Physics 4213/5213
Lecture 13

1 Introduction

This lecture will develop a simple model for weak interactions that is based on the model already used for electromagnetic interactions. The quantity that needs to be defined is the invariant amplitude. In the case of electromagnetic interactions, this required two currents and a propagator to represent the exchange particle. Therefore, to define the weak interaction, the weak current has to be found, and the propagator for the weak exchange particle must also be given.

2 Weak Interactions Review

In order to define the weak current and the propagator, the properties of the weak interaction must be used. The components of the invariant amplitude have to give back the weak interactions properties, or stated differently these properties have to be used to build the invariant amplitude. (Recall, the invariant amplitude for a two vertex diagram is the product of a current a propogator and a current.) The important properties of the weak interaction are:

1. It violates parity;
2. It violates charge conjugation;
3. Must have left hand fermions and right hand anti-fermions at least in the extreme relativistic case;
4. It transforms quark flavors, that is it violates strangeness, charm, beauty, and top quantum numbers;
5. Gauge particles are massive, due to the finite range of the interaction;
6. Charge changing currents.

To built the current, start by specifying the handedness of the Fermions—at this point the flavor violation in the quark sector is ignore, it can be introduced later through quark flavor mixing. If the fermions have a specific handedness (only left handed fermions and right handed anti-fermions), they will automatically violate parity. Now handedness typically refers to helicity, but helicity is a frame dependent quantity, except for massless particles. But since the invariant amplitude is a relativistic invariant, it cannot change from frame to frame. To take care of this problem, recall that in the experiments that saw specific handedness for the fermions, the fermions were either massless or extremely relativistic. In this case the chiral operator is approximately the same as the helicity:

$$\gamma^5 = \frac{\vec{\Sigma} \cdot \vec{p}}{E - \beta m} \approx \frac{\vec{\Sigma} \cdot \vec{p}}{|\vec{p}|} \tag{1}$$

Therefore instead of using left or right handed helicity states, left or right handed chiral states will be used. Ultimately, experiment will determine if the is or is not the right choice.

The definition of left and right handed spinors are given as in an earlier lecture, by noting that in the extreme relativistic case helicity and chirality are the same. The spinors with the z-axis taken as the direction of motion are given by:

$$u^1 = \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ 0 \end{pmatrix} \quad u^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-p_z}{E+m} \end{pmatrix} \quad v^1 = \begin{pmatrix} 0 \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \quad v^2 = \begin{pmatrix} \frac{p_z}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{2}$$

In the limit of $E \gg m$, the spinors become:

$$u^1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad u^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad v^1 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad v^2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{3}$$

which leads to:

$$\gamma^5 u^1 = u^1 \quad \gamma^5 u^2 = -u^2 \quad \gamma^5 v^1 = -v^1 \quad \gamma^5 v^2 = v^2 \tag{4}$$

which leads to the natural interpretation for the fermions of the positive eigen-value state being the right handed chirality, while the negative eigen-value state is left handed chirality, in order to connect up with helicity in the extreme relativistic case. The opposite holds for the for the anti-fermions. Based on these definitions, the projections operators acting on arbitrary spinors are:

$$\begin{aligned} \frac{1}{2}(1+\gamma^5)u = u_R \qquad & \frac{1}{2}(1-\gamma^5)u = u_L \\ \frac{1}{2}(1-\gamma^5)v = v_R \qquad & \frac{1}{2}(1+\gamma^5)v = v_L \end{aligned} \tag{5}$$

The adjoint spinors are given by:

$$\bar{u}_L = \left[\frac{1}{2}(1-\gamma^5)u\right]^\dagger \gamma^0 = \bar{u}\frac{1}{2}(1+\gamma^5) \tag{6}$$

where the anti-commutation of γ^0 and γ^5 was used in the last step.

Having defined the left and right handed chiral states, the various possible currents can now be written down. To be consistent with experiment, left handed fermion currents and right handed anti-fermion currents are allowed (see fig **??**). The left handed fermion current is given by:

$$\bar{u}_L\gamma^\mu u_L = \bar{u}\frac{1}{2}(1+\gamma^5)\gamma^\mu\frac{1}{2}(1-\gamma^5)u = \bar{u}\gamma^\mu\frac{1}{2}(1-\gamma^5)\frac{1}{2}(1-\gamma^5)u = \bar{u}\gamma^\mu\frac{1}{2}(1-\gamma^5)u \tag{7}$$

where the anti-commutation of γ^μ and $gfive$ was used. The right handed anti-fermion current is similarly found:

$$\bar{v}_R\gamma^\mu v_R = \bar{v}\gamma^\mu\frac{1}{2}(1+\gamma^5)v \tag{8}$$

The only remaining currents are those where a fermion-antifermion pair are either created or annihilated (see fig **??**). To determine the currents, use the following Feynman rules:

1. Each incoming fermion has a spinor $u(p, s)$;
2. Each outgoing fermion has a spinor $\bar{u}(p, s)$;
3. Each incoming anti-fermion has a spinor $\bar{v}(p, s)$;
4. Each outgoing anti-fermion has a spinor $v(p, s)$;

This leads to the annihilation current being:

$$\bar{v}_R \gamma^\mu u_L = \bar{v}\gamma^\mu \frac{1}{2}(1 - \gamma^5)u \tag{9}$$

where the procedure is similar to that of equation 7. Finally the creation current is given by:

$$\bar{u}_L \gamma^\mu v_R = \bar{u}\gamma^\mu \frac{1}{2}(1 + \gamma^5)v \tag{10}$$

At this point all the possible currents that are needed for two vertex diagrams, have been defined.

