Physics 4213/5213
Lecture 12

1 Introduction

In the lectures discussing the structure functions, it was pointed out that for the case of hadrons being composed of free point Dirac particles, the structure functions should be independent of any scale parameter—the scale in particle physics is always set by the wavelength of the probe, which in the case discussed is the 4-momentum transfer Q^2 of the virtual photon. On the other hand, once interactions are introduced virtual particle-antiparticle pairs are produced. These pairs live for very short times (this can also be viewed as existing over very short distance). As Q^2 increases, the number of these virtual pairs seen will increase. This introduces a Q^2 (scale) dependence into the structure functions.

This lecture will examine the origins of this scale (Q^2) dependence by looking at the effect of higher order diagrams on electron-muon scattering. It will be shown that the higher order terms lead to a cross section that diverges. It will be found that this divergence can be hidden in the coupling constant at the cost of it no longer being a constant, but a variable depending on the the scale (Q^2) being probed. The idea will then be extended to other variables.

One point needs to be made before starting. The subject of renormalization is a very complex with many subtilties. No attempt will be made to be cover every detail of this topic, this set of lectures are meant only as an overview. Further, many of the calculations involved in this discussion require a significant amount of algebra, plus techniques that are no always straight forward. Therefore to keep the lecture focused on the relevant physics, many details will be omitted as well as any detailed discussion of different renormalization schemes.

2 Polarization of the Vacuum

To understand the origins of the scale dependent coupling constant (running coupling constant) consider the classical experiment of placing a point charge in a dielectric media. If the value of the charge is e when measured outside the dielectric, and the dielectric constant is ϵ, the measured charge inside the dielectric is:

$$Q = \int_{\mathcal{S}} \vec{E} \cdot d\vec{A} = e/\epsilon \tag{1}$$

recall that $\epsilon > 1$ and this result is valid for a measurement made many intermolecular spacings away. As the measurement is made closer to the charge, the screening from the polarization of the media will decrease causing the measured charge to increase. Once the measurement is made inside the last set of molecules surrounding the charge, the full charge is seen. Therefore, the effective charge depends on how closely the charge is probed. The charge is scale dependent.

A similar effect occurs in QED (in fact all quantum field theories). Consider a point charge, the field that it generates can be thought of as collection of photons. The photons can produce particle-antiparticle pairs. These pairs are polarized due to the charge of the point charge, which creates a screening effect. This produces a scale dependent charge, where the intermolecular spacing is replaced by the compton wavelength of the lightest particle that couples to the photon, this being the electron.

3 Loops

To see where the polarization (the running coupling constant) comes from, electron-muon scattering will be used as the starting point. First recall that the properties of the two vertex electron-muon cross section are defined by the invariant amplitude:

$$M_{fi} = e^2 \bar{u}(k_f)\gamma^\mu u(k_i) \frac{-ig_{\mu\nu}}{q^2} \bar{u}(p_f)\gamma^\nu u(p_i) \tag{2}$$

Even though this provides a very good description of electron-muon scattering, this is certainly not the complete picture. In fact, any other diagram where an electron (or any other charged fermion) couples to a photon, and is not part of the initial or final states must be included—notice that if photons are radiated by the initial or final state particles, and they are below the detection threshold of the detector, they will also contribute to the e-μ cross section. Figure 1 shows examples of the lowest order corrections to e-μ scattering, that have added only internal lines. The diagram that is responsible for the running coupling constant, is the loop in the photon line.

