Physics 4213/5213
Lecture 17

1 Introduction

In the previous lecture, the concept of a non-Abelian gauge theory was introduced. This was done as an extension to the Abelian $U(1)$ case of electromagnetism. The basic idea behind gauge theory is to find a symmetry group under which the Lagrangian (interaction) is invariant and use this to generate the dynamics of the system. The system that was of interest in the last lecture was the symmetry or near symmetry between the proton and the neutron. Based on their similar masses and their different charges, it was assumed that under strong interactions, the proton and neutron are identical particles. On the other hand, it was assumed that the electromagnetic interaction broke the symmetry. Therefore under strong interactions, the proton and neutron can be placed in the doublet (isospin equals 1/2) representation of $SU(2)$. If this represents a global invariance of the Lagrangian, then the transformation can be turned into a local phase transformation and used to generate interactions. This lead to a triplet (isospin equals 1) representation of $SU(2)$ for the vector gauge bosons. These objects are massless and do not represent any observable particles.

Of course, it could also be argued that maybe this interaction could be used to represent weak interactions, since based on the Fermi model at least two vector bosons are needed; but these vector bosons are massless. Therefore this also does not appear to represent weak interactions, unless the vector bosons can be given a mass in a way that preserves the gauge invariance of the Lagrangian.

First of all the problem of approximate symmetries needs to be accounted for, that is are these real symmetries that are hidden. One way to hide a symmetry, is as stated earlier. That is, the symmetry is only valid for one of the interactions, strong for example, while the other interaction causes a shift in the degeneracy. An example of this is a ferromagnetic material above the Curie temperature. The magnetic dipoles in this case are randomly aligned due to the lack of any long range forces. The material then exhibits an $SO(3)$ symmetry; invariant to rotations in three dimensions. Once an new interaction is introduced, an external magnetic field, the spins are aligned along the field and the symmetry is broken to an $SO(2)$ symmetry; invariant to rotations about the field lines. Of course once the field is switched off the original $SO(3)$ symmetry is recovered; this can not happen for internal symmetries.

A second example of a broken symmetry, is one that occurs spontaneously. Here there are infinitely many possible ground states that can occur due to the original symmetry. Yet only one of those will occur. Again using the ferromagnet as an example, lower the temperature of the material until it drops below the Curie temperature. At the Curie temperature a phase transition occurs and long range forces between the various magnetic dipoles occur causing them to line up in some random direction. This again causes the original $SO(3)$ symmetry to be broken to an $SO(2)$ symmetry. In this case as stated above, the symmetry is broken spontaneously due to a phase transition. Notice that, this phenomena can not be investigated perturbativly since the Hilbert space describing the two states are very different.

One final scheme that not only breaks a symmetry but also causes the quanta of long range forces to generate masses is the Higgs mechanism. Aitchison and Hey [1] give a very nice description of this for a superconductor. The basic argument is that once the ground state is reached, the Cooper pairs form a resistancesless gas of bosons. The introduction of a magnetic field creates currents that are expelled from the material. This can be view as the photon (magnetic field) acquiring a mass which then limits its range; remember a massless quanta corresponds to a force of infinite range, while a massive quanta corresponds to a force of finite range. It is this last mechanism that will be of most use in particle physics. But first the other mechanism will be investigated.

2 Symmetry Breaking—Discrete Symmetry

As a first example of a physical system where the Lagrangian exhibits a symmetry that is not exhibited by the vacuum, take a Lagrangian which describes the interaction of real scalar fields and is invariant under a parity transformation ($\phi \to -\phi$). Such a Lagrangian is given by

$$\mathcal{L} = (\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{2}\mu^2\phi^2 - \frac{1}{4}\lambda\phi^4, \tag{1}$$

where the parameter λ is positive in order that the energy be bounded from below and terms higher order than ϕ^4 are excluded to make the theory renormalizable. Figure 1 show the form of the effective potential, $V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4$, for two possible choices of the parameter μ^2; one where $\mu^2 > 0$ and one where $\mu^2 < 0$.

