Physics 4213/5213
Lecture 4

1 Introduction

This lecture takes a look at the Lorentz properties of the Dirac equation and its solutions. First the Dirac equation is written in a form that puts space and time coordinates on an equal footing, this was in fact done in the last lecture, but is repeated here for completeness. Then, from the requirement that the Dirac equation be covariant—same form in all frames—the transformation properties of the solutions to the Dirac equation are derived. The transformations between Lorentz frames will be used in the next lecture to determine the solutions in all frames.

2 Covariance of the Dirac Equation

The principle of relativity requires that the laws of physics remain the same in all Lorentz frames. Given that the laws of physics are typically given as equations, this means that the equations of physics must have the same form in all Lorentz frames. This is referred to as the covariance of a particular equation. To determine if the Dirac equation is covariant or not—satisfies the principle of relativity—the equation has to be put into a form that puts the space and time coordinates on an equal footing.

As a reminder, the Dirac equation as specified in the previous lecture is

$$\left[i\frac{\partial}{\partial t} + i\vec{\alpha}\cdot\vec{\nabla} - \beta m\right]\psi = 0. \tag{1}$$

This equation clearly does not put the space and time coordinates on an equal footing; the space and time derivatives should be dotted into a vector that has the various matrices as its components. To put all the coordinates on an equal footing, requires multiplying the Dirac equation on the left by β. This then gives

$$[i\gamma^\mu\partial_\mu - m]\,\psi = 0 \tag{2}$$

where $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha^i$. Notice that the γ matrices still satisfy the anti-commutation relations that the α and β matrices did

$$[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu] = \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{3}$$

Further γ^0 is both unitary, $\gamma^{0\dagger} = \left(\gamma^0\right)^{-1}$, and Hermitian, $\gamma^0 = \gamma^{0\dagger}$, while γ^i is unitary but anti-Hermitian $\gamma^i = -\gamma^{i\dagger}$ as compared to the α_i matrices that are both unitary and Hermitian.

Using the γ matrices, the Dirac equation can be written in a vary compact form using the transformation $\gamma^\mu p_\mu \rightarrow i\gamma^\mu\partial_\mu$ and the definition $\not{p} = \gamma^\mu p_\mu$

$$[\not{p} - m]\,\psi = 0. \tag{4}$$

2.1 Condition for Covariance

The Dirac equation is now in a form that puts the space and time coordinates on an equal footing. The next step is to show the covariance of this equation. In fact what will be done, is to impose the covariance condition on the equation, then use this to determine the Lorentz transformation of the solutions—note that the solutions are not in the form of a 4-vector, or combinations of 4-vectors, and therefore do not transform under the usual transformation equation $x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$. None the less, the transformation will have to be related to the well known Lorentz transformation for boosts between frames. Further, it will be shown that there are two types of transformations, those that can be arrived at by infinitesimal transformations from the identity element ($\det[\Lambda] = +1$) and discrete transformation (space and time reflections $\det[\Lambda] = -1$). This lecture will concentrate on the continuous transformations.

As a reminder, the condition for covariance is that the equation have the same form in any Lorentz frame, with the equation acting on the appropriately transformed wave function;

$$[i\gamma^{\mu}\partial_{\mu} - m]\,\psi(x) = 0 \Longrightarrow \left[i\gamma'^{\mu}\partial'_{\mu} - m\right]\psi'(x') = 0. \tag{5}$$

A rather long algebraic proof exists to show that $\gamma'_{\mu} = \gamma_{\mu}$ (see ref. [1]), but by the principle of relativity it can be argued that this must be the case, otherwise it would be possible to distinguish different Lorentz frames. This being strictly forbidden, since all frames must be equivalent. Therefore the covariance of the Dirac equation is proven, if a transformation can be found to transform the wave function from one frame to another; $\psi(x) \longrightarrow \psi'(x')$.