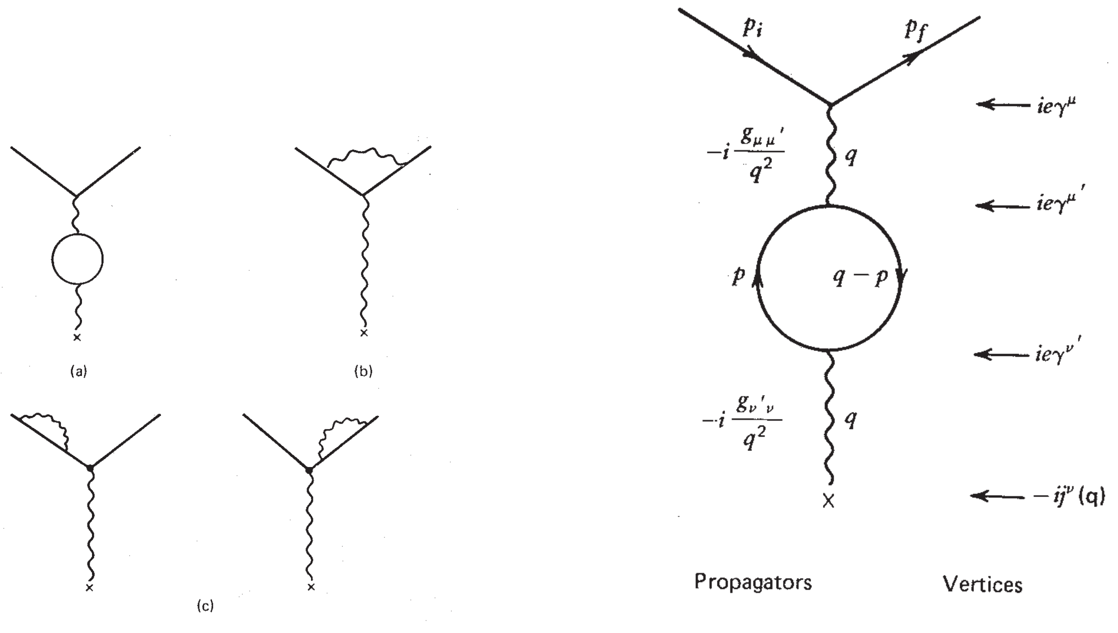


Figure 1: Examples of one loop QED diagrams. Also the Feynman rules for this diagram.

To calculate loop correction to the e-μ cross section, a couple of new Feynman rules have to be introduced. These rules are:

1. The propagator for a fermion (internal fermion line) is given by:

$$\frac{i(\not{k} + m)}{k^2 - m^2} \tag{3}$$

2. Even though momentum has to be conserved at each vertex, the actual value is not constrained. Therefore it must be integrated over for each closed loop:
$$\int \frac{d^4k}{(2\pi)^4} \tag{4}$$
3. A factor of (-1) has to be introduced for each loop;
4. The trace of the factors associated with the loop is taken.

By putting these rules together, the invariant amplitude for the one loop diagram is:
$$M_{fi}^{(1)} = \bar{u}(p_f)\gamma^\mu u(p_i)\frac{-ig_{\mu\alpha}}{q^2} i\Pi^{\alpha\beta}\frac{-ig_{\nu\beta}}{q^2}\bar{u}(k_f)\gamma^\nu u(k_i) \tag{5}$$
where
$$\Pi^{\mu\nu} = -\int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[ie\gamma^\mu \frac{i(\not{k}+m)}{k^2-m^2} ie\gamma^\nu \frac{i(\not{q}-\not{k}+m)}{(q-k)^2-m^2}\right] \tag{6}$$
notice that this integral is expected to diverge since it is being integrated over all momenta.

The integral itself is rather involved to carry out, and therefore the details will be excluded. After carrying out the algebra, the integral itself is found to have the following form:
$$\Pi^{\mu\nu} = -ig_{\mu\nu}I(q^2) + q_\mu q_\nu J(q^2) \tag{7}$$
which is expected since the only 4-vector left is q^μ, and it must have the form of a second rank tensor. The second term becomes zero when applied to the current $(q^\nu \bar{u}(k_f)\gamma^\nu u(k_i) = 0)$. The term $I(q^2)$ is given by:
$$I(q^2) = \frac{e^2}{12\pi^2}\left[\int_{m^2}^{\infty}\frac{dx}{x} - 6\int_0^1 z(z-1)\ln\left(1 - \frac{q^2}{m^2}z(z-1)\right)dz\right] \tag{8}$$
the variables x and y come from changes of variables in evaluating the integrals, and have no specific physical meaning. At this point a problem arises, this being that the first integral diverges logarithmically. This will be handled by introducing a cut-off mass in the integral, and then letting the mass go to infinity at the end of the calculation:
$$\int_{m^2}^{\infty}\frac{dx}{x} \to \int_{m^2}^{M^2}\frac{dx}{x} = \ln\frac{M^2}{m^2} \tag{9}$$
The second integral is finite, but cannot be evaluated analytically over all values of q^2. None the less, in the limit of small and large values of q^2, the integral has the following form:
$$f\left(\frac{-q^2}{m^2}\right) = \begin{cases} \frac{-q^2}{m^2} & q^2 \ll 1 \\ \ln\frac{-q^2}{m^2} & q^2 \gg 1 \end{cases} \tag{10}$$

