Physics 6213
Lecture 9

1 Introduction

This lecture looks at electron proton elastic scattering, with the goal of interpreting the gross structure of the proton. The scattering will occur at small values of q^2, such the the remains intact; that is the underlying (quark) structure is not explored. The starting point for this study, is the electron-muon differential cross section, which describes the scattering of a light fermion off a heavy fermion. In the electron-muon case, both particles are point-like and therefore structureless. This is will require the differential cross section to be modified, to take the extended size of the proton into account.

2 Electron-Proton Elastic Scattering

The simplest approximation to electron-proton scattering, is to treat the proton as a point particle. In this case the electron-muon differential cross section can be used with the proton mass replacing the muon mass:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha^2}{4E_i^2 \sin^4 \frac{\theta}{2}}\right) \frac{E_f}{E_i} \left(\cos^2 \frac{\theta}{2} - \frac{q^2}{2M^2} \sin^2 \frac{\theta}{2}\right) \tag{1}$$

For very low values of the 4-momentum transfer this works, but for any reasonable value, it will not work since the proton is not a point particle.

To determine a form for the differential cross section, start with the transition amplitude (note this is not the only approach, for deep inelastic scattering a different approach will be used):

$$T_{fi} = -i \int j^\mu \left(-\frac{g_{\mu\nu}}{q^2}\right) J^\nu d^4x \tag{2}$$

where the electron current is given by:

$$j^\mu = -e\bar{u}(k_f)\gamma^\mu u(k_i) e^{i(k_f - k_i)\cdot x} \tag{3}$$

and the proton current, is given by:

$$J^\mu = e\bar{u}(p_f)\Gamma^\mu u(p_i) e^{i(p_f - p_i)\cdot x} \tag{4}$$

with the quantity in the brackets being determined from general considerations. The quantity in brackets has to have the following properties:

1. must be a 4-vector built from the 16 matrices linearly independent 4×4 matrices and the various 4-momenta (only two of are independent, with the others being constrained by energy-momentum conservation);

2. each term must be linearly independent of the others;

3. the current must be conserved ($\partial_\mu J^\mu = 0$).

The only parity conserving (this excludes terms with γ^5) terms that can be built from the 4-momenta and the 16 matrices are: γ^μ, $\sigma^{\mu\nu}q_\nu$, q^μ, p^μ, and $\sigma^{\mu\nu}p_\nu$. These terms are then multiplied by a function of all independent invariants to form the vertex factor. The only invariant is q^2, since $p \cdot q = -q^2/2$ due to the mass of the initial and final hadronic systems be the same (elastic scattering preserves the particles). The vertex factor is therefore:

$$\Gamma^\mu(q^2) = \left[\Gamma_1(q^2)\gamma^\mu + \Gamma_2(q^2)\sigma^{\mu\nu}q_\nu + \Gamma_3(q^2)q^\mu + \Gamma_4(q^2)p^\mu + \Gamma_5(q^2)\sigma^{\mu\nu}p_\nu\right] \tag{5}$$

The first criteria has been used to write this equation. Now the second criteria, that the terms are linearly independent, has to be verified. The first four terms are linearly independent, since they cannot be built from each other. However the last term is shown to not be linearly independent as follows:

$$\left.\begin{array}{l}\sigma^{\mu\nu} = (i/2)\left[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right] \\ \left[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right] = 2g^{\mu\nu}\end{array}\right\} \quad \Rightarrow \quad \sigma^{\mu\nu}p_\nu = i(p_\nu\gamma^\mu\gamma^\nu - p^\mu) \tag{6}$$

using the Dirac equation leads to:

$$(\gamma^\mu p_\mu - m)u = 0 \quad \Rightarrow \quad \sigma^{\mu\nu}p_\nu = i(m\gamma^\mu + p^\mu) \tag{7}$$

which are proportional to terms one and four, therefore not linearly independent.

