Physics 4213/5213
Lecture 11

1 Introduction

The

2 Physical Interpretation

As of this point, three differential cross sections have been calculated that fundamentally represent the same process, an electron scattering off a heavier object. The three cross section in order of increasing complexity are (at this point, the variable $Q^2 = -q^2$ is introduced for convenience):

1. Electron-muon scattering:
$$\frac{d\sigma}{dE'd\Omega} = \frac{4\alpha^2 E'^2}{q^4}\left[\cos^2\frac{\theta}{2} + \frac{Q^2}{2M^2}\sin^2\frac{\theta}{2}\right]\delta\left(\nu - \frac{Q^2}{2M}\right) \tag{1}$$

2. Electron-proton elastic scattering:
$$\frac{d\sigma}{dE'd\Omega} = \frac{4\alpha^2 E'^2}{q^4}\left[\frac{G_E^2 + \tau G_M^2}{1+\tau}\cos^2\frac{\theta}{2} + 2\tau G_M^2\sin^2\frac{\theta}{2}\right]\delta\left(\nu - \frac{Q^2}{2M}\right) \tag{2}$$
where $\tau = -q^2/(4M^2)$

3. Electron-proton inelastic scattering:
$$\frac{d\sigma}{dE'd\Omega} = \frac{4\alpha^2 E'^2}{q^4}\left[W_2(\nu, Q^2)\cos^2\frac{\theta}{2} + 2W_1(\nu, Q^2)\sin^2\frac{\theta}{2}\right] \tag{3}$$

The first two cross sections have been written before the integration over the energy, while the last cross section has the delta-function hidden in the W-functions. The delta-function itself, is the energy conserving portion $\delta(E'_e - E_e + E'_p - M)$, recalling that the proton is at rest in this frame, and M is the proton mass. To see that this is in fact the case, calculate each of the terms in the delta-function. First ν:
$$\nu = \frac{p\cdot q}{M} = \frac{M(E_e - E'_e)}{M} = E_e - E'_e \tag{4}$$
while the other term is:
$$\frac{q^2}{2M} = \frac{(p'-p)^2}{2M} = \frac{2M^2 - 2E'M}{2M} = M - E' \tag{5}$$

The important point in the writing down the three cross sections, is to note that the first two cross sections are special cases of the third, with appropriate values of the W-functions selected. To arrive at the first cross section, the W-functions are chosen as:
$$2W_1(\nu, Q^2) = \frac{Q^2}{2M^2}\delta\left(\nu - \frac{Q^2}{2M}\right) \qquad W_2(\nu, Q^2) = \delta\left(\nu - \frac{Q^2}{2M}\right) \tag{6}$$

Through the use of the relation $\delta(ax) = 1/a\delta(x)$, the structure functions can be written as dimensionless structure functions:

$$2MW_1(\nu, Q^2) = \frac{Q^2}{2M\nu}\delta\left(1 - \frac{Q^2}{2M\nu}\right) \qquad \nu W_2(\nu, Q^2) = \delta\left(1 - \frac{Q^2}{2M\nu}\right) \tag{7}$$

Note that these functions are now dependent on a single dimensionless variable $Q^2/(2m\nu)$, this will become important in the interpretation of electron-proton inelastic scattering.

Before looking at inelastic scattering in more detail, the structure functions for the elastic case are examined. These functions are:

$$W_1(Q^2) = 2\tau G^2(Q^2)\delta\left(\nu - \frac{Q^2}{2M}\right) \qquad W_2(Q^2) = G^2(Q^2)\delta\left(\nu - \frac{Q^2}{2M}\right) \tag{8}$$

Notice that these functions can not be written in a dimensionless form. These depend on a length scale given by $1/Q^2$. This has to do with the finite size of the proton. In the case of the muon, since it is a point particle, no length scale should appear. This leads to the conclusion, that as long as the proton is probed at Q^2 less than or on the order of the size of the proton, elastic scattering occurs. Once the value of Q^2 is large compared to the proton dimensions $Q \approx 0.7\,\text{GeV}$, the internal structure is probed, and the proton starts to break up. If the proton is composed of point particles, then the structure functions for large values of Q^2 start to behave like those in equation 7, that is they depend on a single dimensionless variable:

$$MW_1(\nu, Q^2) \stackrel{Q^2 \gg .7\,\text{GeV}}{\longrightarrow} F_1(\omega) \qquad \nu W_2(\nu, Q^2) \stackrel{Q^2 \gg .7\,\text{GeV}}{\longrightarrow} F_2(\omega) \tag{9}$$

where the dimensionless variable $\omega = 2p \cdot q/Q^2 = 2M\nu/Q^2$. Figure **??** shows that in fact the structure functions are independent of Q^2 as would be expected if scattering off a point particle.

So far all that can be said, is that an electron scattering off the proton, is behaving as if it is scattering off a point fermion (particle). The next question to ask, is how to interpret the dimensionless variable in the case of electron-proton inelastic scattering. The structure functions given in equation 9, correspond to the proton being a single point particle. Using the quark model, the proton is built from three quarks, it can be assumed that each quark carries a fraction of the total momentum of the proton x. The identification of x with the fractional momentum holds only in the Breit frame, otherwise the each quark would have a variable mass $(xM)^2 = (xE)^2 - (x\vec{p})^2$. Also in the rest frame, x as a fractional momentum makes no sense. To reflect the fractional momentum carried by an individual parton, the structure functions have to be written in terms of the variable x:

$$F_1(\omega) = \frac{Q^2}{2m\nu x}\delta\left(1 - \frac{Q^2}{2m\nu}\right) = \frac{1}{2x^2\omega}\delta\left(1 - \frac{1}{\omega x}\right) = \frac{1}{2x\omega}\delta\left(x - \frac{1}{\omega}\right) \tag{10}$$

$$F_1(\omega) = \frac{\omega}{2}F_2(\omega) \tag{11}$$

where the variable x comes from substituting m for M ($m = xM$), and the definition of ω which is given in terms of M. These expressions also state that the variable $x = 1/\omega$. Notice that $x = Q^2/(2p \cdot q)$, where $2p \cdot q = W^2 - M^2 + Q^2$, making the maximum value equal to 1 when $W = M$ and the minimum value 0 when $q^2 = 0$. This fits in with its interpretation as a fractional momentum. Note also, that when $x = 1$, the quark carries the full momentum of the proton, that is the proton acts as if it is a single quark, at least as far as the momentum goes.

The structure functions given in equation 11 do not represent the full picture, since they only describe one particle. To represent the full picture, the proton is assumed to be composed of quarks, and possibly other particles (partons). This is done by writing the structure functions as sums of individual parton structure functions:

$$F_2(\omega) = \sum_i \int dx e_i^2 f_i(x) x \delta\left(x - \frac{1}{\omega}\right) \qquad F_1(\omega) = \frac{\omega}{2} F_2(\omega) \tag{12}$$

where the sum is over all partons, and the e_i are the electric charges of the individual parton types. So finally, in terms of the original W's, the structure functions become:

$$\nu W_2(\nu, Q^2) \to F_2(x) = \sum_i e_i^2 x f_i(x) \qquad M W_1(\nu, Q^2) \to 2x F_1(x) = F_2(x) \tag{13}$$