Taking the two lowest order amplitudes (two vertex, and one loop) for e-μscattering, and using the value of the integral calculated above leads to the invariant amplitude:
$$M_{fi} = -ie^2\left\{1 - \frac{e^2}{12\pi^2}\left[\ln\left(\frac{M^2}{m^2}\right) - f\left(\frac{-q^2}{m^2}\right)\right]\right\}\bar{u}(p_f)\gamma^\mu u(p_i)\frac{-ig_{\mu\nu}}{q^2}\bar{u}(k_f)\gamma^\nu u(k_i) \tag{11}$$

This amplitude has the form of the original two vertex diagram, but with a charge that depends on q^2, but the infinity none the less remains. To remove the infinity (the mass cut-off), redefine charge at zero zero q^2 as:

$$e_R = e\sqrt{1 - \frac{e^2}{12\pi^2}\ln\left(\frac{M^2}{m^2}\right)} \tag{12}$$

which leads to:

$$M_{fi} = -ie_R^2\left\{1 + \frac{e^2}{12\pi^2}f\left(\frac{-q^2}{m^2}\right)\right\}\bar{u}(p_f)\gamma^\mu u(p_i)\frac{-ig_{\mu\nu}}{q^2}\bar{u}(k_f)\gamma^\nu u(k_i) \tag{13}$$

where in the braces the charge can be replaced with e_R to this level of approximation (only terms to e^4 matter):

$$M_{fi} = -ie_R^2\left\{1 + \frac{e_R^2}{12\pi^2}f\left(\frac{-q^2}{m^2}\right)\right\}\bar{u}(p_f)\gamma^\mu u(p_i)\frac{-ig_{\mu\nu}}{q^2}\bar{u}(k_f)\gamma^\nu u(k_i) \tag{14}$$

At this point the infinity, and therefore the cut-of have disappeared. They are now hidden in the renormalized charge, which is a finite quantity. Finally, putting equation 14 into the same form as the two vertex amplitude leads to the introduction of a q^2 dependent charge:

$$e_R(q^2) = e_R\sqrt{1 + \frac{e_R^2}{12\pi^2}f\left(\frac{-q^2}{m^2}\right)} \quad \Rightarrow \quad \alpha(q^2) = \alpha_0\left[1 + \frac{\alpha_0}{3\pi}f\left(\frac{-q^2}{m^2}\right)\right] \tag{15}$$

this is the charge that is measured in the lab. In high energy experiments, Q^2 is typically much larger than the mass of the electron. In this case, the coupling constant is given by:

$$\alpha(q^2) = \alpha_0\left[1 + \frac{\alpha_0}{3\pi}\ln\left(\frac{Q^2}{m^2}\right)\right] \tag{16}$$

This is not the complete story, since there can be two loops, or three loops etc. Each additional term is given by:

$$\left[\frac{\alpha_0}{3\pi}\ln\frac{-q^2}{m^2}\right]^n \tag{17}$$

where n is the number of loops in the specific diagram. Using this expression, the coupling constant becomes:

$$\alpha(q^2) = \alpha_0\left\{1 + \frac{\alpha_0}{3\pi}\ln\frac{-q^2}{m^2} + \left[\frac{\alpha_0}{3\pi}\ln\frac{-q^2}{m^2}\right]^2 + \left[\frac{\alpha_0}{3\pi}\ln\frac{-q^2}{m^2}\right]^2 + \cdots\right\} \tag{18}$$

which can be summed using the relation $\sum_i x^i = \frac{1}{1-x}$, giving:

$$\alpha(q^2) = \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi}\ln\frac{-q^2}{m^2}} \tag{19}$$