The only remaining quantity that needs to be defined, is the propagator. This can be done by extending the propagator to one that has a non-zero mass. Recall that the photon propagator is given by $-ig^{\mu\nu}/q^2$ which has a pole at $q^2 = 0$, the mass of the photon. To extend it to a different mass, the pole must occur at that new mass $-ig^{\mu\nu}/(q^2 - M_W^2)$. Notice that this propagator is approximately constant for $q^2 \ll M_W^2$ and is approximately the same as the photon propagator for $q^2 \gg M_W^2$. In the case of very small q^2, the propagator is combined with the coupling constant to form a new coupling constant that has the dimensions of an inverse mass squared:

$$\frac{G_F}{\sqrt{2}} = \frac{g}{8M_W^2} \tag{11}$$

where g plays the same role that e does in electrodynamics; in the units being used it is dimensionless.

3 Transformation Properties of Currents

When electromagnetism was discussed, the current $\bar{u}\gamma^\mu u$ was found to transform like a 4-vector under the Lorentz transformation ($\bar{u}\gamma^\mu u \to \Lambda^\mu{}_\nu \bar{u}\gamma^\nu u$). The transformation properties under parity were never examined, these will be discussed later in this lecture. To determine how the weak current transforms under the Lorentz transformation, the current is split into two pieces $j_w^\mu = 1/2(\bar{u}\gamma^\mu u - \bar{u}\gamma^\mu\gamma^5 u)$. The first piece is the same as the electromagnetic current, therefore transforms the same. The second piece transforms as follows:

$$\bar{u}'\gamma^\mu\gamma^5 u' = \bar{u}S^{-1}(\Lambda)\gamma^\mu\gamma^5 S(\Lambda)u = \bar{u}S^{-1}(\Lambda)\gamma^\mu S(\Lambda)S^{-1}(\Lambda)\gamma^5 S(\Lambda)u = \Lambda^\nu{}_\mu \bar{u}\gamma^\mu S^{-1}(\Lambda)\gamma^5 S(\Lambda)u \tag{12}$$

To simplify the last term, recall that the transformation $S(\Lambda)$ has the infinitesimal form:

$$\Sigma(\Delta\lambda^{\mu\nu}) = 1 - \frac{i}{4}\sigma_{\mu\nu}\Delta\lambda^{\mu\nu} \tag{13}$$

where

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{14}$$

Since γ^5 anti-commutes with γ^μ, it will therefore commute with $\sigma_{\mu\nu}$ and ultimately with $S(\Lambda)$. Equation 12 can now be simplified as follows:

$$\bar{u}'\gamma^\nu\gamma^5 u' = \Lambda^\mu{}_\nu \bar{u}\gamma^\nu S^{-1}(\Lambda)\gamma^5 S(\Lambda)u = \Lambda^\nu{}_\mu \bar{u}\gamma^\mu\gamma^5 u \tag{15}$$

it has the same transformation properties as the first term. Therefore the weak current as a whole transforms like a 4-vector under the Lorentz transformation.

This is of course not the complete story. The coordinates, which are what define a 4-vector, also have a specific transformation under parity. Given that weak interactions violate parity, it would be expected that the parity of the current has an indefinite parity. Before determining the parity operator for the spinors, recall that the coordinates transform under parity as follows:

$$\left.\begin{array}{c} t \xrightarrow{\mathcal{P}} t \\ \vec{x} \xrightarrow{\mathcal{P}} -\vec{x} \end{array}\right\} \quad \Rightarrow \quad x^\mu \xrightarrow{\mathcal{P}} x_\mu \tag{16}$$

which implies that the operator has the following form

$$a^\nu{}_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g^{\mu\nu}. \tag{17}$$

To determine how the spinors transform, the procedure that was followed for the Lorentz transformations is used. This starts with the transformed Dirac equation:

$$(\gamma^\mu \partial'_\mu - m)\psi' = 0 \quad \Rightarrow \quad (\gamma^\mu P \partial'_\mu - mP)\psi = 0 \tag{18}$$

which can be further simplified:

$$(\gamma^\mu P \partial'_\mu - mP)\psi = 0 \quad \Rightarrow \quad (P^{-1}\gamma^\mu P a^\nu{}_\mu \partial_\mu - m)\psi = 0 \tag{19}$$

At this point impose the same condition on the γ^μ matrices as when $S(\Lambda)$ was derived, that is that γ^μ are invariant to this transformation:

$$\gamma^\mu = P^{-1}\gamma^\nu P a^\mu{}_\nu \quad \Rightarrow \quad a^\nu{}_\mu \gamma^\mu = P\gamma^\nu P^{-1} \tag{20}$$

where S is replaced by P for this case. Equation 20 can be converted into the following form:

$$a^\nu_\mu \gamma^\mu = g^{\nu\mu}\gamma^\mu = P a^\sigma_\nu \gamma^\nu p^{-1} \tag{21}$$

