Physics 6213
Lecture 6

1 Introduction

This lecture shows how to add the electromagnetic field into the Dirac equation. In fact, the method will be presented in a general manner such that it can be used for the Schrödinger and Klein-Gordon equations. In the case of non-relativistic quantum mechanics, the potential is easily inserted into the Schrödinger , the method requires the insertion of the Coulomb potential in to the Hamiltonian. In the case of relativistic quantum mechanics, not just the electric potential has to be accounted for, but the magnetic fields generated by the currents has to be taken into account. In fact, in order to maintain the covariance of the relativistic equations, the 4-vector A^{μ} has to be used. The introduction of the electromagnetic fields will be done through the use of Lagrangian mechanics.

In addition, this lecture will introduce the conditions imposed on the Dirac equation due to the requirement that it remain invariant under local phase transformations. This condition will be shown to be connected to the requirement that the equations be invariant to gauge transformations.

2 Gauge Invariance of Electromagnetic Fields

Maxwell's equations define both the electric and magnetic fields completely. That is, these equations can be used to calculate the field or these equations can be used to verify that a given field is correct. An alternative method to solving the Maxwell equations directly, is to rewrite them in terms of potentials. In many cases, it is easier to solve for the potential first and then determine the fields later.

The Maxwell equations are given by:

$$\begin{array}{ll} \nabla \cdot \mathbf{B} = 0 & \nabla \times \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0 \\ \nabla \times \mathbf{B} - \dfrac{\partial \mathbf{E}}{\partial t} = \mathbf{j} & \nabla \cdot \mathbf{E} = \rho \end{array} \tag{1}$$

The two equations on the first line lead to the definition of the potentials

$$\begin{array}{lll} \nabla \cdot \mathbf{B} = 0 & \Rightarrow & \mathbf{B} = \nabla \times \mathbf{A} \\ \nabla \times \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0 & \Rightarrow & \mathbf{E} = -\nabla \Phi - \dfrac{\partial \mathbf{A}}{\partial t} \end{array} \tag{2}$$

Notice, in the case of no time dependent magnetic fields, the relation between the electrostatic potential is recovered. It is important to notice that the fields are given by taking derivatives of the potentials. That is the potential are not uniquely defined, only potential differences matter. The potential $\mathbf{A}$ can be changed by $\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi$ and the $\mathbf{B}$ field remains the same. This transformation imposes the condition that Φ transform as follows:

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi \quad \Rightarrow \quad \Phi \rightarrow \Phi - \frac{\partial \chi}{\partial t} \quad \Rightarrow \quad A^{\mu} \rightarrow A^{\mu} - \partial^{\mu}\chi \tag{3}$$

This is referred to as a gauge transformation.

3 Charged Particle in Electromagnetic Field—The Minimal Substitution

****Before starting the discussion of gauge invariance, the introduction of the electromagnetic field into the Dirac equation is discussed. Since the Dirac equation originated from the Hamiltonian, the fields must be introduced into the Dirac equation in terms of the potentials. Starting with The simplest starting point is the Lorentz force:

$$\left.\begin{array}{l}\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} = \nabla \times \mathbf{A}\end{array}\right\} \Rightarrow \mathbf{F} = q\left(-\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times \nabla \times \mathbf{A}\right) \tag{4}$$

The triple cross product is simplified using the vector relation:

$$\mathbf{B} \times (\nabla \times \mathbf{C}) = \nabla(\mathbf{B} \cdot \mathbf{C}) - (\mathbf{B} \cdot \nabla)\mathbf{C} - \mathbf{C} \times (\nabla \times \mathbf{B}) - (\mathbf{C} \cdot \nabla)\mathbf{B} \tag{5}$$

which gives:

$$\Rightarrow \mathbf{v} \times (\nabla \times \mathbf{A}) = \nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla)\mathbf{A} \tag{6}$$

where the fact that the velocity does not depend explicitly on the position is used. Next the total time derivative of the $\mathbf{A}$ is calculated:

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{A} \tag{7}$$

where the first terms comes from the explicit time dependents of $\mathbf{A}$, while the second term comes from the value of the potential at the position of the particle comes from the motion of the particle. Putting these equations together leads to:

$$\mathbf{F} = e\left[-\nabla\Phi + \nabla(\mathbf{v} \cdot \mathbf{A}) - \frac{d\mathbf{A}}{dt}\right] \tag{8}$$

which represents a generalized force due to a velocity dependent potential.

