Physics 4213/5213
Lecture 16

1 Introduction

To get to a theory of weak interaction, the concept of non-Abelian gauge transformations have to be introduced. To do this without adding the complexity of weak interactions, a simple theory of strong interactions will be derived. The theory itself does not work for strong interactions, but it will be found to work for weak interactions with two modifications. This lecture will begin with a very quick review of the QED Lagrangian, and how gauge invariance is used to create it from the Dirac Lagrangian.

2 Review of Gauge Invariance in Electrodynamics

3 A Theory of Strong Interactions

In strong interactions, it is observed that the proton and neutron behave the same. The differences in mass are attributed to the electromagnetic interactions. Therefore it is assumed that if the electromagnetic interactions could be switched off, the proton and neutron would be indistinguishable. The symmetry is call isospin, and it operates similar to spin in ordinary space. To see how isospin works, start by writing the Dirac Lagrangian for the proton and neutron:

$$\mathcal{L} = \bar{p}\,(i\,\partial\!\!\!/ - m)\,p + \bar{n}\,(i\,\partial\!\!\!/ - m)\,n \tag{1}$$

in order for the Lagrangian to be invariant under a transformation that converts a neutron to a proton and visa versa, the two masses must be equal. The Lagrangian can be put into a more compact form by defining the isospin doublet:

$$\psi = \begin{pmatrix} n \\ p \end{pmatrix} \tag{2}$$

it is important to note that both n and p are Dirac spinors, therefore the state ψ is a iso-spinor of Dirac spinors. The Lagrangian can now be written in the form:

$$\mathcal{L} = \bar{\psi}\,(i\,\partial\!\!\!/ - m)\,\psi \tag{3}$$

Since isospin is conserved in strong interaction, the Lagrangian must be invariant to any rotation in isospin space ($\psi \to \mathcal{R}(\alpha)\psi$). The rotation operator is the same as for ordinary spin, and is given by:

$$\mathcal{R}(\alpha) = e^{i\vec{\tau}\cdot\vec{\alpha}/2} \tag{4}$$

where $\vec{\tau}$ are the Pauli spin matrices.

For the Lagrangian to invariant to a rotation in isospin space, it must satisfy the following requirement:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\partial_\mu\psi + \frac{\partial\mathcal{L}}{\partial\bar{\psi}}\delta\bar{\psi} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\delta\partial_\mu\bar{\psi} = 0 \tag{5}$$

where the infinitesimal transformation of ψ is given by:

$$\psi \to \psi + \frac{i}{2}\vec{\alpha}\cdot\vec{\tau}\psi \quad \Rightarrow \quad \begin{cases} \delta\psi = \frac{i}{2}\vec{\alpha}\cdot\vec{\tau}\psi \\ \delta\partial_\mu\psi = \frac{i}{2}\vec{\alpha}\cdot\vec{\tau}(\partial(\partial_\mu\psi)) \end{cases} \tag{6}$$

Now combine equations 5 and 6 to yield:

$$\partial_\mu\vec{\alpha}\cdot\left[\frac{i}{2}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right] = 0 \tag{7}$$

where the Euler-Lagrange equations are used to derive the expression:

$$\frac{\partial\mathcal{L}}{\partial\psi} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} \tag{8}$$

Now, given that $\vec{\alpha}$ is arbitrary and independent of the coordinates, the following must be hold:

$$\partial_\mu\left[\frac{i}{2}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\vec{\tau}\psi\right] = 0 \tag{9}$$

which has the form of a current conservation equation $\partial_\mu J^\mu = 0$ with the current given by:

$$J^\mu = \left[\frac{i}{2}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\vec{\tau}\psi\right] \tag{10}$$

The current can be written in terms of the iso-spinors as:

$$J^\mu = \frac{1}{2}\bar{\psi}\gamma^\mu\vec{\tau}\psi \tag{11}$$

which is referred to as the isospin current, and leads to the conservation of J^0 the isospin. Note that J^0 can be treated as a charge.