The right hand side of equation 21 has the *obvious* solution

$$P = e^{i\varphi}\gamma^0, \qquad\qquad P^{-1} = e^{-i\varphi}\gamma^0. \tag{22}$$

where this comes from the anti-commutation properties of γ^μ with γ^0 $\gamma^0, \gamma^\mu = g^{\mu\mu}\gamma^\mu\gamma^0$ This operator is unitary, $P^{-1} = P^\dagger$, and also satisfies relation $P^{-1} = \gamma^0 P^\dagger \gamma^0$.

To determine the parity of spin $\frac{1}{2}$ particles start with the Dirac equation. First of all, in order for the wave function to be an eigenfunction of both the the Dirac equation and the parity operator, the two must commute. Since the parity operator is basically γ^0, which is equal to β, it will only commute with the Dirac Hamiltonian if the momentum is zero, since $\vec{\alpha}\beta = -\beta\vec{\alpha}$ where as a reminder=the Dirac Hamiltonian is:

$$H = \vec{\alpha} \cdot \vec{p} + \beta m \tag{23}$$

This also implies that the wave function is independent of the space coordinates. The solution to the Dirac equation in this case is given by solving the following differential equation

$$\left[\gamma^0 \partial_0 - m\right] \psi = 0. \tag{24}$$

At this point a specific representation of the γ^0 matrix is needed. Returning to the first lecture, it was shown that the matrices have eigenvalues of ± 1 with equal numbers of each. If the representation for γ^0 is given by a diagonal matrix, then it can be written as

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{25}$$

With this representation, the Dirac equation is given by

$$\left[i \begin{pmatrix} \partial_0 & 0 & 0 & \\ 0 & \partial_0 & 0 & 0 \\ 0 & 0 & \partial_0 & 0 \\ 0 & 0 & 0 & \partial_0 \end{pmatrix} - \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & -m & 0 \\ 0 & 0 & 0 & -m \end{pmatrix} \right] \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = 0. \tag{26}$$

This represents four independent differential equations, which have the solution

$$\psi^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt} \qquad \psi^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt} \tag{27}$$

$$\psi^{(3)} = N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{imt} \qquad \psi^{(4)} = N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{imt}. \tag{28}$$

Based on these solutions and the form of the parity operator, the parity of spin $\frac{1}{2}$ particles is given by

$$\begin{aligned} P\psi^{(1)} &= \psi^{(1)} & \qquad P\psi^{(2)} &= \psi^{(2)} \\ P\psi^{(3)} &= -\psi^{(3)} & \qquad P\psi^{(4)} &= -\psi^{(4)} \end{aligned} \tag{29}$$

where it should be noted that the positive and negative energy solutions have opposite intrinsic parity, and the phase factor has been drop since the phase is completely arbitrary.

3.1 Parity of 4-Current Density

At this point the parity of the weak current can be calculated. As in the case of the Lorentz transformation, each term will be examined separately. Start with the current $\bar{u}\gamma^\mu u$ which has already been shown to transforms like a 4-vector under Lorentz transformations. Here it will be shown that this piece of the weak current also transforms like a vector under parity.

The current density in the primed frame (reflected frame) is given by

$$\bar{u}'\gamma^\mu u' \tag{30}$$

Expressing it in terms of the unprimed coordinates gives

$$\bar{u}P^{-1}\gamma^\mu P u = \bar{u}a^\nu_\mu \gamma^\mu u = g^{\mu\mu}\bar{u}\gamma^\mu u = \bar{u}\gamma_\mu u \tag{31}$$

This is exactly of the form of equation 16, which defines how a vector should transform—note that the second equality is equation 20.

Now for the other term $\bar{u}\gamma^\mu\gamma^5 u$. Again start in the primed frame:

$$\bar{u}'\gamma^\mu\gamma^5 u' \tag{32}$$

Expressing this term in the unprimed coordinates gives:

$$\bar{u}P^{-1}\gamma^\mu\gamma^5 P u = -\bar{u}P^{-1}\gamma^\mu P\gamma^5 u \tag{33}$$

which comes from the anti-commuting of γ^5 with γ^μ and that fact that $P = \gamma^0$. From this point the steps are the same as for equation 31, this leads to:

$$\bar{u}P^{-1}\gamma^\mu\gamma^5 P u = -g^{\mu\mu}\bar{u}\gamma^\mu\gamma^5 u = -\bar{u}\gamma_\mu\gamma^5 u \tag{34}$$

which is the opposite of a 4-vector, this transforms like a pseudo-vector. Therefore, the current does not have a specific parity:

$$\bar{u}\gamma^\mu \frac{1}{2}(1-\gamma^5)u \xrightarrow{\mathcal{P}} \bar{u}\gamma_\mu \frac{1}{2}(1+\gamma^5)u \tag{35}$$

or stated differently, the weak current is not an eigen-state of parity. Therefore parity is not conserved by this current.