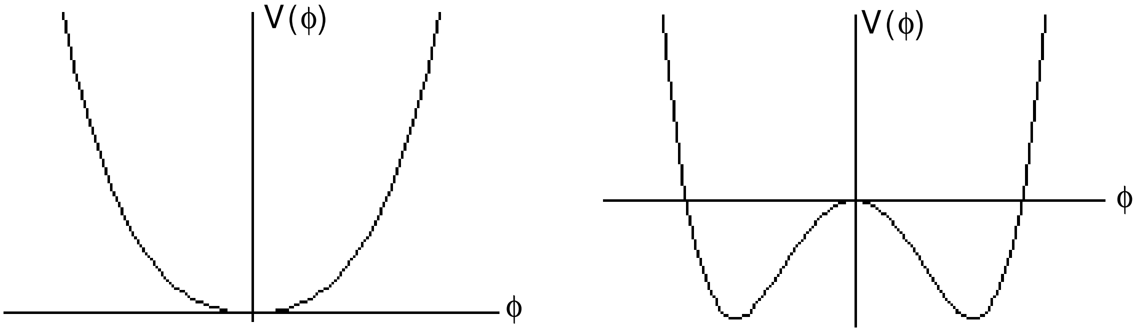

Figure 1: The figure on the left corresponds to the scalar potential for the case where $\mu^2 > 0$. The figure on the right corresponds to the case where $\mu^2 < 0$. The minimum energy states correspond to $\langle\phi\rangle_0 = \pm v$.

For the first choice $\mu^2 > 0$, the vacuum expectation value is given by $\langle\phi\rangle_0 \equiv \langle 0|\phi|0\rangle = 0$, with the vacuum still exhibiting the the original symmetry. To determine the mass spectrum of the theory—in this case it is obvious what it is, but for more complicated cases it may not be, so a procedure needs to be developed—expand about the vacuum keeping only the lowest order terms. This makes the Lagrangian

$$\mathcal{L} \approx (\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{2}\mu^2\phi^2, \tag{2}$$

which describes a free particle of mass μ. In this case the Lagrangian and the vacuum exhibit the same symmetry, so that the parameter μ has the expected interpretation of a mass.

Now consider the case where the parameter $\mu^2 < 0$; this makes the interpretation of μ being a mass untenable. The vacuum, being the lowest energy state, are now given by $\langle\phi\rangle_0 = \pm\sqrt{-\mu^2/\lambda} \equiv$

$\pm v$ which has a two fold degeneracy. Notice the Lagrangian is still invariant to a parity transformation, at least until a specific vacuum state is selected. Either of the two possible vacuum states is acceptable and will lead to the same physics due to the original symmetry. Once a specific vacuum state is selected, the symmetry of the Lagrangian will disappear; be hidden.

To look at a specific case, take the vacuum state on which to construct a quantum field theory as $\langle\phi\rangle_0 = +v$. To determine the particle spectra of the theory, expand the Lagrangian for small oscillations about the vacuum state. The state $\langle\phi\rangle_0 = 0$ is no longer a valid point about which to expand, since this point is a local maximum and any fluctuation will cause the particle to fall into one of the two valleys (vacuum states). To facilitate the expansion, take the shifted field $\xi(x) = \phi(x) - v$, which is centered on the selected vacuum. The Lagrangian is now given by

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\xi)(\partial_\mu\xi) - \lambda v^2\xi^2 - \lambda v\xi^3 - \frac{1}{4}\lambda\xi^4, \tag{3}$$

where the original symmetry under parity is no longer evident for the ξ field. Expanding the Lagrangian about the vacuum expectation value and keeping only the lowest order terms, the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\xi)(\partial_\mu\xi) - \lambda v^2\xi^2, \tag{4}$$

which as before corresponds to a free particle Lagrangian, but this time the particle mass is given by $m = \sqrt{-2\mu^2}$; remember that $\mu^2 < 0$. Therefore due to the breaking of the symmetry, the mass is shifted and the symmetry is hidden.

3 Symmetry Breaking—Continuous Symmetry

Next consider a Lagrangian that exhibits a continuous symmetry. A possible example is a Lagrangian composed of complex fields, which is invariant under a $U(1)$ transformation. This Lagrangian is just that for a complex Klein-Gordon field

$$\mathcal{L} = (\partial^\mu\phi)^\dagger(\partial_\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi\phi^\dagger)^2. \tag{5}$$

The Lagrangian can be seen to be invariant to a global $U(1)$ transformation

$$\phi \to e^{i\theta}\phi. \tag{6}$$

For simplicity, define $\rho = \phi^\dagger\phi$, thereby making the effective potential $V(\rho) = \mu^2\rho + \lambda(\rho)^2$. Again $\lambda > 0$ in order for the energy to be bounded from below and μ^2 can be either positive or negative (see fig. 2). As in the case of discrete symmetries, the effective potential for $\mu^2 > 0$ continues to exhibit the $U(1)$ symmetry and the Lagrangian corresponds to that of a free particle of mass μ.