It was shown above that a global phase transformation leads to the conservation of charge. This holds for all forms of global transformations, they always lead to charge conservation (a conservation law). Now in electromagnetism it was also shown that by imposing the requirement that the phase transformation be local, interactions were generated. The same condition will therefore be imposed on the isospin transformation. The Lagrangian must now be invariant to the transformation:

$$\psi \to e^{i\vec{\tau}\cdot\vec{\alpha}(x)/2}\psi \tag{12}$$

Now as in the case of electromagnetism, the derivative must simultaneously undergo the following transformation:

$$\partial_\mu \to \partial_\mu + ig\vec{\tau}\cdot\vec{b}_\mu \tag{13}$$

notice that B_μ is a three component iso-vector of 4-vectors ($\vec{b}_\mu = (b^1_\mu, b^2_\mu, b^3_\mu)$). At this point the covariant derivative is defined as $\mathcal{D}_\mu \equiv \partial_\mu + ig\vec{\tau}\cdot\vec{b}_\mu$.

Now as in the case of electromagnetism, the local phase transformation is required to act on the covariant derivative term in the Lagrangian as follows:

$$\mathcal{D}_\mu\psi \to \mathcal{D}'_\mu\psi' = \mathcal{R}_x\mathcal{D}_\mu\psi \tag{14}$$

where $\mathcal{R}_x = \mathcal{R}[\alpha(x)]$ Applying the transformation directly on the explicitly form of the covariant derivative yields:

$$\begin{aligned}\mathcal{D}'_\mu \psi' &= (\partial_\mu + igB'_\mu)\psi' \\ &= \mathcal{R}_x[\partial_\mu \psi] + [\partial_\mu \mathcal{R}_x]\psi + igB'_\mu[\mathcal{R}_x \psi]\end{aligned} \tag{15}$$

where $B_\mu = \vec{\tau} \cdot \vec{b}_\mu$. Next combine equations 14 and 15 to get a condition on B'_μ:

$$igB'_\mu[\mathcal{R}_x \psi] = ig\mathcal{R}_x[B_\mu \psi] - [\partial_\mu \mathcal{R}_x]\psi \tag{16}$$

Since this equation must hold for any arbitrary ψ, this leads to:

$$B'_\mu = \mathcal{R}_x \left[B_\mu + \frac{i}{g}\mathcal{R}_x^{-1}(\partial_\mu \mathcal{R}_x) \right] \mathcal{R}_x^{-1} \tag{17}$$

In the case of electromagnetism, this is just the gauge condition:

$$A'_\mu = A_\mu - \partial\alpha(x) \tag{18}$$

To determine the gauge condition for the isospin transformation, start with an infinitesimal transformation:

$$\mathcal{R}_x \approx 1 + \frac{i}{2}\vec{\tau} \cdot \vec{\alpha} \tag{19}$$

Applying this to equation 17 leads to:

$$B'_\mu = B_\mu + \frac{i}{2}\vec{\alpha} \cdot \vec{\tau} B_\mu - \frac{i}{2} B_\mu \vec{\alpha} \cdot \vec{\tau} - \frac{1}{2g}\partial_\mu(\vec{\alpha} \cdot \vec{\tau}) \tag{20}$$

notice that if B_μ and $\vec{\alpha} \cdot \vec{\tau}$ commute, the relation is the same as in electromagnetism. This equation can be written in terms of $\vec{b}_\mu$ as:

$$\vec{\tau} \cdot \vec{b}'_\mu = \vec{\tau} \cdot \vec{b}_\mu + \frac{i}{2}(\vec{\tau} \cdot \vec{\alpha} \vec{\tau} \cdot \vec{b}_\mu - \vec{\tau} \cdot \vec{b}_\mu \vec{\tau} \cdot \vec{\alpha}) - \frac{1}{g}\partial_\mu(\vec{\alpha} \cdot \vec{\tau}) \tag{21}$$