Finally criteria three, current conservation, must be satisfied. To do this, notice that the temporal and spatial parts of the current are given by $e^{i(p_f - p_i)\cdot x} = e^{iq\cdot x}$. Since this is the only piece of the current that is acted on by the derivative, the condition for current conservation is given by $\partial_\mu J^\mu = iq_\mu J^\mu$. The first term is:

$$\begin{aligned} q_\mu \bar{u}(p_f)\gamma^\mu u(p_i) &= \bar{u}(p_f)[p^f_\mu \bar{u}\gamma^\mu - \gamma^\mu p^i_\mu]u(p_i) \\ &= \bar{u}(p_f)[M - M]u(p_i) = 0 \end{aligned} \tag{8}$$

where the Dirac equation and its adjoint were used in the last step. The second term is zero, since it is the product of a symmetric term, and an anti-symmetric term $q_\mu\sigma^{\mu\nu}p_\nu = 0$. The third term is the mass of the virtual photon $q_\mu q^\mu = q^2 \neq 0$. Finally the fourth term $q_\mu p^\mu = -q^2/2 \neq 0$. Therefore, to conserve current, $\Gamma_3 = \Gamma_4 = 0$, leaving the vertex factor to equal (using standard notation):

$$\Gamma(q^2) = \left[F_1(q^2)\gamma^\mu + \frac{i\kappa F_2(q^2)}{2M}\sigma^{\mu\nu}q_\nu\right] \tag{9}$$

where κ is the anomalous magnetic moment; this is a measure of how much it deviates from a point particle, for the proton $\kappa = 1.79$, while for the neutron it is $\kappa = -1.91$. In addition, the form factors $F_i(q^2)$ define the structure of the proton.

Using the currents defined above, and following the procedure for calculating the cross section previously given, the differential cross section is given by:

$$\begin{aligned} \frac{d\sigma}{d\Omega} = \left(\frac{\alpha^2}{4E_i^2\sin^4\frac{\theta}{2}}\right)\frac{E_f}{E_i}\Bigg[&\left(F_1(q^2) - \frac{\kappa q^2}{4M^2}F_2(q^2)\right)\cos^2\frac{\theta}{2} \\ &+ \frac{q^2}{2M^2}\left(F_1(q^2) + \kappa F_2(q^2)\right)^2\sin^2\frac{\theta}{2}\Bigg] \end{aligned} \tag{10}$$

In the limit of $q^2 \to 0$ (very long wave-length), this formula must have the form of the electron-muon cross section with a magnetic dipole moment given by $e(1+\kappa)/2M$. This condition leads to $F_1(0) = 1$ and $F_2(0) = 1$, and by setting $\kappa = 0$ the original formula is recovered.

To compare the cross section to experiment, the coefficients are written in a form where there are no cross terms. This leads to the following definitions:

$$G_E(q^2) \equiv F_1(q^2) + \frac{\kappa q^2}{4M^2} F_2(q^2) \qquad G_M(q^2) \equiv F_1(q^2) + \kappa F_2(q^2) \tag{11}$$

Substituting these into equation 10 leads to the differential cross section being written as:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha^2}{4E_i^2 \sin^4 \frac{\theta}{2}} \right) \frac{E_f}{E_i} \left[\left(\frac{G_E^2(q^2) + \tau G_M^2(q^2)}{1+\tau} \right) \cos^2 \frac{\theta}{2} + 2\tau G_M^2(q^2) \sin^2 \frac{\theta}{2} \right] \tag{12}$$

where $\tau = -q^2/4M^2$.

2.1 Interpretation of $G_M(q^2)$ and $G_E(q^2)$

To interpret the $G_M(q^2)$and $G_E(q^2)$, it is easiest to work in the Breit (brick wall) frame. In this frame, the momentum of the proton after the interaction is equal in magnitude, but opposite in direction to the incident momenta; ($\mathbf{p}_f = -\mathbf{p}_i$) hence the name brick wall frame. To simplify the interpretation, it is best to rewrite the proton current as follows:

$$J^\mu = e\bar{u}(p_f) \left[\gamma^\mu (F_1(q^2) + \kappa F_2(q^2)) - \frac{p_f^\mu + p_i^\mu}{2M} \kappa F_2(q^2) \right] u(p_i) \tag{13}$$

where the current is the same as in equation 4, but rewritten through the use of the Gordon-Decomposition:

$$\bar{u}(p_f)\gamma^\mu u(p_i) = \frac{1}{2m} \bar{u}(p_f) \left[(p_f + p_i)^\mu + i\sigma^{\mu\nu} q_\nu \right] u(p_i) \tag{14}$$

