Physics 4213/5213
Lecture 5

1 Introduction

The purpose of this lecture is to derive a specific form for the Dirac spinors (since these are column vectors with four elements, the are actually referred to as bi-spinors, but for this course they will be referred to as spinors). The spinors will first be derived in the rest frame of the particle, and by applying a Lorentz transformation, the general form of the spinor will be determined. The same approach will be used to calculate the space-time portion of the wave-function (field).

Once the spinors are given, there properties will be examined. This includes orthogonality relations, normalization, interpretation as particle and anti-particle states.

2 Solutions to the Dirac Equation

As stated in the introduction, the solutions to the Dirac equation will be found first in the rest frame of the particle, and then transformed to an arbitrary frame. Since the Dirac equation is a 4×4 matrix equation that can be written in either a 4-momentum or 4-coordinate representation:

$$(i\gamma^\mu \partial_\mu + m)\psi \quad \text{or} \quad (\not{p} - m)\psi = 0 \tag{1}$$

The solutions are expected to be 4-element column vectors, times a space-time function, which determines its mode of propagation. The reason to expect a column vector has to do with the interpretation of the conserved 4-current. This current was found to transform as a 4-vector in the previous lecture, leading the 4-current be a collection of four numbers. The only way to do so is if the solutions are column vectors:

$$j^\mu = \bar{\psi}\gamma^\mu\psi \tag{2}$$

2.1 Solution in the Rest Frame

The solutions in the rest frame are calculated by writing the matrix equation explicitly:

$$(E - m)\psi = (i\gamma^0\partial_0 - m)\psi = 0 \quad \Rightarrow \quad \begin{pmatrix} i\partial_0 - m & 0 & 0 & 0 \\ 0 & i\partial_0 - m & 0 & 0 \\ 0 & 0 & -i\partial_0 - m & 0 \\ 0 & 0 & 0 & -i\partial_0 - m \end{pmatrix}\psi = 0 \tag{3}$$

This equation leads to the solutions:

$$\psi_1 = N\begin{pmatrix}1\\0\\0\\0\end{pmatrix}e^{-imt} \quad \psi_2 = N\begin{pmatrix}0\\1\\0\\0\end{pmatrix}e^{-imt} \quad \psi_3 = N\begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{imt} \quad \psi_4 = N\begin{pmatrix}0\\0\\0\\1\end{pmatrix}e^{imt} \tag{4}$$

where N is an overall normalization factor yet to be determined. Notice that the first two equations have energy eigen-values that are $+m$ (positive energy), while the second two solutions have energy eigen-values that are $-m$ (negative energy). Therefore as in the Klein-Gordon equation, there are both positive and negative energy solutions.

Before proceeding to find the transformation equation between frames, the spin of the spinors above is examined. In the previous lecture, the infinitesimal transformation between frames was given as:

$$S(\Lambda) = 1 + \frac{i}{4}\sigma^{\mu\nu}\Delta\lambda_{\mu\nu} \tag{5}$$

The σ^{0i} and σ^{i0} were found to be associated with the boosts, but the σ^{ij} were found to be associated with rotation. These operators are referred to as the generators of the group of transformations, in this case three are associated with boosts and three with rotations. The three that are associated with rotations are the angular momentum operators, with σ^{12} being the angular momentum operator along the 3-axis (Σ_3), etc. This particular operator can be written as a 4×4 matrix in terms of the Pauli spin matrices:

$$\Sigma_3 = \frac{i}{2}\left(\gamma^1\gamma^2 + \gamma^2\gamma^1\right) = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6}$$

Applying this matrix on the four spinors of equation 4 gives

$$\Sigma_3\psi_1 = +1\psi_1 \quad \Sigma_3\psi_2 = -1\psi_2 \quad \Sigma_3\psi_3 = +1\psi_3 \quad \Sigma_3\psi_4 = -1\psi_4 \tag{7}$$

which is equivalent to two states having their spins up and two down. Note that the Σ_3 operator must be multiplied by 1/2 to get the correct spins.

