Physics 6213
Lecture 7

1 Introduction

Having gone through the properties of the Dirac equation and its solutions, it is time to start studying the case of fermion-fermion scattering, through photon exchange. This lecture will derive the form of the differntial cross section, starting from first order perturbation theory. This will lead to the Feynman rules for QED (Quantum ElectroDynamics). This method is not completely rigorous, but it get the correct result quickly.

The first problem that will be treated, is that of electron-muon scattering. This is the easist problem to solve, and will be used as the starting point for the study of electron-proton (quark) scattering. This will lead to the the understanding of the proton, and other hadrons, in terms of the quark model.

2 The Transition Amplitude

To calculate differential cross section for fermion-fermion scattering, the transition amplitude has to first be determined. The basic form of the transition amplitude used, will be that derived from non-relativistic quantum mechanics, which corresponds to born scattering—lowest order perturbation theory:

$$T_{fi} = -i \int \psi_f^\dagger V \psi_i d^4x \tag{1}$$

notice that a potential V has to be introduced, and must enter the Dirac equation in the same form as in the Schrödinger equation, that is associated with the energy.(One additional point needs to be made, that is that the wave-functions for the Dirac equation involve spinors, therefore instead taking the complex conjugate of ψ_f the Hermitian conjugate is taken.) Therefore to proceed, the electromagnetic field must be inserted into the Dirac equation. Then the Dirac equation must be put into a form that is similar to that of the Schrödinger and Klein-Gordon equations; that is a form with a scalar term that resembles the electromagnetic potential. This can be done by remembering that the momentum in the presence of an electromagnetic field becomes $p^\mu \to p^\mu + eA^\mu$, where the charge has bee taken as $-e$. Inserting this into the Dirac equation gives:

$$[\gamma^\mu(p_\mu + eA^\mu) - m]\,\psi = 0 \quad \Rightarrow \quad (\gamma^\mu p_\mu - m)\,\psi = \gamma^0 V \psi \tag{2}$$

where $\gamma^0 V = -e\gamma_\mu A^\mu$ and the purpose of the γ^0 is to put this term on the same footing as the potential in the Schrödinger equation, that is associate it with the energy—notice that $\gamma^\mu A_\mu$ is a Lorentz scalar.

Now using the form of the transition amplitude as used in the case of both the Schrödinger and Klein-Gordon equations:

$$T_{fi} = -i \int \psi_f^\dagger(x) V(x) \psi(x) d^4x = ie \int \bar{\psi}_f \gamma_\mu A^\mu \psi_i d^4x = -i \int j_\mu^{fi} A^\mu d^4x \tag{3}$$

where the current is that determined earlier $j_\mu^{fi} = -e\bar{\psi}_f \gamma_\mu \psi_i$ and the second term above comes from the identity $\gamma^0\gamma^0 = 1$. The current is expanded into the spinor portion, and the space-time portion as:

$$j_\mu^{fi} = -e\bar{\psi}_f \gamma_\mu \psi_i = -e\bar{u}_f \gamma_\mu u_i e^{-i(p_f - p_i)^\mu x_\mu} \tag{4}$$

in order to distinguish those pieces that are intergarted over. Recall, an integration over the exponential term will lead to the a delta function, which will impose energy-momentum conservation.

The electromagnetic field is derived through the use of the Maxwell equations written in terms of the potentials:

$$\partial^\nu \partial_\nu A^\mu = j^\nu \tag{5}$$

note that the form of the above equation depends on the gauge selected, as will A^μ, but the final result will not. The current in equation 5, corresponds to the particle generating the field, which is not the particle in equationeq:pone. None the less, the paticle generating the field must have the same form for the current:

$$j_\mu^{fi} = -e\bar{u}_f \gamma_\mu u_i e^{-i(p_f - p_i)^\mu x_\mu} \tag{6}$$

Inserting this current into equation 5 yields:

$$\partial^\nu \partial_\nu A^\mu = j^n u \quad \Rightarrow \quad A^\mu = -\frac{g^{\mu\nu}}{q^2} j_\nu^{fi} \tag{7}$$