To determine the potential, the Lagrangian formalism is introduced. First recall that the Lagrangian is the difference in kinetic and potential energies. Then recall that the generalized force, is the gradient of the potential that is sought:

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \Rightarrow L = T - U \tag{9}$$

To put the generalized force into a more useful form, the total time derivative of the vector potential is rewritten as:

$$\frac{d\mathbf{A}}{dt} = \frac{d}{dt}\nabla_v(\mathbf{A} \cdot \mathbf{v}) \tag{10}$$

where ∇_v is the gradient in terms of the velocity, and the fact that the vector potential is independent of the velocity is used. The force is now written as:

$$\mathbf{F} = q\left\{-\nabla\left(\Phi - \mathbf{v} \cdot \mathbf{A}\right) - \frac{d}{dt}\left(\nabla_v(\mathbf{v} \cdot \mathbf{A})\right)\right\} \tag{11}$$

and since the scalar potential is independent of the velocity, the generalized potential is identified as:

$$U = q(\Phi - \mathbf{v} \cdot \mathbf{A}) \tag{12}$$

At this point everything that is necessary to include the electromagnetic potentials into the Dirac equation is available. The final step comes in recognizing that the relation $\mathbf{p} \to -i\nabla$ the quantum condition is imposed on the canonical momentum and not the mechanical momentum (mv). This comes from the Lagrangian, and is given by:

$$\frac{\partial L}{\partial \dot{q}_j} = m\mathbf{v} + q\mathbf{A} \tag{13}$$

In the original derivation of the Dirac equation, the mechanical momentum is used; in this case both momenta are equal to each other. Therefore, to include the electromagnetic potentials, and maintain the quantum condition, the mechanical momentum is replace with the canonical momentum and the vector potential:

$$m\mathbf{v} \to \mathbf{p} - q\mathbf{A} \quad \Rightarrow \quad mu^\mu \to p^\mu - qA^\mu \quad \Rightarrow \quad i\partial^\mu \to i(\partial^\mu + iqA^\mu) \tag{14}$$

where u^μ is the 4-velocity, $\mathbf{p}$ represents the canonical momentum, the scalar potential is added to the energy (here the energy represents the rest mass energy plus kinetic energy), and the covariant derivative is defined as $\mathcal{D}^\mu \equiv \partial^\mu + iqA^\mu$.

4 Gauge Invariance in Quantum Mechanics

Now that the procedure for incorporating the fields into the Dirac equation is known, and the transformation properties of the potential under which the fields remain invariant, the properties of the Dirac equation can be studied. Again, as in the case of the electromagnetic potentials, this will be worked out in the general case.

From quantum mechanics, it is known that the measurement of an observable, is associated with a real number, and that the expectation value for the measurement is given by:

$$\langle \mathcal{O} \rangle = \int \psi^* \mathcal{O} \psi dV \tag{15}$$

Therefore, a global phase transformation of the wave-function ($\psi \to \psi e^{i\phi}$) leaves the the expectation value unchanged. The wave-equation (the wave-equation can be any one of the Schrödinger , Klein-Gordon or Dirac equations) is also invariant to a global phase transformation. Even though this invariance property leads to the conservation of charge, it would make more sense to apply a local phase transformation, since relativity says that no information can be transmitted faster than the speed of light. That is, the phase can not change everywhere instantaneously.

Consider a local phase transformation $\psi \to e^{iq\phi(x)}\psi$, then recall that all the wave-equation have a term of the form $\partial^\mu \psi$. This term will clearly lead to the wave-equation being non-invariant under a local phase transformation:

$$\partial^\mu \left[e^{iq\phi(x)} \psi \right] = e^{iq\phi(x)} \partial^\mu \psi + iqe^{iq\phi(x)} \psi \partial^\mu \phi(x) = e^{iq\phi(x)} \left[\partial^\mu + iqA^\mu + iq\partial^\mu \phi(x) \right] \psi \tag{16}$$

To make this term invariant, and therefore the various wave-equations, the last term has to be removed. But the last term is exactly the quantity that would come about under a gauge transformation of the electromagnetic potential $A^\mu \to A^\mu - \partial^\mu \phi(x)$. If the minimal substitution applied to the derivative equation:

$$\partial^\mu \psi \to [\partial^\mu + iqA^\mu]\,\psi \tag{17}$$

the equation is not invariant to a gauge transformation:

$$[\partial^\mu + iqA^\mu - iq\partial^\mu \phi(x)]\,\psi \tag{18}$$

But, if the wave-function is required undergo a local phase transformation, in addition to the gauge transformation, the equation is then invariant:

$$[\partial^\mu + iqA^\mu - iq\partial^\mu \phi(x)]\, e^{iq\phi(x)}\psi = e^{iq\phi(x)}\,[\partial^\mu + iqA^\mu]\,\psi \tag{19}$$

where the the parameter q is identified with the charge, and it must be recalled that this term is part of the each of the wave-equations, with each term in the wave-equation containing a ψ factor leading to the exponential being canceled. Therefore, requiring the wave-equation being invariant to a local phase transformation requires the electromagnetic potential being introduced into the wave equation. On the other hand, requiring the wave-equation being invariant to a gauge transformation, requires the wave-function undergoing a local phase transformation.