The transformation from one frame to another must be linear, since both the Dirac equation and the Lorentz transformations are linear. This implies the condition

$$\psi'(x') = \psi'(\Lambda x) = \mathrm{S}(\Lambda)\psi(x) = \mathrm{S}(\Lambda)\psi(\Lambda^{-1}x'). \tag{6}$$

where Λ refers to the matrix of the Lorentz transformation and $\mathrm{S}(\Lambda)$ is 4×4 matrix that operates on the wave function $\psi(x)$. The transformation $\mathrm{S}(\Lambda)$ depends on the relative velocity of the two frames and relative orientation through its dependence on Λ. Further the principle of relativity requires that the inverse transformation $\mathrm{S}^{-1}(\Lambda)$ exist. This transformation has the properties

$$\psi(x) = \mathrm{S}^{-1}(\Lambda)\psi'(x') = \mathrm{S}^{-1}(\Lambda)\psi'(\Lambda x) \tag{7}$$

Equation 6 could also have been written as

$$\psi(x) = \mathrm{S}(\Lambda^{-1})\psi'(x') = \mathrm{S}(\Lambda^{-1})\psi(\Lambda x), \tag{8}$$

which implies that $\mathrm{S}^{-1}(\Lambda) = \mathrm{S}(\Lambda^{-1})$. This last result should have been expected, since it simply states that the inverse transform is the transform itself as a function of the inverse Lorentz transformation.

The transformation $\mathrm{S}(\Lambda)$ will now be used to transform the Dirac equation from one frame to another. Start with the Dirac equation and then write the wave function in terms of the primed coordinates

$$[i\gamma^{\mu}\partial_{\mu} - m]\,\psi(x) = [i\gamma^{\mu}\partial_{\mu} - m]\,\mathrm{S}^{-1}(\Lambda)\psi'(x') = 0. \tag{9}$$

Next multiply equation 9 from the left by $\mathrm{S}(\Lambda)$

$$\mathrm{S}(\Lambda)\,[i\gamma^{\mu}\partial_{\nu} - m]\,\mathrm{S}^{-1}(\Lambda)\psi'(x') = \left[i\mathrm{S}(\Lambda)\gamma^{\mu}\mathrm{S}^{-1}(\Lambda)\partial_{\mu} - m\right]\psi'(x') = 0. \tag{10}$$

Finally express the derivative in terms of the primed coordinates

$$\left[i\mathrm{S}(\Lambda)\gamma^{\mu}\mathrm{S}^{-1}(\Lambda)\Lambda^{\nu}_{\mu}\partial'_{\nu} - m\right]\psi'(x') = 0. \tag{11}$$

Given the earlier condition that the γ^{μ} do not change on being transformed to another frame, the following condition is imposed

$$\mathrm{S}(\Lambda)\gamma^{\mu}\mathrm{S}^{-1}(\Lambda)\Lambda^{\nu}_{\mu} = \gamma^{\nu} \quad \Rightarrow \quad \mathrm{S}(\Lambda)\gamma^{\mu}\mathrm{S}^{-1}(\Lambda) = \Lambda^{\mu}_{\nu}\gamma^{\nu} \tag{12}$$

which will be used to determine $\mathrm{S}(\Lambda)$.

2.2 Determination of $\mathrm{S}(\Lambda)$

Equation 12 will now be used to determine $\mathrm{S}(\Lambda)$. This will be done by first calculating the infinitesimal transformation and then repeatedly applying the transformation. The infinitesimal Lorentz transformation is given by

$$\Lambda^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \Delta\lambda^{\mu}_{\nu} \tag{13}$$

where the condition $\Lambda\Lambda^{-1} = 1 \Longrightarrow \Lambda^{\mu}_{\nu}\Lambda^{\nu}_{\sigma} = \delta^{\mu}_{\sigma}$, which further implies that $\Delta\lambda^{\mu\nu} = -\Delta\lambda^{\nu\mu}$ neglecting quadratic terms. This immediately imposes the condition that there are only six independent non-vanishing parameters $\Delta\lambda^{\mu\nu}$—this comes about, since those terms where the μ and ν indices are the same have to be zero, leaving only six independent parameters. This should make sense from the point of view that there are three boost directions and three rotation angles (the Euler angles).