Putting everything together, gives the transition amplitude for fermion-fermion scattering:

$$T_{fi} = ie^2 \int u_{f_1} \gamma^\mu u_{i_1} \frac{g^{\mu\nu}}{q^2} u_{f_2} \gamma^\mu u_{i_2} \tag{8}$$

with figure 1 representing pictorially the same quantity as equation 8

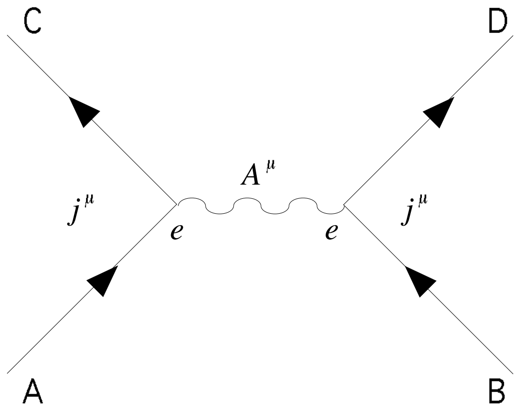

Figure 1: Feynman diagram representing $2 \to 2$ fermion-fermion scattering. The vertical axis represents time.

At this point the 4-dimensional integration is carried out yielding (see fig. 1 for particle indices):

$$T_{fi} = (2\pi)^4 \delta^4(p_C + p_D - p_A - p_B) M_{fi} \tag{9}$$

where the invariant amplitude is given by:

$$-iM_{fi} = (ie\bar{u}_C \gamma^\mu u_A) \left(-i\frac{g_{\mu\nu}}{q^2} \right) (ie\bar{u}_D \gamma^\mu u_B) \tag{10}$$

To get to the cross section, the magnitude square of the T_{fi}has to be calculated. This is not as simple as would first appear, since the expression involves the square of a delta-function. To handle the delta-function correctly, the intergrals in the product have to be carried out seperatly, and one at a time:

$$\begin{aligned}
&\int e^{-i(p_{f_C}+p_{f_D}-p_{i_A}-p_{i_B})} d^4x \int e^{-i(p_{f_C}+p_{f_D}-p_{i_A}-p_{i_B})} d^4x = \qquad (11)\\
&(2\pi)^4 \delta(p_{f_C} + p_{f_D} - p_{i_A} - p_{i_B}) \int e^{-i(p_{f_C}+p_{f_D}-p_{i_A}-p_{i_B})} d^4x = \\
&(2\pi)^4 \delta(p_{f_C} + p_{f_D} - p_{i_A} - p_{i_B}) \int d^4x = \\
&(2\pi)^4 \delta(p_{f_C} + p_{f_D} - p_{i_A} - p_{i_B}) \lim_{TV\to\infty} TV
\end{aligned}$$

Therefore, instead of using the transition amplitude, the transition rate per unit volume is used instead:

$$W_{fi} = \frac{|T_{fi}|^2}{TV} = (2\pi)^4 \delta^4(p_C + p_D - p_A - p_B) |M_{fi}|^2 \tag{12}$$

The final steps to the cross section, involve normalizing the transition rate to the initial flux, and counting the number of final states. The initial flux comes from the normalization condition of j^0:

$$\int j^0 d^3x = 2E \tag{13}$$

where $2E$ represents the number of particles in a volume V. The flux itself is the number of particles crossing a given surface per unit time. Looking at a single beam of particles, the flux is $2E\mathbf{v}$. The overlapping flux, is then given by the two rates $|\mathbf{v}_A - \mathbf{v}_B| 2E_A 2E_B$.

The only remaining term is the density of final states, which is give by:

$$\frac{d^3p}{(2\pi)^3 2E} \tag{14}$$

where it has been normalized to the number of particles in the final state. The cross section is given by:

$$\sigma = \frac{W_{fi}}{\text{initial flux}} (\text{number of final states}) \tag{15}$$

Finally putting everything together, the differential cross section is given by:

$$d\sigma = \frac{(2\pi)^4 \delta^4(p_C + p_D - p_A - p_B)}{2E_A 2E_B |\vec{v}_f - \vec{v}_i|} |M_{fi}|^2 \frac{d^3p_C}{(2\pi)^3 2E_C} \frac{d^3p_D}{(2\pi)^3 2E_D} \tag{16}$$

Therefore the evaluation of the differential cross section involves the same techniques as before to handle the phase space factors. The only thing that is different is the evaluation of the square of the invariant amplitude.