5 Covariant Lagrangian

Before concluding this set of lectures, and moving on to calculating cross sections, various properties of the Lagrangian will be examined In addition the Lagrangian for the Dirac equation will be given. The reason for this diversion has to do with the Lagrangian being the starting point for all theories that we know. This being the case, its basic properties should be understood in order to understand the theories that we have and how to extend them.

The first of these properties is that it is a Lorentz invariant. That is it transforms under Lorentz transformations like a scalar (or pseudo-scalar). This comes from the fact the any variation of the action is zero in all frames:

$$\delta A = 0 \quad \text{where} \quad A = \int_{\tau_1}^{\tau_2} L d\tau \tag{20}$$

otherwise it would be possible to distinguish between different frames. Note in the integral, since $d\tau$ is an invariant, L must also be—recall τ is the proper time. This is in fact the reason that all relativistic theories are written in terms of the Lagrangian, and not the Hamiltonian (*H is the energy, there a component of a 4-vector that changes from frame to frame*).

As a first example of this invariance of the Lagrangian, a Lagrangian will be constructed for a charged particle in an electromagnetic field. In the non-relativistic case this Lagrangian would be:

$$L = \frac{1}{2}mv^2 - q\phi + q\mathbf{v}\cdot\mathbf{A} \tag{21}$$

The standard procedure to convert from a non-relativistic theory to a relativistic theory, is to convert 3-vectors to 4-vectors:

$$L = \frac{1}{2} m u^\mu u_\mu + q u^\mu A_\mu \tag{22}$$

This equation has the correct non-relativistic limit, that is equation 21 is recovered. Also, by using the Euler-Lagrange equations, the Lorentz force is recovered:

$$\frac{d}{d\tau}\frac{\partial L}{\partial u_\mu} - \frac{\partial L}{\partial x_\mu} = 0 \quad \Rightarrow \quad m\frac{du^\mu}{d\tau} = q\left[\partial^\mu (u_\nu A^\nu) + \frac{dA^\mu}{d\tau}\right]$$
$$\Rightarrow \quad \mathbf{F} = e\left[-\nabla\Phi + \nabla(\mathbf{v}\cdot\mathbf{A}) - \frac{d\mathbf{A}}{dt}\right] \tag{23}$$

where the last step takes the space part of the force 4-vector.

As another example of the use of the relativistic Lagrangian, it is very straight forward to use it to derive the canonical momentum:

$$p^\mu = \frac{\partial L}{\partial u_\mu} = m u^\mu + q A^\mu \tag{24}$$

6 The Dirac Lagrangian and the Electromagnetic Current

Many properties of the Dirac equation are easily arrived at by starting with the Dirac Lagrangian. Since the Dirac equation is given as a wave-equation, in the language of Lagrangians it is represented as a field equation, where the solutions ψ represent the fields. Also, instead of use the Lagrangian, the Lagrangian density is used. In this case the Euler-Lagrange equations are given in term of derivatives that involve the fields:

$$\frac{\partial L}{\partial \psi} - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu \psi)}\right) = 0 \tag{25}$$

The Lagrangian density will be used to show a couple of properties of the Dirac equation.

Up to this point the conserved current has not been identified with any physical quantity, as in the case of the Schrödinger or Klein-Gordon equations. Recall, in the case of the Schrödinger equation, the current density is given a probabilistic interpretation, while in the case of the Klein-Gordon equation, to avoid negative probabilities, it is interpreted as a charge current. To determine what j^μ corresponds to physically, it is best to start with the Lagrangian for the Dirac equation:

$$\mathcal{L} = \bar{\psi}\left[i\gamma^\mu \partial_\mu - m\right]\psi \tag{26}$$

the Dirac equation can be retrieved by applying the Euler-Lagrange equations. Writing the Lagrangian in gauge invariant form leads to:

$$\mathcal{L} = \bar{\psi}\left[i\gamma^\mu \partial_\mu + iq\gamma^\mu A_\mu - m\right]\psi = \bar{\psi}\left[i\gamma^\mu \partial_\mu - m\right]\psi - q\bar{\psi}\gamma^\mu\psi A_\mu \tag{27}$$

where the conserved current which found earlier is $j^\mu = \bar{\psi}\gamma^\mu\psi$. Therefore the last term can be written in the suggestive form:

$$q\bar{\psi}\gamma^\mu\psi A_\mu = q j^\mu A_\mu \tag{28}$$

This last equation leads to the natural interpretation of the current being associated with a charged current. Therefore the current is redefined as $j^\mu = q\bar{\psi}\gamma^\mu\psi$

Another property that is easily arrived at through the Lagrangian, is the that the photon must be massless. This comes about by imposing the condition on the Lagrangian that it must be invariant to gauge transformations. This is best seen by using the Lorentz covariant formulation of electrodynamics. Recall that the field strength tensor (this is composed of all the components of the electric and magnetic fields) is related to the vector potential as follows:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{29}$$

The Lagrangian that defines Maxwell's equations is given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{30}$$

(to prove that this is the correct Lagrangian, use the Euler-Lagrange equation to show that the Maxwell equations are recovered). This equation is both Lorentz invariant and gauge invariant. To add a mass term, recall that in the Dirac Lagrangian this corresponds the self-interacting field ($\bar{\psi}\psi$). In the case of this equation, the field is the vector potential, so a self-interacting term that is also Lorentz invariant gives a Lagrangian of the form:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A^\mu A_\mu \tag{31}$$

The first term in the Lagrangian is gauge invariant, since it leads to Maxwell' equations (also very straight forward to show). On the other hand, the mass term changes under this transformation

$$A^\mu \to A^\mu + \partial^\mu\chi \quad \Rightarrow \quad A^\mu A_\mu \to A^\mu A_\mu + 2A^\mu\partial_\mu\chi + (\partial^\mu\chi)(\partial_\mu\chi) \tag{32}$$

clearly not gauge invariant unless $m = 0$.

7 Non-Relativistic Limit of Dirac Equation

To get the non-relativistic limit of the Dirac equation, start by expanding it out into a set of two matrix equations:

$$\begin{aligned} \sigma\cdot(\mathbf{p}+q\mathbf{A})u_B &= (E - qA^0 - m)u_A \\ \sigma\cdot(\mathbf{p}+q\mathbf{A})u_A &= (E + qA^0 + m)u_B \approx 2mu_B \end{aligned} \tag{33}$$

where the approximation is due to the momentum being small ($E \approx m$). The second equation above solves for u_B, which is then substituted into the first equation:

$$\frac{1}{2m}\sigma\cdot(\mathbf{p}+q\mathbf{A})\sigma\cdot(\mathbf{p}+q\mathbf{A}) = (E + qA^0 - m)u_A \tag{34}$$

Next use the identity

$$(\sigma\cdot\mathbf{A})(\sigma\cdot\mathbf{B}) = \mathbf{A}\cdot\mathbf{B} + i\sigma(\mathbf{A}\times\mathbf{B}) \tag{35}$$

to simplify the equation 34:

$$\frac{1}{2m}\left[(\mathbf{p}+q\mathbf{A})^2 + i\sigma\cdot(\mathbf{p}+q\mathbf{A})\times(\mathbf{p}+q\mathbf{A})\right] = (E+qA^0-m)u_A \tag{36}$$

The next step is to make the substitution $\mathbf{p}\to -i\nabla$, which after working through the cross product gives:

$$\frac{1}{2m}\left[(\mathbf{p}+q\mathbf{A})^2 + q\sigma\cdot(\nabla\times\mathbf{A})\right] = (E+qA^0-m)u_A \tag{37}$$

Finally, recall that $\mathbf{B}=\nabla\times\mathbf{A}$ and $\mathbf{S}=1/2\sigma$. This leads to the final result:

$$\left[\frac{(\mathbf{p}+q\mathbf{A})^2}{2m} + \frac{q}{m}\mathbf{S}\cdot\mathbf{B} - qA^0\right]u_A = (E-m)u_A \tag{38}$$

In the last equation, the magnetic momentum is identified as $\mu_B = e/m$, where the charge of an electron is taken ($q=e$). Typically in quantum mechanics, the magnetic moment is given as $\mu_B = ge/2m$ where g, the gyromagnetic ratio is given by experiment as $g=2$. In the case of the Dirac equation, it automatically appears being a consequence of special relativity.