Physics 6213
Lecture 10

1 Introduction

In the previous lecture, the formalism for probing the structure of the proton by scattering electrons off it was setup. This formalism assumed that the proton remained intact after the collision. In this case, both the electron and proton could be treated as vector currents, with the proton current being built from the most general 4-vectors available. Each term in the current was required to be built from the 16 Dirac matrices, and the two independent 4-momenta associated with the proton vertex, each term was required to be linearly independent, and current formed must be conserved. This lead to a differential cross section that depends on two scalar functions of q^2 (the 4-momentum transfer). These two functions, are shown to be associated with the electric charge distribution, and the magnetic dipole distribution of the proton, with the functions coming from experiment.

This formalism allows the overall gross structure of the proton to be examined. Typically values of $q^2 \approx 1/r^2_{\mathrm{proton}}$ associated with the size of the proton (1 fm) or less, leave the proton intact. For values larger than this, the proton breaks up (see fig 1). For this case, the formalism no longer works, since the current is not easily defined due to the variable number of final state particles in each event.

This lecture will examine how to describe processes where the scattering is not elastic. This case being far more important than the elastic case, since large values of q^2 are required to probe the inner structure of the proton.

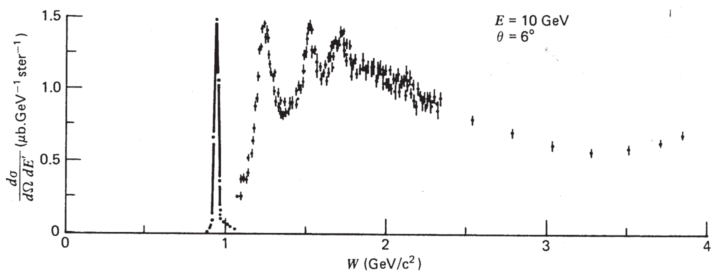

Figure 1: The electron-proton inelastic cross section, as a function of the mass of the hadronic system.

2 Deep Inelastic Cross Section

In inelatic scattering, the proton either goes into an excited state that decays to some number of particles, or it breaks up. Even though it is of interest what happens to the proton, to study its structure only the kinematics of the electron have to be observed. When only some of the final state particles are observed, and the remainder are ignored, the process is referred to as an inclusive process. In this case the reaction becomes $e^- + p \to e^- + X$, where X represents the proton fragments.

To represent the electron-proton inelastic cross section, recall in that in the lecture on electron-muon scattering, the differential cross section being written in terms of two second rank tensors:

$$d\sigma \propto L_e^{\alpha\beta} L^{\mu}_{\alpha\beta} \tag{1}$$

where one of the tensor represents the electron, and the other the represents the muon. The lepton tensor itself is given by:

$$L_e^{\mu\nu} = 2\left[k_f^{\mu} k_i^{\nu} + k_f^{\nu} k_i^{\mu} - \left(k_f^{\sigma} k_{\sigma}^{i} - m^2\right) g^{\mu\nu}\right] \tag{2}$$

To describe electron-proton inelastic scattering, replace the muon tensor with the most general second rank tensor possible; this being the same technique that was used in elastic scattering except for the use of the currents in that case. Before proceeding one more point has to be made, the final state X is not unique, so somehow this has to be taken into account. To determine how to take this into account, recall that the differential cross section has the form:

$$d\sigma = \frac{|M_{fi}|^2}{F} dQ \quad \text{with} \quad |M_{fi}|^2 = L^e_{\mu\nu} K^{\mu\nu} \tag{3}$$

where $K^{\mu\nu}$ represents the hadronic tensor, F is the incident flux, and dQ, the phase space factor, is given by:

$$dQ = (2\pi)^4 \delta^4(k + p - k' - p_1' - p_2' - \cdots p_n') \frac{dk'^3}{(2\pi)^3 2E_e'} \frac{dp_1'^3}{(2\pi)^3 2E_1'} \frac{dp_2'^3}{(2\pi)^3 2E_2'} \cdots \frac{dp_n'^3}{(2\pi)^3 2E_n'} \tag{4}$$

where n is the number of particles that the proton fragments into, which varies from event to event making it difficult to deal with. But since only the protopeties of the electron are important from the point of view of the physics, only the phase space factor for the electron is needed. Therefore, the delta-function and the remaining phase space factors are incorporated into the hadronic tensor, which is redefined as:

$$W^{\mu\nu} = \frac{1}{4\pi M} \sum_X \int \cdots \int K^{\mu\nu}(X) \frac{dp_1'^3}{(2\pi)^3 2E_1'} \frac{dp_2'^3}{(2\pi)^3 2E_2'} \cdots \frac{dp_n'^3}{(2\pi)^3 2E_n'} \tag{5}$$

note the sum is over all possible final states.