3 Electron-Muon Scattering

In this section, the invariant amplitude for electron-muon scattering is calculated. The amplitude is given by:

$$-iM_{fi} = (ie\bar{u}_C\gamma^\mu u_A)\left(-i\frac{g_{\mu\nu}}{q^2}\right)(ie\bar{u}_D\gamma^\nu u_B) \tag{17}$$

which can be used to define a set of general rules for calculating any other Feynman diagram. These rules are given by:

1. Each fermion-photon vertex gives a factor of $ie\gamma^\mu$;
2. Each fermion coming into a vertex gives a factor of $u(p_i)$;
3. Each fermion leaving a vertex gives a factor of $\bar{u}(p_f)$;
4. Each virtual photon gives a factor of $-ig_{\mu\nu}/q^2$, where q^2 is the 4-momentum transfer at the vertex, which is the virtual mass of the photon.

Even though not given in the amplitude above, the following rules apply to real photons and anti-particles:

1. Each external photon going into a vertex is given by a polarization ϵ_μ;
2. Each external photon leaving a vertex is given by a the complex conjugate of the polarization ϵ^*_μ;
3. Each anti-fermion leaving a vertex gives a factor of $v(p_i)$:
4. Each anti-fermion coming into a vertex gives a factor of $\bar{v}(p_f)$.

For the anti-fermions it is important to draw the arrows going backwards in time to get the assignments given above right.

Having defined the Feynman rules above, it is now time to calculate the magnitude squared of the invarinat amplitude. The first step before the actual calculation is to realize that in most experiment the incident beams are not polarized and that the final state spins are not measured. Therefore, the standard procedure is to average over inital state spins, and sum over final state spins. If polarized cross section is needed, it can arrived at by taking the representation already given for the spinors and multiplying out the various terms. To calculate the spin averaged cross section, various tricks are used.

At this point, the cross section is ready to be calculated. Start with the magnitude squared of the invariant amplitude. This is given in the following form, which clearly shows that it is a Lorentz scalar:

$$|M_{fi}|^2 = \frac{e^4}{q^4} L_{\mathrm{e}}^{\mu\nu} L^{\mathrm{muon}}_{\mu\nu} \tag{18}$$

where $L_{\mathrm{e}}^{\mu\nu}$, the lepton tensor, is given by:

$$L_{\mathrm{e}}^{\mu\nu} = \frac{1}{2}\sum_{\mathrm{spin}} \left[\bar{u}(k_f)\gamma^\mu u(k_i)\right]\left[\bar{u}(k_f)\gamma^\nu u(k_i)\right]^* \tag{19}$$

the factor of 1/2 comes from the averaging process and the muon is given by a term similar to the electron. The purpose for using the lepton tensor, rather than directly doing the calculation, is that this form lends itself to generalizing the cross section. For instance, when a electon is scattered off a proton, the lepton tensor can still be used to describe the electron, while the most general second rank tensor, consistent with all conservation laws, is used to describe the proton. In that case, the invariant amplitude is given by $|M_{fi}|^2 \propto L_e^{\mu\nu} K_{\mu\nu}^p$. Of course $K_{\mu\nu}^p$ is unknown, but it can be built from the known 4-vectors associated with the proton vertex.

