Physics 6213
Lecture 8

1 Introduction

This lecture is a continuation of the previous lecture, looking into electron-muon scattering. In this lecture the differential and total cross sections in the center of mass frame are calculated. The cross section is then interpreted in terms of helicity conservation. A discussion of helicity, chirality and spin is included.

2 Center of Mass Cross Section

The starting point for the calculation of the cross section, is the magnitude squared of the invariant amplitude:

$$|M_{fi}|^2 = \frac{8e^4}{|k_f - k_i|^4}\left[(k_f \cdot p_f)(k_i \cdot p_i) + (k_f \cdot p_i)(k_i \cdot p_f) - m^2(p_f \cdot p_i) - M^2 k_f \cdot k_i + 2m^2M^2\right] \tag{1}$$

As in the case of the lab frame cross section, this calculation will be carried out under the approximation of extremely relativistic electrons. In this case both the mass of the electron and muon can be ignored:

$$|M_{fi}|^2 = \frac{8e^4}{|k_f - k_i|^4}\left[(k_f \cdot p_f)(k_i \cdot p_i) + (k_f \cdot p_i)(k_i \cdot p_f)\right] \tag{2}$$

Even though this form of the invariant amplitude is correct, and contains all the needed information, it is usually more illuminating to write it in terms of the Mandelstam variables. First of all, the Mandelstam variables in the relativistic limit are given by:

$$\begin{aligned} s &= (p_i + k_i)^2 \approx 2k_i \cdot p_i & s &= (p_f + k_f)^2 \approx 2k_f \cdot p_f \\ t &= (k_f - k_i)^2 \approx -2k_f \cdot k_i & t &= (p_f - p_i)^2 \approx -2p_f \cdot p_i \\ u &= (p_f - k_i)^2 \approx -2p_f \cdot k_i & u &= (k_f - p_i)^2 \approx -2k_f \cdot p_i \end{aligned} \tag{3}$$

The invariant amplitude is then trivially seen to be:

$$|M_{fi}|^2 = 2e^2\frac{s^2 + u^2}{t^2} \tag{4}$$

From this amplitude, and the use of the crossing relations, the cross section for $e^-e^+ \to \mu^-\mu^+$ is given by:

$$|M_{fi}|^2 = 2e^2\frac{t^2 + u^2}{s^2} \tag{5}$$

where the crossing relation interchanges $k_f \leftrightarrow -p_i$, which implies $s \leftrightarrow t$. The difference between the two cross section, is the exchange particle goes from a time-like photon with an imaginary mass, to a space-like photon with a real mass.

Having alraedy calculated the differential cross section for electron-muon scattering in the lab frame where it will be useful, the differential cross section for $e^-e^+ \to \mu^-\mu^+$ will now be calculated. The first step is to insert equation 5 into the fromula for the differential cross section:

$$d\sigma = \frac{(2\pi)^4\delta^4(p'_f + p_f - k'_i - k_i)}{2E^e_i 2E^e_i|\vec{v}^e_i - \vec{v}^e_i|}|M_{fi}|^2 \frac{d^3p'_f}{(2\pi)^3 2E'^{\mu}_f} \frac{d^3p_f}{(2\pi)^3 2E^{\mu}_f} \tag{6}$$

First make the substitution $|\vec{v}_i - \vec{v}_i| = 2|k_i|/E^e_i$ The next step is to integrate out the three-momenta the first p_f, then rewrite the remaining differential as $d^3p'_f = p'^2_f dp_f d\Omega$, and use the relation $E^e_f dE^e_f = p_f dp_f$, which comes from differentiating $E^2 = |\vec{p}|^2 + m^2$. The differetial cross section (angular distribution) is:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1 + \cos^2\theta) \tag{7}$$

where $\alpha = e^2/4\pi$. Finally the total cross section is:

$$\sigma = \frac{4\pi\alpha^2}{3s} \tag{8}$$

3 Spin, Helicity and Chirality

To understand the cross sections derived above, the spin for relativistic Fermions needs to be understood. Start in the rest frame of the Fermion, where the spinors are given by:

$$u^1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad u^2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad v^1 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \quad v^2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \tag{9}$$

The spin operators associated with these spinors is generalized from the non-relativistic case to be:

$$\vec{\Sigma} = \begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix} \tag{10}$$

where the $\vec{\sigma}$ are the Pauli spin matrices. As in the Schrödinger case, the spinors given in equation 9 are eigen-states of Σ_3, they represent spin along the positive and negative z-axis:

$$\Sigma_3 u^1 = u^1 \quad \Sigma_3 u^2 = -u^3 \quad \Sigma_3 v^1 = -v^1 \quad \Sigma_3 v^2 = v^2 \tag{11}$$

recalling that

$$\sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{12}$$

So far, this is the same as for the spinors used in the Schrödinger equation. Now boost the spinors in an arbitrary direction, and apply Σ_3 to them (since the following holds for all spinors, only the case of u^1 is examined):

$$\Sigma_3 u^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{-p_3}{E+m} \\ \frac{p_1+ip_2}{E+m} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \frac{-p_z}{E+m} \\ 0 \end{pmatrix} \quad \Rightarrow \quad \Sigma_3 u^1 \neq s u^1 \tag{13}$$

implying that the spinors are not eigen-states of the spin operator unless the axis is selected to be in the direction of motion—in this case $p_1 = p_2 = 0$. The reason for this is that neither the spin or orbital angular momentum commute with the Hamiltonian, only the total angular momentum commutes.