Even this is not the complete story. Equation 18 still depends on the cutoff mass that has to be set to infinity at this point of the calculation. So even though the infinity is hidden, it is not very well hidden at this point. To get around this and take into account that the true coupling constant is Q^2 dependent, a measured value of the coupling constant will be used and then reinserted into the equation. Make the measurement at $q^2 = -\mu^2$:

$$\alpha(\mu^2) = \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi}\ln\left(\frac{\mu^2}{m^2}\right)} \quad \Rightarrow \quad \frac{\alpha_0}{\alpha(\mu^2)} = 1 - \frac{\alpha_0}{3\pi}\ln\left(\frac{\mu^2}{m^2}\right) \tag{20}$$

Now rewrite equation 18 in the following form:

$$\alpha(q^2) = \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi}\ln\left(\frac{-q^2\mu^2}{m^2\mu^2}\right)} = \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi}\ln\left(\frac{\mu^2}{m^2}\right) - \frac{\alpha_0}{3\pi}\ln\left(\frac{-q^2}{\mu^2}\right)} \tag{21}$$

Next substitute equation 20 into this equation to get:

$$\alpha(q^2) = \frac{\alpha_0}{\frac{\alpha_0}{\alpha(\mu^2)} - \frac{\alpha_0}{3\pi}\ln\left(\frac{-q^2}{\mu^2}\right)} = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi}\ln\left(\frac{-q^2}{\mu^2}\right)} \tag{22}$$

which is finally an answer without the cut-off parameter included. This expression contains only the measured value of the coupling constant at some scale μ^2, and is a function of the scale $-q^2$. This expression is still not complete in that the other fermions must be included. But only those fermions that have values of the mass that is much smaller than the scale being probed need be included—recall the approximation used earlier.

To include the remaining fermions, one factor for each of the available fermion, and the charge has to be corrected at each vertex. The correction factor is $N_c = n_\ell + 3(2/3)^2 n_u + 3(1/3)^2 n_d$, making the running coupling constant:

$$\alpha(q^2) = \frac{\alpha_0}{\frac{\alpha_0}{\alpha(\mu^2)} - \frac{\alpha_0}{3\pi}\ln\left(\frac{-q^2}{\mu^2}\right)} = \frac{\alpha(\mu^2)}{1 - N_c\frac{\alpha(\mu^2)}{3\pi}\ln\left(\frac{-q^2}{\mu^2}\right)} \tag{23}$$

To get a feel for change in the coupling constant, consider the case of $-q^2 = M_W^2$. For this value of q^2, all the charged fermions contribute to the loops except for the top-quark making $N_c = 20/3$. Using $\mu = 100\,\mathrm{MeV}$ yields $\alpha(M_W) \approx 1/128$, which is what has been measured.

3.1 Renormalization Group Equation

The coupling constant still depends on the value of the parameter μ, and how it is selected. Yet the physics, that is any observable, must be independent of the of it. This simple statement can be written mathematically as:

$$\mu\frac{d}{d\mu}R = \left(\mu\left.\frac{\partial}{\partial\mu}\right|_\alpha + \mu\frac{\partial\alpha}{\partial\mu}\frac{\partial}{\partial\alpha}\right)R = 0 \tag{24}$$

where R is an observable, such as the invariant amplitude M_{fi}. Through the use of this equation, the evolution of the observable can be calculated once the evolution of the coupling constant is known.