The first step in the process of evaluating the sum, is to realize that the terms in brackets are 1×1 matrices. Therefore the complex conjugate of the second term, is equivalent to taking its Hermitian conjugate. The second term is therefore rewritten as:

$$\begin{aligned} &[\bar{u}(k_f)\gamma^\nu u(k_i)]^* = [\bar{u}(k_f)\gamma^\nu u(k_i)]^\dagger = \left[u^\dagger(k_f)\gamma^0\gamma^\nu u(k_i)\right]^\dagger = \left[u^\dagger(k_i)\gamma^{\dagger\nu}\gamma^0 u(k_f)\right] \\ &\Rightarrow \left[u^\dagger(k_i)\gamma^0\gamma^\nu u(k_f)\right] = [\bar{u}(k_i)\gamma^\nu u(k_f)] \end{aligned} \tag{20}$$

where the Hermiticity of $\gamma^0\gamma^\mu$ is used in the second to the last step. The complex conjugate inverts the two spinors. The lepton tensor now becomes, with the sums explicitly:

$$L_{\mathrm{e}}^{\mu\nu} = \frac{1}{2}\sum_{s,s'} \left[\bar{u}^s(k_f)\gamma^\mu u^{s'}(k_i)\right] \left[\bar{u}^{s'}(k_i)\gamma^\nu u^s(k_f)\right] \tag{21}$$

Given that these terms correspond to the products of matrices, the lepton tensor can be expressed in terms of the individual matrix elements:

$$L_{\mathrm{e}}^{\mu\nu} = \frac{1}{2}\sum_{s,s'} \left[\bar{u}_\alpha^s(k_f)\gamma_{\alpha\beta}^\mu u_\beta^{s'}(k_i)\right] \left[\bar{u}_\gamma^{s'}(k_i)\gamma_{\gamma\delta}^\nu u_\delta^s(k_f)\right] \tag{22}$$

where the purpose of this, is to convert the original sum over objects that may or may not commute into components that do commute; note repeated indices are summed over. The equation above can then be rewritten as:

$$L_{\mathrm{e}}^{\mu\nu} = \frac{1}{2}\sum_{s,s'} \left\{[u_\delta^s(k_f)\bar{u}_\alpha^s(k_f)]\,\gamma_{\alpha\beta}^\mu \left[u_\beta^{s'}(k_i)\bar{u}_\gamma^{s'}(k_i)\right]\gamma_{\gamma\delta}^\nu\right\} \tag{23}$$

This gives sums over spin for the product of the spinor and its adjoint:

$$\sum_s u^s(k)\bar{u}^s(k) = \gamma^\mu k_\mu + m \tag{24}$$

which can be worked out by inserting the explicit representations of the spinors. These sums give:

$$L_{\mathrm{e}}^{\mu\nu} = \frac{1}{2}\left(\gamma^\mu k_\mu^f + m\right)_{\delta\alpha} \gamma_{\alpha\beta}^\mu \left(\gamma^\nu k_\nu^i + m\right)_{\beta\gamma} \gamma_{\gamma\delta}^\nu \tag{25}$$

remember that like indices are summed over. Rearranging the terms gives:

$$\begin{aligned} L_{\mathrm{e}}^{\mu\nu} &= \frac{1}{2}\left[\left(\gamma^\mu k_\mu^f + m\right)_{\delta\alpha}\gamma_{\alpha\beta}^\mu\right]\left[\left(\gamma^\nu k_\nu^i + m\right)_{\beta\gamma}\gamma_{\gamma\delta}^\nu\right] \\ &= \frac{1}{2}\left[\left(\gamma^\mu k_\mu^f + m\right)\gamma^\mu\right]_{\delta\beta}\left[\left(\gamma^\nu k_\nu^i + m\right)\gamma^\nu\right]_{\beta\delta} \\ &= \frac{1}{2}\left[(\gamma^\mu k_\mu^f + m)\gamma^\mu(\gamma^\nu k_\nu^i + m)\gamma^\nu\right]_{\delta\delta} \\ &= \frac{1}{2}\mathrm{Tr}\left[(\gamma^\mu k_\mu^f + m)\gamma^\mu(\gamma^\nu k_\nu^i + m)\gamma^\nu\right] \end{aligned} \tag{26}$$

The key to understanding the step between the last two lines, is to realize that a sum over β is performed leaving a term that is diagonal, $\mathbf{M}_{\delta\delta}$, where a sum is performed over δ which is the sum over the diagonal terms. This is the definition of the trace. Therefore the problem has come down to one of determining the trace of the matrix products.