Given that the spinors are not in general eigen-states of the spin operator, but only when the one of the axis is along the direction of motion, it makes sense to define the operator $\vec{\Sigma} \cdot \vec{p}/|\vec{p}|$, which is the spin projection along the momentum. This is called the helicity operator, where the helicity is a vector either along or opposit the direction of motion. This operator is easily shown to commute with the with the Hamiltonian:

$$\left[\vec{\Sigma} \cdot \vec{p}, \vec{\alpha} \cdot \vec{p} + \beta m\right] = (\vec{\sigma} \cdot \vec{p})^2 \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = 0 \tag{14}$$

where the βm term commutes with the helicity, since they are both diagonal, also recall that:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \tag{15}$$

Therefore, the spinors are eigen-states of helicity, but keep in mind that the helicity is a frame dependent quantity.

An alternate method of writing the helicity, is in terms of the chirality operator, which has the following definition:

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{16}$$

where the matrix form corresponds to the representation being used in these lectures. If the spinors are given for massless particles, then γ^5 would be given in a diagonal representation. The γ^5 matrix has the following properties:

$$\gamma^{5\dagger} = \gamma^5 \qquad \left(\gamma^5\right)^2 = I \qquad \left\{\gamma^5, \gamma^\mu\right\} = 0 \tag{17}$$

To make the connection to helicity, write the Dirac equations in terms of γ^5:

$$\left.\begin{array}{l} (\vec{\alpha} \cdot \vec{p} + \beta m)u = Eu \\ \vec{\alpha} = \gamma^5 \vec{\Sigma} = \vec{\Sigma}\gamma^5 \end{array}\right\} \quad \Rightarrow \quad \vec{\Sigma} \cdot \vec{p} = \gamma^5 (E - \beta m) \tag{18}$$

This leads to the following definition for the helicity operator:

$$\Lambda = \gamma^5 \frac{E - \beta m}{|\vec{p}|} \tag{19}$$

which in either of the following two limits, $m = 0$ or $E \gg m$, is:

$$\Lambda = \gamma^5 \tag{20}$$

In either of these two limits, the Fermion spinors become:

$$u^1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad u^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad v^1 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad v^2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{21}$$

Applying γ^5 to these spinors gives:

$$\gamma^5 u^1 = u^1 \qquad \gamma^5 u^2 = -u^2 \qquad \gamma^5 v^1 = -v^1 \qquad \gamma^5 v^2 = v^2 \tag{22}$$

The propose for all of this, is to define left and right handed chiral states; these are helicity states of massless particles, therefore have the same chirality (helicity) in all frames since there is no rest frame. The left and rifht handed chiral states are defined in terms of the projection operators with the following properties:

$$\begin{array}{ll} \frac{1}{2}(1+\gamma^5)u = u_R & \frac{1}{2}(1-\gamma^5)u = u_L \\ \frac{1}{2}(1-\gamma^5)v = v_R & \frac{1}{2}(1+\gamma^5)v = v_L \end{array} \tag{23}$$

A general spinor is written as the sum of a left handed chiral state and a right handed chiral state $u = u_R + u_L$. The conjugate states are given by:

$$\bar{u}_L = \left[\frac{1}{2}(1-\gamma^5)u\right]^\dagger \gamma^0 = \bar{u}\frac{1}{2}(1+\gamma^5) \tag{24}$$

where the anticommutation of γ^5 and γ^0 is used in the last step. Writing the current in terms of left and right handed states leads to:

$$\bar{u}\gamma^\mu u = \bar{u}_R \gamma^\mu u_R + \bar{u}_L \gamma^\mu u_L \tag{25}$$

where the term that couples left and right handed states is zero:

$$\bar{u}_R \gamma^\mu u_L = \frac{1}{4}\bar{u}(1-\gamma^5)\gamma^\mu(1-\gamma^5)u = \frac{1}{4}\bar{u}\gamma^\mu(1+\gamma^5)(1-\gamma^5)u = 0 \tag{26}$$

where the anticommutation of γ^5 and γ^μ is used in the second to the last step and $(\gamma^5)^2 = 1$ is used in the last step. In the case of particles and anti-particles, the opposite holds. Here opposite chiral states couple.

From all of this, it should be obvious that the helicity in $e^+e^- \to \mu^+\mu^-$ for both particles in the initial state are aligned in the same direction, and the final state particles are likewise aligned.

This casn be described by the transition amplitude $d^j_{\lambda'\lambda}(\theta) = \langle j\lambda'|e^{-i\theta J_y}|j\lambda\rangle$ where λ is the net spin direction in the inital state j the total angular momentum, while the primed quantities correspond to the final state. The exponential is a rotation operator, that rotates the helicities from the inital to final state. There are four possible combinations of helicities:

$$\begin{aligned} d^1_{11}(\theta) = d^1_{-1-1}(\theta) &= \frac{1}{2}(1+\cos\theta) \approx -\frac{u}{s} \\ d^1_{1-1}(\theta) = d^1_{-11}(\theta) &= \frac{1}{2}(1-\cos\theta) \approx -\frac{u}{s} \end{aligned} \tag{27}$$

where the $d^1_{\lambda'\lambda}(\theta)$ are tabulated in tables. Since the different spin processes are distinguishable, the cross section is found by adding the four cross sections together. Averaging over spins (this gives a factor of 1/4 which is ignored in the expression below) gives:

$$|M_{fi}|^2 \propto \frac{u^2+t^2}{s^2} \propto (1-\cos^2\theta) \tag{28}$$