \vec{\mathbf{F}}dt$, where dp is typically defined to be the impulse. The third case is when the net force acting on the object is zero ($\sum_i \vec{\mathbf{F}}_i = 0$). This in fact is the most interesting case, because it leads to the momentum being a constant of the motion; momentum conservation. This in fact becomes a principle of physics, any system of particles that experience a zero net force will have a constant momentum in time. This principle can be expanded to include angular momentum.

Even though the time integral of the equations of motion is useful, there are cases where it is difficult to apply. Take the case where a particle traverses a region of space with a position dependent force. These problems are easier to solve, either analytically or numerically, by calculating

[2]By motion we mean either the velocity as a function of time or position or the position as a function of time.

the space integral of the force. Since we want the space integral to correspond to the path taken by the particle, the space integral is defined as a dot product. This allows the direction of the force relative to path direction to be taken into account. Therefore, the space integral of Newton's second law takes on the following form

$$\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = m \int \frac{d\vec{\mathbf{v}}}{dt} \cdot d\vec{\mathbf{r}} = m \int \frac{d\vec{\mathbf{v}}}{dt} \cdot \frac{d\vec{\mathbf{r}}}{dt}\, dt = m \int \vec{\mathbf{v}} \cdot \frac{d\vec{\mathbf{v}}}{dt}\, dt = \frac{1}{2} m \int \frac{d|\vec{\mathbf{v}}|^2}{dt}\, dt = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 \quad (11)$$

The left hand side of this equation is defined as the work done on the particle by the force during the displacement. The right-hand side is the kinetic energy, which is just a different way of writing $m\vec{\mathbf{a}}$; we account for the acceleration through the change in velocity in the kinetic energy difference. In order to interpret this equation and to extract any new physics from it, let's start with a simple case where the force can be written as the gradient of a scalar function[3] $\vec{\mathbf{F}} = -\boldsymbol{\nabla} U$, therefore the integrand is a total derivative $\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -dU$. In this case, the integral of the left hand side is the difference of the scalar function (potential) at the endpoints of the trajectory (*it is independent of the path between the endpoints*

$$U_1 - U_2 = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 \quad \Rightarrow \quad \frac{1}{2} m v_1^2 + U_1 = \frac{1}{2} m v_2^2 + U_2 = E \quad (12)$$

where the second equation has grouped the kinetic and potential energies at the appropriate endpoints together. Notice that the sum of the kinetic and potential energies (total energy) at the two endpoints are the same. Since the endpoints are arbitrary, this implies that the total energy is conserved for all points along the path. Therefore, we have a new principle of physics; energy conservation for systems where the force is derived from a potential. These are defined as conservative systems, with the following being examples of such potentials:

$\vec{\mathbf{F}} = -mg$	$U = mgy$	Constant force
$\vec{\mathbf{F}} = -kx$	$U = \frac{1}{2} k x^2$	Linear (spring) force
$\vec{\mathbf{F}} = -\frac{GMm}{r^2}$	$U = -\frac{GMm}{r}$	Gravitational force

When applying a new theory or principle, one has to make sure that the situation fits the assumptions of the theory. None-the-less we always try to expand the theory beyond its limits. As an example of a non-conservative system, consider a ball rolling down the parabolic slide under the influence of gravity as shown in Fig. 1. If we assume that the ball rolls over the entire path, we know that the ball will not return to its original height even though gravity is a conservative force. That is, energy is lost. From earlier courses, we know that the lost energy is due to friction between the surface and the ball. In fact, it is friction that causes the ball to roll. Mathematically we can account for the lost energy as follows

$$\Delta U = -\int |\vec{\mathbf{f}}| d\ell \quad (13)$$

[3] the minus sign is used to insure that the force points in the direction of smaller potential.

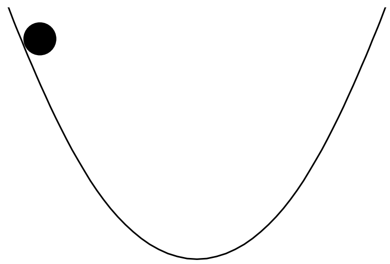

Figure 1: Ball on a parabolic slide.

where $\vec{\mathbf{f}}$ is the frictional force, $d\ell$ is a differential path length, the minus sign is due to the force and path being opposite direction, and ΔU is the difference in potential energy between two endpoints of the motion. Based on this, the lost energy goes into work done by friction on the ball, and ultimately it generates heat in the system. In the end, we still apply the concept of energy conservation except that we expand the definition of energy to include sources that are not mechanical. Therefore, the concept that we started has been generalized due to its usefulness.

References

[1] The Variational Principles of Mechanics, *C. Lanczos*pgs. xxi, xxix

[2] Theoretical Mechanics, *E.J. Saletan & A.H. Cromer*, chap. 1

[3] The Foundations of Physics, *R.B. Lindsay & H. Margenau*, chap. 3

[4] On the Classical Laws of Motion, *L. Eisenbud*, Am. J. Phys, **26**. 144 (1958)

[5] Science of Mechanics, *E. Mach*,