2.2 Solution in an Arbitrary Frame

To calculate the solutions in an arbitrary frame, the Lorentz transformation ($S(\Lambda)$) has to be applied to the equation. But before doing so, the general form of the argument of the exponential is determined. This is done by noting that in the rest frame it is given by the product of the rest mass times the proper time. Both these quantities are invariants, making their product an invariant. The natural choice in all frames, is to take the invariant quantity $p^\mu x_\mu$ as the phase of the exponential term—*this reduces to the correct form in the rest frame.* Therefore the exponential terms in an arbitrary frame are:

$$e^{-\epsilon_r p^\mu x_\mu} \quad \text{where} \quad \epsilon_r = \begin{cases} +1 & r = 1,2 \\ -1 & r = 3,4 \end{cases} \tag{8}$$

Now comes the difficult part, the calculation of $S(\Lambda)$ for a transformation of the spinor to a frame moving in an arbitrary direction. Recall that the transformation is given by:

$$S(\Lambda) = e^{-i/4\lambda\sigma_{\mu\nu}I_n^{\mu\nu}} \tag{9}$$

The transformation will be from the rest frame to a frame moving with a velocity $-\vec{v}$ (frame where the observer sees the particle moving with velocity $+\vec{v}$).

The first step is to determine the components of the unit rotation matrix I_n. Recall that the values with an index of 0 correspond to boosts while the others correspond to rotations. The velocity 3-vector written in terms of the direction cosines is:

$$\vec{v} = v\vec{n} = v(\cos\alpha, \cos\beta, \cos\gamma) \tag{10}$$

Recalling that in the case of an infinitesimal boost along the negative x-axis, $\Delta\lambda^{\mu}{}_{\nu}$ was given by:

$$\Delta\beta \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{11}$$

In that case, $\Delta\beta$ was both the velocity and the boost parameter (rapidity), extending this to the finite case makes $\lambda^{\mu}{}_{\nu}$ equal to:

$$\lambda^{\mu}{}_{\nu} = \lambda \begin{pmatrix} 0 & -\cos\alpha & -\cos\beta & -\cos\gamma \\ -\cos\alpha & 0 & 0 & 0 \\ -\cos\beta & 0 & 0 & 0 \\ -\cos\gamma & 0 & 0 & 0 \end{pmatrix} \tag{12}$$

Note that for an infinitesimal transformation of the coordinates, this gives:

$$\begin{aligned} x'^0 &= x^0 - x^1\Delta\beta\cos\alpha - x^2\Delta\beta\cos\beta - x^3\Delta\beta\cos\gamma \\ x'^1 &= -x^0\cos\alpha - x^1 \\ x'^2 &= -x^0\cos\beta - x^2 \\ x'^3 &= -x^0\cos\gamma - x^3 \end{aligned} \tag{13}$$

as would be expected in the non-relativistic case. In fact the correct way to do this, is to start with the infinitesimal transformation, to derive the unit rotation matrix.

Now expand the two tensor terms in the exponent, using the relation $I^{\mu\nu} = g^{\mu\alpha} I^{\mu}_{\alpha} \Rightarrow I^{\mu\nu} = -I^{\mu}{}_{\nu}$:

$$\sigma_{\mu\nu} I_n^{\mu\nu} = 2\sigma_{0\nu} I_n^{0\nu} = 2(\sigma_{01}\cos\alpha + \sigma_{02}\cos\beta + \sigma_{03}\cos\gamma) = -2i\vec{\alpha}\cdot\frac{\vec{v}}{v} \tag{14}$$

where use is made of the following:

$$\sigma_{0i} = \frac{i}{2}[\gamma_0, \gamma_i] = i\gamma_0\gamma_i = i\gamma_0 g^{00}\gamma_i g^{ii} = -i\gamma^0\gamma^0\vec{\alpha} = -i\vec{\alpha} \tag{15}$$