4 Evaluation of Lepton Current

At this point, the lepton current (tensor) has been put into a form that requires the evaluation of a trace; recall a trace is the sum of the diagonal elements of a matrix. Since the trace is the sum of several terms, the trace of each term can be evaluated separately. Expanding out the terms, the following is found:

$$\begin{aligned} L_{\mathrm{e}}^{\mu\nu} &= \frac{1}{2}\mathrm{Tr}\left[\left(\gamma^\sigma k_\sigma^f + m\right)\gamma^\mu\left(\gamma^\beta k_\beta^i + m\right)\gamma^\nu\right] \\ &= \frac{1}{2}\left\{\mathrm{Tr}\left[\gamma^\mu k_\mu^f \gamma^\mu \gamma^\beta k_\beta^i \gamma^\nu\right] + \mathrm{Tr}\left[\gamma^\sigma k_\sigma^f \gamma^\mu \gamma^\nu + \gamma^\mu \gamma^\beta \gamma^\nu k_\beta^i\right] m + \mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right] m^2\right\} \end{aligned} \tag{27}$$

Start with the last term, use the anti-commutation relation and the trace relation $\mathrm{Tr}[\mathbf{AB}] = \mathrm{Tr}[\mathbf{BA}]$ to get the following:

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right] m^2 = \frac{1}{2}\mathrm{Tr}\left[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right] m^2 = \frac{1}{2}\mathrm{Tr}\left[2g^{\mu\nu}\mathbf{I}\right] m^2 = 4g^{\mu\nu}m^2 \tag{28}$$

Next consider the first term; remember that the momenta are just numbers not matrices and can therefore be pulled out of the trace:

$$\begin{aligned} \mathrm{Tr}\left[\gamma^\sigma k_\sigma^f \gamma^\mu \gamma^\beta k_\beta^i \gamma^\nu\right] &= k_\sigma^f k_\beta^i \mathrm{Tr}\left[\gamma^\sigma\gamma^\mu\gamma^\beta\gamma^\nu\right] = k_\sigma^f k_\beta^i \mathrm{Tr}\left[\left(2g^{\sigma\mu} - \gamma^\mu\gamma^\sigma\right)\gamma^\beta\gamma^\nu\right] \\ &= 2k_\sigma^f k_\beta^i g^{\sigma\mu}\mathrm{Tr}\left[\gamma^\beta\gamma^\nu\right] - k_\sigma^f k_\beta^i \mathrm{Tr}\left[\gamma^\mu\gamma^\sigma\gamma^\beta\gamma^\nu\right] \end{aligned} \tag{29}$$

The first term in the previous equation, when simplified, gives:

$$2k_\sigma^f k_\beta^i g^{\sigma\mu}\mathrm{Tr}\left[\gamma^\beta\gamma^\nu\right] = 2k_\sigma^f k_\beta^i g^{\sigma\mu}\frac{1}{2}\mathrm{Tr}\left[\gamma^\beta\gamma^\nu + \gamma^\nu\gamma^\beta\right] = k_\sigma^f k_\beta^i g^{\sigma\mu} 2g^{\beta\nu}\mathrm{Tr}\left[\mathbf{I}\right] = 8k_f^\mu k_i^\nu \tag{30}$$

The second term in equation 29 simplifies to:

$$\begin{aligned} \mathrm{Tr}\left[\gamma^\sigma k_\sigma^f \gamma^\mu \gamma^\nu \gamma^\beta k_\beta^i\right] &= 8k_f^\mu k_i^\nu - k_\sigma^f k_i^\sigma \mathrm{Tr}\left[\gamma^\mu\gamma^\sigma\gamma^\beta\gamma^\nu\right] \\ &= 8k_f^\mu k_i^\nu - k_\sigma^f k_i^\sigma \mathrm{Tr}\left[\gamma^\mu\left(2g^{\sigma\beta} - \gamma^\beta\gamma^\sigma\right)\gamma^\nu\right] \\ &= 8k_f^\mu k_i^\nu - 8k_\sigma^f k_i^\sigma g^{\mu\nu} + k_\sigma^f k_\beta^i \mathrm{Tr}\left[\gamma^\mu\gamma^\beta\gamma^\sigma\gamma^\nu\right] \\ &= 8k_f^\mu k_i^\nu - 8k_\sigma^f k_i^\sigma g^{\mu\nu} + k_\sigma^f k_\beta^i \mathrm{Tr}\left[\gamma^\mu\gamma^\beta\left(2g^{\sigma\nu} - \gamma^\nu\gamma^\sigma\right)\right] \\ &= 8k_f^\mu k_i^\nu - 8k_\sigma^f k_i^\sigma g^{\mu\nu} + 8k_f^\nu k_i^\mu - k_\sigma^f k_\beta^i \mathrm{Tr}\left[\gamma^\mu\gamma^\beta\gamma^\nu\gamma^\sigma\right] \\ &= 8k_f^\mu k_i^\nu - 8k_\sigma^f k_i^\sigma g^{\mu\nu} + 8k_f^\nu k_i^\mu - k_\sigma^f k_\beta^i \mathrm{Tr}\left[\gamma^\sigma\gamma^\mu\gamma^\beta\gamma^\nu\right] \end{aligned} \tag{31}$$

The last term on the right hand side is the same as the term on the left hand side, therefore moving it to the left hand side leads to the trace being equal to:

$$\mathrm{Tr}\left[\gamma^\sigma k^f_\sigma \gamma^\mu \gamma^\nu \gamma^\beta k^i_\beta\right] = 4k^\mu_f k^\nu_i + 4k^\nu_f k^\mu_i - 4k^\beta_f k^i_\beta g^{\mu\nu} \tag{32}$$

Finally the terms with the three gamma matrices are evaluated. Start with the first one:

$$\begin{aligned}
\mathrm{Tr}\left[\gamma^\sigma k^f_\sigma \gamma^\mu \gamma^\nu\right] &= k^f_\sigma \mathrm{Tr}\left[\gamma^\sigma \gamma^\mu \gamma^\nu\right] = k^f_\sigma \mathrm{Tr}\left[\left(2g^{\sigma\mu} - \gamma^\mu\gamma^\sigma\right)\gamma^\nu\right] \\
&= 8k^\mu_f \mathrm{Tr}\left[\gamma^\nu\right] - k^f_\sigma \mathrm{Tr}\left[\gamma^\mu \gamma^\sigma \gamma^\nu\right] \\
&= 8k^\mu_f \mathrm{Tr}\left[\gamma^\nu\right] - k^f_\sigma \mathrm{Tr}\left[\gamma^\mu \left(2g^{\sigma\nu} - \gamma^\sigma\gamma^\nu\right)\right] \\
&= 8k^\mu_f \mathrm{Tr}\left[\gamma^\nu\right] - 8k^\nu_f \mathrm{Tr}\left[\gamma^\mu\right] + k^f_\sigma \mathrm{Tr}\left[\gamma^\mu \gamma^\nu \gamma^\sigma\right] \\
&= 8k^\mu_f \mathrm{Tr}\left[\gamma^\nu\right] - 8k^\nu_f \mathrm{Tr}\left[\gamma^\mu\right] + k^f_\sigma \mathrm{Tr}\left[\gamma^\sigma \gamma^\mu \gamma^\nu\right]
\end{aligned} \tag{33}$$

but the trace of a gamma matrix is zero and the last term on the left hand side is equal to the term on the right hand side. This leads to the trace being equal to zero. In fact the trace of the product of any odd number of gamma matrices, is zero.