The lack of knowledge of the final state will be hidden in the various form factors that will be introduced in generating the most general second rank tensor. Therefore, the integrals and sums are only there for formal considerations. The tensor itself, is composed of the independent momenta and $g^{\mu\nu}$; notice it will not depend on the various Dirac matrices, since by this point the terms have been spin averaged etc. The most general second rank tensor that is derived from these quantities is:

$$\begin{aligned} W^{\mu\nu} = -W_1 g^{\mu\nu} + \frac{W_2}{M^2} p^{\mu} p^{\nu} + i\frac{W_3}{M^2} \epsilon^{\mu\nu\alpha\beta} p_{\alpha} p_{\beta} \\ + \frac{W_4}{M^2} q^{\mu} q^{\nu} + \frac{W_5}{M^2}(p^{\mu} q^{\nu} + p^{\nu} q^{\mu}) + \frac{W_6}{M^2}(p^{\mu} q^{\nu} - p^{\nu} q^{\mu}) \end{aligned} \tag{6}$$

where $\epsilon^{\mu\nu\alpha\beta}$ is the total anti-symmetric tensor, with $\epsilon^{0123} = -1$, implying even permutations are -1, while odd permuations are $+1$, and zero if any two or more indices are the same. At this point two terms can be removed, the W_3 term violates parity (this is due to it coming from a trace that has a γ^5), and the W_6 term since it is antisymmetric in the Lorentz indices, while the lepton tensor is symmetric, this term will be zero.

This leaves only the W_1, W_2, W_4, and W_5. Current conservation still has to be applied to the tensor, but before doing this the functional dependence of the W_i is discussed. In the case of elastic scattering, the form factors depended on the only independent Lorentz scalar q^2, with the only other Lorentz scalar associated with the hadronic vertex $p \cdot q$ being equal to $-q^2/2$. The reason for this, has to do with the mass of the proton (hadronic system) being the same before and after the interaction. In the inelastic case, the mass condition is no longer valid, therefore $p \cdot q$ is independent of q^2:

$$W^2 = (p+q)^2 = M^2 + q^2 + 2p \cdot q \quad \Rightarrow \quad p \cdot q = \frac{W^2 - M^2 - q^2}{2} \tag{7}$$

where W^2 is the square of the mass of the final hadronic system. The W_i are then functions of two Lorentz scalars, q^2, and $\nu = p \cdot q/M$ (this being the variable that is typically used).

The last condition that has to be applied to the hadronic tensor, is that of current conservation. To apply this condition, recall that the leptonic and therefore the hadronic tensor are the products of the current and its complex conjugate $W^{\mu\nu} \propto J^\mu J^{*\nu}$. Current conservation is then applied in the same manner as in the elastic case, that is $q_\mu W^{\mu\nu} = 0$. Applying the condition of current conservation to the hadronic tensor gives:

$$q_\mu W^{\mu\nu} = [-W_1 M^2 + W_4 q^2 + W_5 (p \cdot q)] q^\nu + [W_2 (p \cdot q) + W_5 q^2] p^\nu = 0 \tag{8}$$

since p^ν and q^ν are independent, each of the terms in brackets must be equal to zero separately, leading to:

$$W_5 = -\frac{p \cdot q}{q^2} W_2 \qquad W_4 = \frac{M^2}{q^2} W_1 + \frac{(p \cdot q)^2}{q^4} W_2 \tag{9}$$

This finally leads to:

$$W^{\mu\nu} = W_1(q^2, \nu) \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + \frac{W_2(q^2, \nu)}{M^2} \left(p^\mu - \frac{p \cdot q}{q^2} q^\mu \right) \left(p^\nu - \frac{p \cdot q}{q^2} q^\nu \right) \tag{10}$$

Dividing out the M^2 term to put this into a more common form, and taking the tensor product with equation 2 gives:

$$L^e_{\mu\nu} W^{\mu\nu} = 4W_1 (k \cdot k') + \frac{2W_2}{M^2} \left[2(p \cdot k)(p \cdot k') - M^2 k \cdot k' \right] \tag{11}$$

which in the lab frame (proton rest frame), and extremely relativistic electron, this becomes:

$$L^e_{\mu\nu} W^{\mu\nu} = 4E_e E'_e \left[W_2 \cos^2 \frac{\theta}{2} + 2W_1 \sin^2 \frac{\theta}{2} \right] \tag{12}$$

With the value of the magnitude square of the matrix element, the differential cross section can finally be calculated. This is given by:

$$d\sigma = \frac{1}{2E2M|v_e|}\left[4\pi M\frac{e^4}{q^4}L^e_{\mu\nu}W^{\mu\nu}\right]\frac{d^3k'}{(2\pi)^3 2E'_e} \tag{13}$$

Making the transformation $d^3k' = k'^2 dk' d\Omega$ and realizing that in the exterme relativistic region for the electron $E' = |k'|$ the differential cross section becomes:

$$\frac{d\sigma}{dE'd\Omega} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\left[W_2(q^2,\nu)\cos^2\frac{\theta}{2} + 2W_1(q^2,\nu)\sin^2\frac{\theta}{2}\right] \tag{14}$$