The second case $\mu^2 < 0$ again gives a vacuum expectation value that is not equal to zero. This time the expectation value is given by a circular trough of radius

$$\langle\phi\rangle_0 = \sqrt{\frac{-\mu^2}{2\lambda}} \equiv \frac{v}{\sqrt{2}}. \tag{7}$$

Therefore the possible vacuum states are now infinitely degenerate, since it exhibits the original $U(1)$ symmetry. By selecting one of the infinitely many possible vacua, note they are all equivalent

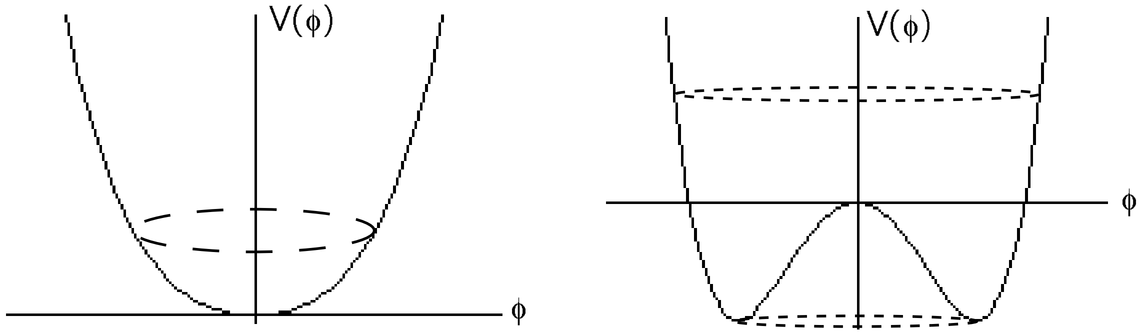

Figure 2: The figure on the left corresponds to the scalar potential for the case where $\mu^2 > 0$. The figure on the right corresponds to the case where $\mu^2 < 0$. The minimum energy states correspond to $\langle\phi\rangle_0 = \pm v$.

and lead to the same physics, the symmetry is said to be spontaneously broken. To determine the particle spectra, the Lagrangian is expanded about the vacuum state selected. The easiest means of doing this, is to shift the fields, centering them on the vacuum state. For simplicity select the vacuum on the real ϕ axis. Then expand the fields about this point to investigate their behavior:

$$\phi = \frac{1}{\sqrt{2}}(v + \xi(x))e^{i\chi(x)/v} \approx (v + \xi(x) + i\chi(x)). \tag{8}$$

The Lagrangian then becomes

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \xi)^2 + \frac{1}{2}(\partial_\mu \chi)^2 - \lambda v^2 \xi^2 - \lambda v \xi(\xi^2 + \chi^2) - \frac{1}{4}\lambda(\xi^2 + \chi^2)^2 + \text{constants}. \tag{9}$$

To determine the particle spectra only the lowest order terms are examined (free particle Lagrangian)

$$\mathcal{L} = \left[\frac{1}{2}(\partial^\mu \xi)^2 - \lambda v^2 \xi^2\right] + \frac{1}{2}(\partial_\mu \chi)^2. \tag{10}$$

The Lagrangian, which originally described a single massive particle, now describes a particle of mass $m_\xi = \sqrt{-2\mu^2}$ and a second massless particle. This second massless particle is referred to as a Goldstone boson—sometimes referred to as a Nambu-Goldstone boson. A general theorem states that *for each broken group generator of a continuous global symmetry, there will appear in the theory one massless boson.*

To see the origin of the Goldstone boson, look at equation 10 and notice that the field ξ can thought of as "radial oscillations" against a restoring force that generates the mass. The field χ corresponds to angular motion along the trough, where there is no restoring force so that no inertia (mass) is generated.