The next step is to expand out the exponential. Before doing so, notice that dot product $(\vec{\alpha}\cdot\vec{v})^2$ can be simplified significantly:

$$(\vec{\alpha}\cdot\vec{v})^2 = \alpha^i\alpha^j v_i v_j = \gamma^0\gamma^i\gamma^0\gamma^j v_i v_j = -\gamma^i\gamma^j v_i v_j = -\frac{1}{2}(\gamma^i\gamma^j + \gamma^j\gamma^i)v_i v_j = -\frac{1}{2}(2g^{ij}v_i v_j) \tag{16}$$

where the third equality comes from the v_i's commuting with each other and the fact that the i and j indices are being summed over. Now expand the exponential transform separating out even and odd powers in $\vec{\alpha}\cdot\vec{v}$:

$$S(-\vec{v}) = \left[1 + \frac{1}{2!}\left(\frac{\lambda}{2}\right)^2\left(\frac{\vec{v}}{v}\right)^2 + \frac{1}{4!}\left(\frac{\lambda}{2}\right)^4\left(\frac{\vec{v}}{v}\right)^4 + \cdots\right] - \frac{\vec{\alpha}\cdot\vec{v}}{v}\left[\left(\frac{\lambda}{2}\right) + \frac{1}{3!}\left(\frac{\lambda}{2}\right)^3\left(\frac{\vec{v}}{v}\right)^2 + \cdots\right] \tag{17}$$

which is simplified as:

$$S(-\vec{v}) = \cosh\frac{\lambda}{2} - \frac{\vec{\alpha}\cdot\vec{v}}{v}\sinh\frac{\lambda}{2} \tag{18}$$

The hyperbolic functions can be written in terms of the kinematic variables using the following relations:

$$\cosh\lambda = \gamma \qquad \sinh\lambda = \gamma\beta \tag{19}$$

$$2\cosh\frac{\lambda}{2}\sinh\frac{\lambda}{2} = \sinh\lambda \qquad 2\cosh^2\frac{\lambda}{2} = \cosh\lambda + 1$$

with

$$\gamma = \frac{E}{m} \qquad \gamma\beta = \frac{p}{m} \tag{20}$$

From these expressions it follows that:

$$\cosh\frac{\lambda}{2} = \sqrt{\frac{E+m}{2m}} \qquad \sinh\frac{\lambda}{2} = \sqrt{\frac{E-m}{2m}} = \sqrt{\frac{E+m}{2m}}\sqrt{\frac{E-m}{E+m}} = \sqrt{\frac{E+m}{2m}}\frac{p}{E+m} \tag{21}$$

where the last equality uses the relation $E^2 - m^2 = p^2$ and is given as it simplifies the final result. Noting that $v_i/v = p_i/p$, that $\vec{\alpha} = \gamma^0\gamma^i$, and that there is a unit matrix multiplying the cosh term in equation 18, the transformation is:

$$S(-\vec{v}) = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} 1 & 0 & \frac{p_z}{E+m} & \frac{p_x-ip_y}{E+m} \\ 0 & 1 & \frac{p_x+ip_y}{E+m} & \frac{-p_z}{E+m} \\ \frac{-p_z}{E+m} & \frac{p_x-ip_y}{E+m} & 1 & 0 \\ \frac{p_x+ip_y}{E+m} & \frac{-p_z}{E+m} & 0 & 1 \end{pmatrix} \tag{22}$$

With the transformation finally determined in matrix form, it can be applied to the rest frame solutions given in equation 4. This leads to the spinors being the columns of the transformation (see eq. 22):

$$u^{(1)} = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} 1 \\ 0 \\ \frac{-p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \qquad u^{(2)} = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \tag{23}$$

$$v^{(2)} = u^{(3)} = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix} \qquad v^{(1)} = u^{(4)} = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}$$

3 Particles and Anti-particles

Before looking into the properties of the spinors, lets consider the four solutions given above. First expand the Dirac equation into matrix form:

$$\begin{aligned}(\not p - m)\psi &= (\gamma^0 p_0 - \vec{\gamma}\cdot\mathbf{p} - m)\psi \\ &= \begin{pmatrix} E-m & 0 & -p_z & -(p_x - ip_y) \\ 0 & E-m & -(p_x+ip_y) & p_z \\ p_z & -(p_x-ip_y) & -(E-m) & 0 \\ p_x+ip_y & -p_z & 0 & -(E-m) \end{pmatrix}\psi = 0 \end{aligned} \quad (24)$$