Finally, putting all the terms together, leads to the lepton tensor for the electron, being equal to:

$$L^{\mu\nu}_{\mathrm{e}} = 2\left(k^\mu_f k^\nu_i + k^\nu_f k^\mu_i - \left(k^\sigma_f k^i_\sigma - m^2\right) g^{\mu\nu}\right) \tag{34}$$

The lepton tensor for the muon has the same form:

$$L^{\mu\nu}_{\mathrm{muon}} = 2\left(p^\mu_f p^\nu_i + p^\nu_f p^\mu_i - \left(p^\sigma_f p^i_\sigma - M^2\right) g^{\mu\nu}\right) \tag{35}$$

5 Electron-Muon Scattering Amplitude

At this point the square of the electron-muon scattering amplitude can be calculated. This involves substituting in the form of the lepton tensors into the invariant amplitude, gives the expression:

$$\begin{aligned}
|M_{fi}|^2 = \frac{8e^4}{|k_f - k_i|^4} &\left[(k_f \cdot p_f)(k_i \cdot p_i) + (k_f \cdot p_i)(k_i \cdot p_f)\right. \\
&\left. - m^2(p_f \cdot p_i) - M^2 k_f \cdot k_i + 2m^2M^2\right]
\end{aligned} \tag{36}$$

This is the exact form of the amplitude. From this expression the differential cross section can be determined.

As an example of the use of this expression, the case of electron-muon scattering in the lab frame will be looked at with the muon at rest. The electron will be taken in the extreme relativistic limit; $|\vec{v}| \approx 1$. In this case all terms involving the electron mass can be neglected giving an approximate value for the amplitude squared as:

$$|M_{fi}|^2 = \frac{8e^4}{|k_f - k_i|^4}\left[(k_f \cdot p_f)(k_i \cdot p_i) + (k_f \cdot p_i)(k_i \cdot p_f) - M^2 k_f \cdot k_i\right] \tag{37}$$

In the extreme relativistic limit, the following approximations hold:

$$k_i^2 = k_f^2 = 0, \qquad q^2 = -2k_i \cdot k_f \tag{38}$$

This approximation along with the conservation of energy and momentum leads to:

$$|M_{fi}|^2 = \frac{8e^4}{q^4}\left[-\frac{1}{2}q^2(k_i \cdot p_i - k_f \cdot p_i) + 2(k_f \cdot p_i)(k_i \cdot p_i) + \frac{1}{2}M^2 q^2\right], \tag{39}$$

where the identity $p_f = k_i - k_f + p_i$ was used. This involves the use of the fact that the muon is initially at rest $p_i = \{M, 0, 0, 0\}$ and the approximation:

$$q^2 = -2k_i \cdot k_f = -2E_i E_f(1 - \cos\theta) = -4E_i E_f \sin^2\left(\frac{\theta}{2}\right) \tag{40}$$

which gives:

$$|M_{fi}|^2 = \frac{8e^4}{q^4} 2M^2 E_i E_f \left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]. \tag{41}$$

At this point the differential cross section can be calculated. This involves multiplying by the density of final states, dividing by the flux and multiplying by the Dirac Delta function:

$$d\sigma = \frac{(2\pi)^4\delta^4(p_i + k_i - p_f - k_f)}{2E_i 2M|\vec{v}|}|M_{fi}|^2 \frac{d^3p_f}{(2\pi)^3 2p_f^0}\frac{d^3k_f}{(2\pi)^3 2E_f} \tag{42}$$

where p_f^0 is the final energy of the muon and $|\vec{v}| \approx 1$. The integral can be evaluated by first integrating over the muon momentum, then making the substitution $d^3k_f = k_f^2 dk_f d\Omega$ and integrating the electron momentum using the relation:

$$E_f dE_f = p_f dp_f \tag{43}$$

This leads to:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha^2}{4E_i^2\sin^4\frac{\theta}{2}}\right)\frac{E_f}{E_i}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right) \tag{44}$$

where the first term is just the Rutherford cross section, the ratio of the energies is a factor corresponding to the recoil of the muon. Finally the two terms in the last parenthesis correspond to the electromagnetic interaction. The first term is related to the exchange of a photon, while the second term corresponds to the interaction of the magnetic dipole moments. This is referred to as the Rosenbluth cross section.

A similar calculation with the same approximations for the scattering of two spinless particles, leads to a differential cross section given by:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha^2}{4E_i^2\sin^4\frac{\theta}{2}}\right)\frac{E_f}{E_i}\left(\cos^2\frac{\theta}{2}\right) \tag{45}$$

This is exactly as would be expected, the photon exchange term is still there, but the magnetic dipole term disappears.