Physics 4213/5213
Lecture 13

1 Introduction

2 Weak Interactions Review

The weak interaction, which is responsible for all beta decays, several peculiar properties that are not found in the other interactions:

1. It violates parity;
2. It violates charge conjugation;
3. It transforms quark flavors, violates strangeness, charm, beauty, and top quantum numbers;
4. Gauge particles are massive.

3 π^- Decay

As an example of the difference between helicity and chirality, consider the decay of the charged pion. The two possible decay modes that satisfy the various conservation laws (charge, energy, momentum, lepton number ...) are $\pi^- \to e^- + \bar{\nu}_e$ and $\pi^- \to \mu^- + \bar{\nu}_\mu$. Given the masses in table 1, the electron decay mode should be expected to dominate based on the much larger available phase space. Yet, experiment shows that the muon decay mode is more likely to occur by a ratio of

$$\frac{\Gamma(\pi^- \to e^- + \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- + \bar{\nu}_\mu)} = (1.23 \pm 0.02) \times 10^{-4}. \tag{1}$$

The principle reason for this comes from the fact that this is a weak decay with the anti-neutrino having a right handed helicity and the electron, being highly relativistic, having a negative helicity, the overall spin of the final state then adds to one. On the other hand, the muon having a mass similar to the pion, will be moving slowly, therefore its helicity and chirality are different. Notice, if the electron were exactly massless, then the pion could not decay to electrons.

Particle	Mass	Spin
π^-	140 MeV	0
μ^-	100 MeV	1/2
e^-	0.511 MeV	1/2
ν	0 MeV	1/2

Table 1: The masses and spins of the particles involved in charged pion decay.

To calculate the decay rate, the Feynman rules for the charged lepton and neutrino are the same as in QED, while the vertex factor needs to be modified to account for the $(1 - \gamma^5)$ term. The Feynman rules are therefore;

1. Each incoming fermion has a spinor $u(p,s)$;
2. Each outgoing fermion has a spinor $\bar{u}(p,s)$;
3. Each incoming anti-fermion has a spinor $\bar{v}(p,s)$;
4. Each outgoing anti-fermion has a spinor $v(p,s)$;
5. Each vertex has a factor $-i(G/\sqrt{2})\gamma^\mu(1-\gamma^5)$.

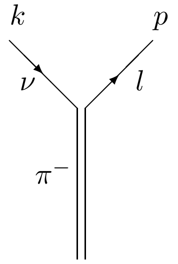

Figure 1: This diagram describes charged pion decay

The Feynman diagram for pion decay is given in figure 1. Applying the rules given above, only accounts for the lepton current, the pion is not taken into account since it is not a point particle. It needs to be taken into account via a form factor that makes the amplitude a Lorentz scalar. Before proceeding with the form factor, the leptonic terms of the amplitude are written down

$$-i\mathcal{M} = -i\frac{G}{\sqrt{2}}\left[\bar{u}_l\gamma^\mu\left(1-\gamma^5\right)v_{\nu_l}\right]F_\mu \tag{2}$$

where F_μ represents the pion form factor. The form of F_μ can be arrived at through the following argument; the pion has zero spin, the only vector associated with the pion is its 4-momentum, further the only other quantity associated with the pion that is invariant is its mass, therefore the factor $F_\mu = \mathcal{F}(m_\pi)p_\mu = \mathcal{F}_\pi p_\mu$. The amplitude can therefore be written as

$$-i\mathcal{M} = -i\frac{G}{\sqrt{2}}\left[\bar{u}_l\gamma^\mu\left(1-\gamma^5\right)v_{\nu_l}\right]\mathcal{F}_\pi p_\mu. \tag{3}$$

Substituting in all the phase space factors and using the amplitude given in equation 3, the differential decay rate is

$$d\Gamma = \frac{(2\pi)^4\delta^4(p_\pi - p_l - k_{\nu_l})}{2m_\pi}|\mathcal{M}|^2\frac{d^3p_l}{(2\pi)^3 2E_l}\frac{d^3p_{\nu_l}}{(2\pi)^3 2E_{\nu_l}}. \tag{4}$$

The only thing left to do before working out the integrals, is to calculate the amplitude. As in the scattering case, the spin averaged amplitude is used; notice that any objects that have only one spin orientation are take care through the $(1-\gamma^5)$ term. Before spin averaging, one simplification can be made by substituting the pion 4-momentum with the sum of the outgoing lepton and neutrino 4-momentum; $p^\pi_\mu = p_\mu + k_\mu$. Then use the relations $\bar{u}(\not{p} - m_l) = 0$ and $\not{k} = 0$ giving

$$\begin{aligned}
\mathcal{M} &= \frac{G}{\sqrt{2}}\left[\bar{u}_l\gamma^\mu\left(1-\gamma^5\right)v_{\nu_l}\right]\mathcal{F}_\pi(p_\mu + k_\mu) \\
&= \frac{G}{\sqrt{2}}\left[\bar{u}(\not{p}+\not{k})(1-\gamma^5)v\right]\mathcal{F}_\pi \\
&= \frac{G}{\sqrt{2}}m_l\left[\bar{u}(1-\gamma^5)v\right]\mathcal{F}_\pi. \qquad (5)
\end{aligned}$$