4 Symmetry Breaking—Abelian Gauge Symmetry

As of now, two examples of broken symmetry have been given, but neither of the two help in resolving the main problem. That of generating masses for the vector bosons of the $SU(2)$ theory.

Further in the case of a theory that contains a global symmetry, more massless particles are generate, giving even more objects that are not seen. In this section, symmetry breaking will be applied to the case of a Lagrangian that exhibits a local symmetry; invariance under a local phase transformation. This is in fact the case that is of most interest, since this is the symmetry operation that generates interactions.

To start with, consider the example of a Lagrangian that is invariant to a local $U(1)$ transformation. For simplicity assume that the matter fields are spin zero bosons. In addition to the electromagnetic interactions add the self interactions as in the previous two cases

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{11}$$

where as before $\lambda > 0$ and

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \tag{12}$$

$$D^\mu = \partial^\mu + iqA^\mu \tag{13}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{14}$$

This Lagrangian is invariant under local $U(1)$ phase transformations and local gauge transformations

$$\phi(x) \to e^{iq\alpha(x)} \tag{15}$$

$$A_\mu(x) \to A_\mu(x) - \partial_\mu\alpha(x). \tag{16}$$

As in the two previous examples of symmetry breaking, there are two possible cases for the parameter μ^2. One where it is positive, making the symmetry of the vacuum and the Lagrangian the same and the second when $\mu^2 < 0$. The later case is the one of interest since it is the one that will lead to a breaking of the symmetry.

First the effective potential is given by

$$V(|\phi|) = \mu^2|\phi| + \lambda|\phi|^2 \qquad \Longrightarrow \qquad \langle|\phi|^2\rangle_0 = -\frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2}, \tag{17}$$

where $\langle|\phi|^2\rangle_0$ corresponds to the radial value of the minimum in the complex ϕ_1-$i\phi_2$ plane. As in the global $U(1)$ case, choose the vacuum to be along the positive real ϕ_1 axis;

$$\langle\phi\rangle_0 = \frac{v}{\sqrt{2}}. \tag{18}$$

To look at the behavior of the free fields (the mass spectra) center the field at the value of the vacuum. The field for small excursions along the imaginary axis is given by

$$\phi(x) \approx \frac{1}{\sqrt{2}}\left[v + \eta(x) + i\xi(x)\right], \tag{19}$$

just as in the global $U(1)$ case. This expression is substituted into the Lagrangian, keeping only lowest order terms gives

$$\mathcal{L} \approx \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) + \mu^2\eta + \frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi) + qvA_\mu(\partial^\mu\xi) + \frac{1}{2}q^2v^2A^\mu A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{20}$$

This Lagrangian has three fields, the η with a mass of $m_\eta = \sqrt{-2\mu^2}$, the ξ which is massless and finally the A^μ field has now acquired a mass $m_A = qv$. A problem also appears to have occurred, the original number of degrees of freedom was 4 (2 for the photon and 1 each for the two scalar fields) now there appear to be 5 degrees of freedom (3 for the massive photon and 1 each for the two scalars). This turns out to be an illusion. By making an appropriate gauge transformation, the physics has to be invariant to gauge transformations, the ξ field can be made to disappear

$$\begin{aligned} \phi(x) &\rightarrow e^{-i\xi(x)/v}\phi(x) = \frac{v+\eta(x)}{\sqrt(2)} \\ A_\mu &\rightarrow A_\mu + \frac{1}{qv}\partial_\mu \xi(x). \end{aligned} \tag{21}$$

Substituting equation 21 into the Lagrangian of equation 20 gives

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + \mu^2 \eta + \frac{1}{2}q^2 v^2 A^\mu A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{22}$$

The extra massless field has disappeared, the number of degrees of freedom is now 4 as expected and the photon has acquired a mass in a manner that keeps the Lagrangian gauge invariant. The scalar fields in this theory are the Higgs field. The massless scalar field that had earlier appeared can be thought of as having been eaten by the photon to gain its mass (longitudinal degree of freedom).

Finally a theory that does not introduce any new massless particles and gives the gauge bosons mass has been found. This theory does introduce a new scalar boson that was not there originally. The acquisition of a mass by the photon required the scalar field and the parameters associated with it.

References

[1] Gauge Theories in Particle Physics *Aitchison and Hey* Chapter 13