The middle term can be written out in component form as follows:

$$\frac{i}{2}\alpha^j b^i_\mu(\tau^j \tau^i - \tau^i \tau^j) = \frac{i}{2}\alpha^j b^i_\mu[\tau^j, \tau^i] \tag{22}$$

where the commutator is given by $[\tau^j, \tau^i] = 2i\epsilon_{ijk}\tau^k$. This leads to:

$$\frac{i}{2}\alpha^j b^i_\mu[\tau^j, \tau^i] = -\vec{\alpha} \times \vec{b}_\mu \cdot \tau \tag{23}$$

which gives the transformation equation:

$$\vec{\tau} \cdot \vec{b}'_\mu = \vec{\tau} \cdot \vec{b}_\mu - \vec{\alpha} \times \vec{b}_\mu \cdot \tau - \frac{1}{g}\partial_\mu(\vec{\alpha} \cdot \vec{\tau}) \tag{24}$$

Finally, since the three components of the iso-vector $\vec{b}_\mu$ are linearly independent, the transformation equation is:

$$\vec{b}'_\mu = \vec{b}_\mu - \vec{\alpha} \times \vec{b}_\mu - \frac{1}{g}\partial_\mu \vec{\alpha} \tag{25}$$

which is the gauge condition for a non-Abelian phase transformation.

The Lagrangian for the local phase and gauge conditions given above is:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \mathcal{D}_\mu - m)\psi = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - g\bar{\psi}\gamma^\mu B_\mu \psi = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{g}{2}\vec{b}_\mu \cdot \bar{\psi}\gamma^\mu \vec{\tau}\psi \tag{26}$$

Having arrived at the Lagrangian for the free Fermion fields, and the interaction of the Fermions with the gauge fields, the piece that describes the free gauge fields must still be computed. To arrive at this piece, recall that in electromagnetism this term is given by $F^{\mu\nu}F_{\mu\nu}$, where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Further recall that both the tensor and the tensor product are gauge invariant, even though only the product of the tensors is required for the Lagrangian itself to be gauge invariant. In the case of isospin, there are three fields that go to form the tensor. The tensor is defined to be:

$$F^{\mu\nu} = \frac{1}{2}\vec{F}^{\mu\nu} \cdot \vec{\tau} \tag{27}$$

with the kinetic energy term given by:

$$-\frac{1}{4}\vec{F}^{\mu\nu} \cdot \vec{F}_{\mu\nu} = -\frac{1}{2}\vec{F}^{\mu\nu} \cdot \vec{\tau}\vec{F}_{\mu\nu} \cdot \vec{\tau} = -\frac{1}{2}\mathrm{tr}[F^{\mu\nu}F_{\mu\nu}] \tag{28}$$

where the relation $\mathrm{tr}[\tau^a\tau^b] = 2\delta^{ab}$ was used.

For electromagnetism, the field strength tensor can be written in terms of the commutator of covariant derivatives as follows:

$$F_{\mu\nu} = -\frac{i}{q}[\mathcal{D}_\mu, \mathcal{D}_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + iq[A_\mu, A_\nu] \tag{29}$$

where for electromagnetism the commutator is zero. Using this analogy, the field strength tensor for isospin (actually any non-Abelian gauge theory using the appropriate gauge field) is given by:

$$F_{\mu\nu} = -\frac{i}{q}[\mathcal{D}_\mu, \mathcal{D}_\nu] = \partial_\mu B_\nu - \partial_\nu B_\mu + iq[B_\mu, B_\nu] \tag{30}$$

which can be written in terms of the three vector fields as:

$$F_{\mu\nu} = \partial_\mu \vec{b}_\mu - \partial_\nu \vec{b}_\nu + g\vec{b}_\mu \times \vec{b}_\nu \tag{31}$$

Using the previously derived form for the gauge transformation in equation 25, the field strength tensor can be shown to be gauge invariant.

Finally the Lagrangian term associated with the fields only is given by:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{32}$$

which has terms that are b^3 and b^4. This leads to terms that have three and four gauge particles at the same vertex. This implies that the gauge particle carries the charge of the associated with the conserved current, which is very different from the electromagnetic case.