4 The Strong Coupling Constant

Having derived an expression for the q^2 dependence of the electromagnetic coupling constant, the section will give an overview of the strong coupling constant—given that the Feynman rules for the strong interaction are significantly different and more complicated to use and that not much new will be learned from doing the algebra, the only the result will be given. First, the one-loop contributions to the gluon propagator includes both quark and gluon loops (see fig 2. The gluon loops are due to the gluon carrying the charge of the strong interaction, which allows gluons to interact with each other. The effect of the quark loops is the same as the fermion loops in electromagnetic interaction, that is to screen the charge. The effect of the gluon loops is the opposite. This can be thought of in the following way, since the field carries the charge of the strong interaction (color charge), the gluon can carry away the strong charge of the quark. The strong charge is therefore diffused over the range of the strong field, meaning that only a Gaussian surface that is infinitely far from the quark will see the entire charge. As the radius of the Gaussian surface decreases, less of the color charge is seen. Whether the screening or anti-screening is larger depends on the details of the calculation. The strong coupling constant for a single loop is given by:

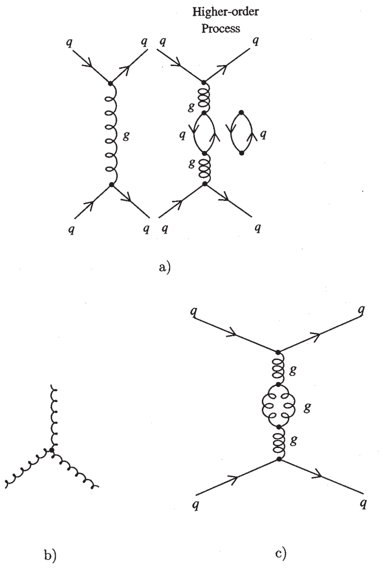
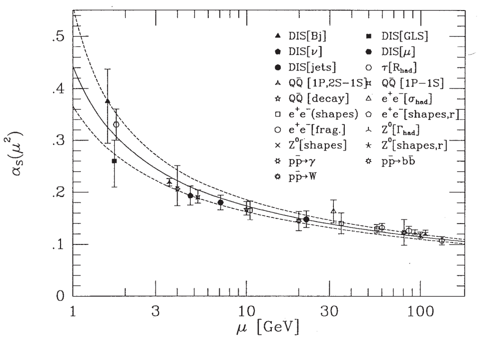


Figure 2: The one loop corrections to the gluon propagator.

$$\alpha_s(Q^2) = \frac{12\pi}{(33 - 2n_f)\ln\frac{Q^2}{\Lambda^2}} \tag{25}$$

where n_f is the number of quarks accessible at the given value of Q^2 and Λ plays the same role as μ in the electromagnetic coupling. First notice that for $n_f = 6$, the total number of quarks believed to exist, the factor $33 - 2n_f > 0$ and therefore the coupling constant decrease with increasing Q^2. The value of Λ is taken taken between 100 MeV and 500 MeV, which correspond to the masses of the lightest mesons. Figure 3 shows measured value of α_s for different values of Q^2.



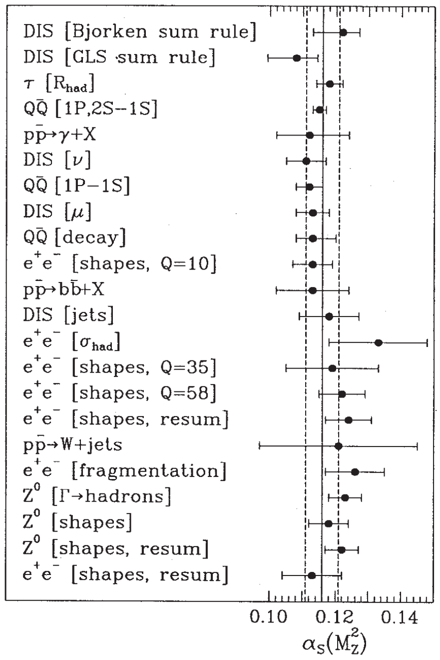


Figure 3: The strong coupling constant as a function of the scale probed. Also the coupling constant at the mass of the Z. In some case the measurement is performed at this scale, in others the coupling constant is evolved form smaller values of q^2.