Multiplying the matrix with $u^{(1)}$ and $u^{(2)}$ gives zero, but applying the matrix to the states $u^{(3)}$ and $u^{(4)}$ does not give zero. As an example, the case of the matrix being applied to $u^{(3)}$ gives:

$$\sqrt{\frac{E+m}{2m}}\begin{pmatrix} (\frac{E-m}{E+m}-1)p_z \\ (\frac{E-m}{E+m}-1)(p_x+ip_y) \\ 0 \\ 0 \end{pmatrix} \quad (25)$$

If the minus sign in the numerator of the first two terms can be converted to a plus sign, then the result would be zero. But in fact this can be done, since the fact that the energy is negative for these solutions has not been taken into account. But inverting the sign on the energy will not solve this alone, the direction of time (momentum) must also be inverted giving:

$$(\not p + m)u^{(2,3)} = 0 \quad \Rightarrow \quad (\not p + \epsilon_r m)u^{(r)} = 0 \qquad \epsilon_r \left\{ \begin{array}{ll} -1 & r = 1,2 \\ +1 & r = 3,4 \end{array} \right. \quad (26)$$

From this, the particle states are $r = 1,2$, while the anti-particle states are $r = 3,4$.

4 Properties of the Spinors

To use the spinors in later calculations, the quantities that occur over and over again in cross section calculations will be determined in this section. Before proceeding, the normalization of the spinors has to be fixed. The normalization is given in terms of the charge density (*the 4-current, as in the Klein-Gordon case, is treated as a charge current density*), which is:

$$j^0 = \bar{\psi}\gamma^0\psi = u^{r\dagger}u^{r'} = |N|^2\frac{E}{m}\delta_{rr'} \quad (27)$$

If $|N|^2 = 1$ then problems can occur later when scattering of a massless particle needs to be calculated—note that this normalization is used often but with extreme care. Before selecting the normalization, recall that in the case of the Klein-Gordon equation, the normalization was given by $\rho = j^0 = 2E$ with the energy dependence making the charge frame independent—the same E appears for the Dirac equation. A more natural normalization, without the mass problem, is to select N such that the normalization is the same as for the Klein-Gordon equation:

$$u^{r\dagger}u^{r'} = 2E \quad \Rightarrow \quad |N|^2 = 2m \quad (28)$$

Notice that this corresponds to $2E$ particles per unit volume.

The next relation $u^{r\dagger}u^r = \epsilon_r 2m$, can be done brute force using the spinors with the normalization given above. An alternate and more elegant method is as follows. Start with the Dirac equation and its Hermitian conjugate:

$$\left.\begin{array}{r} (\not p - \epsilon_r m)u^r = 0 \\ \bar{u}^r(\not p - \epsilon_r m) = 0 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} \bar{u}^r\gamma^0(\not p - \epsilon_r m)u^r = 0 \\ \bar{u}^r(\not p - \epsilon_r m)\gamma^0 u^r = 0 \end{array}\right. \tag{29}$$

Now add the two equations together keeping in mind that $\gamma^0\gamma^i = -\gamma^i\gamma^0$:

$$2\bar{u}^r E u^r - \epsilon_r 2m u^{r\dagger}u^r = 0 \quad \Rightarrow \quad \bar{u}^r u^r = \epsilon_r \frac{m}{E} u^{r\dagger}u^r = \epsilon_r 2m \tag{30}$$

where equation 27 was used.

Finally, the last set of relations are derived by brute force. The algebra will not be shown, but simply start with the spinors, and multiply them out:

$$\sum_{r=1}^{2} u^r\bar{u}^r = \not p + m \qquad \sum_{r=1}^{2} v^r\bar{v}^r = \not p - m \tag{31}$$