The next step is to convert the spin amplitude squared into a trace, where the amplitude squared is

$$|\mathcal{M}|^2 = \frac{G^2}{2}m_l\left[\bar{u}(1-\gamma^5)v\right]\left[\bar{u}(1-\gamma^5)v\right]^* \qquad (6)$$

$$= \frac{G^2}{2}m_l\left[\bar{u}(1-\gamma^5)v\right]\left[\bar{v}(1+\gamma^5)u\right] \qquad (7)$$

where the unitarity of γ^0 and γ^5 are used as is the anti-commuting property of the two matrices. Following the steps used in the cross section calculation, the spin averaged amplitude squared becomes

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{G^2}{2}\mathcal{F}_\pi m_l^2 \mathrm{Tr}\left[(\not{p}+m_l)(1-\gamma^5)\not{k}(1+\gamma^5)\right] \\
&= \frac{G^2}{2}\mathcal{F}_\pi m_l^2 \mathrm{Tr}\left[(\not{p}+m_l)\not{k}(2+2\gamma^5)\right] \\
&= G^2\mathcal{F}_\pi^2 m_l^2 \mathrm{Tr}\left[\not{p}\not{k}+\not{p}\not{k}\gamma^5 + m_l\not{k} + m_l\not{k}\gamma^5\right] \\
&= 4G^2\mathcal{F}_\pi^2 m_l^2(p\cdot k) \qquad (8)
\end{aligned}$$

where the trace of an odd number of γ^μ matrices being zero is used as is the $\mathrm{Tr}\left[\not{p}\not{k}\gamma^5\right] = 0$. The dot product can be simplified further. The decay will be looked at from the rest frame of the π^-, in this case $p\cdot k = \omega(E+\omega)$ where the momenta of the two particles being equal and the energy and momenta of the neutrino being equal were used. The amplitude squared is therefore

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= 4G^2\mathcal{F}_\pi^2 m_l^2\omega(E+\omega) \\
&= 4G^2\mathcal{F}_\pi^2 m_l^2\omega m_\pi. \qquad (9)
\end{aligned}$$

Notice that the amplitude squared depends on the square of the lepton mass, if in fact the lepton mass was zero the amplitude would also be zero.

Given the spin averaged amplitude in equation 9, the integrals can now be carried out. The first step is to integrate over p since this takes care of the delta function. Further, since there is no angular dependence, the angular part of the k integral can be done giving a factor of 4π. This leaves the integral

$$\Gamma = \frac{G^2\mathcal{F}_\pi^2 m_l^2}{(2\pi)^2 2m_\pi}4\pi\int d\omega\omega^2\left(\frac{m_\pi}{E}\right)\delta(m_\pi - E - \omega) \qquad (10)$$

In the delta-function the energy of the charged lepton E depends on the energy of the neutrino ω. This implies that a change of variables needs to be done. The easiest change of variable is to realize that the delta-function is infinite when its argument is equal to zero. This implies that the argument can be divided into ω and everything else ω_0. The change of variables then goes as

$$\int \delta\left(f(\omega)\right) d\omega = \int \frac{\delta(\omega - \omega_0)}{|df(\omega)/d\omega|} d\omega, \tag{11}$$

with the absolute value being due to the fact that the integral of the delta function is always equal to one (positive). The value of ω_0 can be found by substituting $E = \sqrt{m_l^2 + \omega^2}$ into the argument of the delta-function setting it equal to zero and solving for ω. This gives

$$\omega_0 = \frac{m_\pi^2 - m_l^2}{2m_\pi}, \tag{12}$$

and for the derivative of the argument of the delta-function

$$\left|\frac{df(\omega)}{d\omega}\right| = \frac{E + \omega}{E} = \frac{m_\pi}{E}. \tag{13}$$

Finally solving the integral gives

$$\Gamma = \frac{G^2}{8\pi} \mathcal{F}_\pi^2 m_\pi m_l^2 \left(\frac{m_\pi^2 - m_l^2}{m_\pi^2}\right)^2. \tag{14}$$

The same calculation could have been carried out assuming that the amplitude squared was a constant. This would have lead to a decay rate that depends only on phase space. The decay rate would then have been

$$\Gamma = \frac{m_\pi^2 - m_l^2}{8\pi m_\pi^2}. \tag{15}$$

So as expected, if the dynamics of weak interactions are ignored, the decay rate increases with decreasing charged lepton mass.

Finally the ratio of muon and electron decay rates is given by

$$\frac{\Gamma(\pi^- \to e^- \nu_e)}{\Gamma(\pi^- \to \mu^- \nu_\mu)} = \left(\frac{m_e}{m_\mu}\right)^2 \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 = 1.232 \times 10^{-4} \tag{16}$$