First, the charge density (J^0) is extracted at $x = 0$:

$$J^0 = e\bar{u}(p_f) \left[\gamma^0 (F_1(q^2) + \kappa F_2(q^2)) - \frac{2E}{2M} \kappa F_2(q^2) \right] u(p_i) \tag{15}$$

where the fact that $E_f = E_i$ in the Breit frame was used. This expression is further simplified using the relations $\bar{u}u = 2E$, $\bar{u}\gamma^0 u = 2M$, and equation 11. This lead to:

$$\rho = J^0 = 2MeG_E(q^2) \tag{16}$$

note that the relations among the spinors requires them to be equal before and after the interaction, implying the the helicity flips due to the change in proton direction.

The remaining piece of the current is given by:

$$\vec{J} = e\bar{u}(p_f)\vec{\gamma} u(p_i) G_M(q^2) \tag{17}$$

since the second term in equation 13 is zero due to the 3-currents being opposite in sign, and equal in magnitude. The terms can be worked out using the explicit form of the spinors, and γ matrices.

What is found is that the terms with like initial and final spinors are zero, $J^3 = 0$ and the remaining currents give:

$$J^1 \pm iJ^2 = \mp 2e|\vec{q}|G_M(q^2) \qquad \text{for } \lambda_i = \lambda_f = \pm\frac{1}{2} \tag{18}$$

the helicity remains the same after the interaction, since only the terms with unlike spinors are non-zero. Notice that the produces a magnetic moment perpendicular to its plane, along the spin direction.

Given the identification of $G_E(q^2)$with the charge density at $x = 0$, it can be treated as the Fourier transform of the charge distribution for all x:

$$F(|\vec{q}|) = \int \rho(\vec{r}) e^{i\vec{q}\cdot\vec{x}} d^3x \tag{19}$$

this being valid only for the case where $|\vec{q}| \ll M$. For the case where the charge distribution is taken to be an exponential, $F(|\vec{q}|)$ is given by:

$$\rho(r) = e^{-mr} \quad \Rightarrow \quad F(|\vec{q}|) = \left(1 - \frac{q^2}{m^2}\right)^{-2} \tag{20}$$

where $q^2 = -|\vec{q}|^2$.

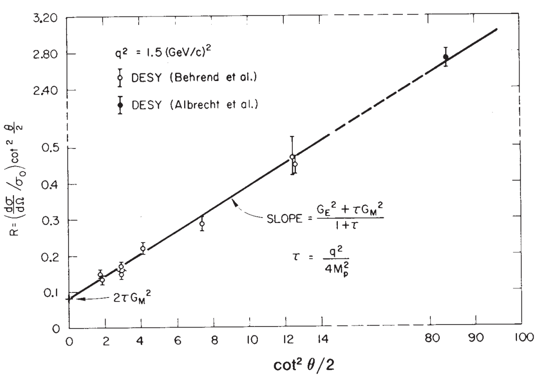

Figure 1: This figure shows elastic electron-proton data plotted in terms of the G_E and G_M. From this plot for fixed values of q^2, the values of both G_E and G_M can be extracted.

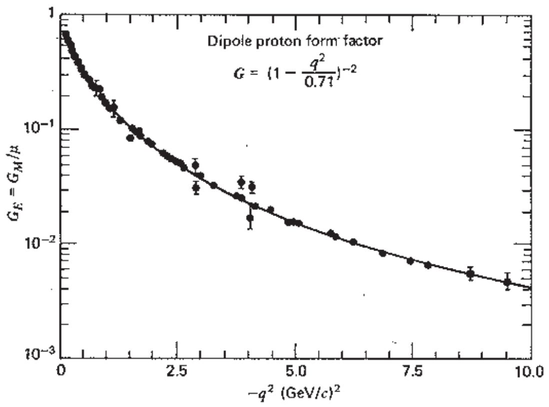

Figure 2: This figure shows elastic electron-proton data as function of $-q^2$. The parameter μ is the magnetic dipole moment of the proton.