Now expanding out $\mathrm{S}(\Delta\lambda^{\mu\nu})$ to first order gives

$$\begin{aligned}
&\mathrm{S}(\Delta\lambda^{\mu\nu}) = 1 - \frac{i}{4}\sigma_{\mu\nu}\Delta\lambda^{\mu\nu} \qquad (14)\\
&\mathrm{S}^{-1}(\Delta\lambda^{\mu\nu}) = \mathrm{S}(\Delta\lambda^{\nu\mu}) = 1 + \frac{i}{4}\sigma_{\mu\nu}\Delta\lambda^{\mu\nu}\\
&\Longrightarrow \sigma_{\mu\nu} = -\sigma_{\nu\mu}
\end{aligned}$$

where the form of the last term in the first equation is due to the transformation being in the form of a Lorentz scalar, the second equation uses the antisymmetric property of $\Delta\lambda^{\mu\nu}$, and the last equation comes from the first equality of the second equation. This imposes the condition of six independent non-zero $\sigma_{\mu\nu}$ as would be expected from the previous condition on the $\Delta\lambda$. Now apply the infinitesimal transformations to equation 12 giving

$$\left(\delta^{\nu}_{\mu} + \Delta\lambda^{\nu}_{\mu}\right)\gamma^{\mu} = \left(1 - \frac{i}{4}\sigma_{\alpha\beta}\Delta\lambda^{\alpha\beta}\right)\gamma^{\nu}\left(1 + \frac{i}{4}\sigma_{\alpha\beta}\Delta\lambda^{\alpha\beta}\right) \tag{15}$$

$$\Delta\lambda^{\nu}_{\mu}\gamma^{\mu} = -\frac{i}{4}\Delta\lambda^{\alpha\beta}\left(\sigma_{\alpha\beta}\gamma^{\nu} - \gamma^{\nu}\sigma_{\alpha\beta}\right) \tag{16}$$

where equation 16 comes from expanding equation 15 and ignoring the quadratic terms, and the $\delta^{\nu}_{\mu}\gamma^{\mu} = \gamma^{\nu}$ cancels with a like term on the right hand side. The left hand side of equation 16 can be written as

$$\begin{aligned}
-\frac{i}{4}\Delta\lambda^{\alpha\beta}\left(\sigma_{\alpha\beta}\gamma^{\nu}-\gamma^{\nu}\sigma_{\alpha\beta}\right) &= \Delta\lambda^{\nu}_{\mu}\gamma^{\mu} \\
&= \Delta\lambda^{\alpha}_{\mu}g^{\nu}_{\alpha}\gamma^{\mu} = \Delta\lambda^{\alpha}_{\beta}g^{\nu}_{\alpha}\gamma^{\beta} = \Delta\lambda^{\beta\alpha}\left(g^{\nu}_{\alpha}\gamma_{\beta}\right) \\
&= \frac{1}{2}\Delta\lambda^{\beta\alpha}\left(g^{\nu}_{\alpha}\gamma_{\beta}-g^{\nu}_{\beta}\gamma_{\alpha}\right) \\
&= -\frac{1}{2}\Delta\lambda^{\alpha\beta}\left(g^{\nu}_{\alpha}\gamma_{\beta}-g^{\nu}_{\beta}\gamma_{\alpha}\right) \qquad (17)
\end{aligned}$$

where the relation $\Delta\lambda^{\beta\alpha} = -\Delta\lambda^{\alpha\beta}$ was used above. Equation 17, finally ends up giving the relation

$$\left(\sigma_{\alpha\beta}\gamma^{\nu}-\gamma^{\nu}\sigma_{\alpha\beta}\right) = 2i\left(g^{\nu}_{\alpha}\gamma_{\beta}-g^{\nu}_{\beta}\gamma_{\alpha}\right). \qquad (18)$$

The problem of constructing S according to the relation 12, has been reduced to the problem of finding the six $\sigma_{\alpha\beta}$ that satisfy equation 18. Given that $\sigma_{\alpha\beta}$ is anti-symmetric in both its indices, an obvious choice is that $\sigma_{\alpha\beta} = (i/2)\left[\gamma_{\alpha},\gamma_{\beta}\right]$; this also makes sense from the form of equation 18. (The antisymmetry of $[\gamma^{\mu},\gamma^{\nu}]$ is easily shown using the anti-commuting property of the gamma matrices.) This can be substituted into equation 18 and found to satisfy the relation. Therefore, the infinitesimal transformation for S is known.

Before going further, the significance of $\Delta\lambda$ will be determined, since this will be needed later. Start by taking $\Delta\lambda^{10} = -\Delta\lambda^{01} = -\Delta\beta \neq 0$ with all other $\Delta\lambda^{\mu\nu} = 0$. Applying the infinitesimal Lorentz transformation to the coordinates gives

$$\begin{aligned}
\left(x'\right)^{\nu} &= \Lambda^{\nu}_{\mu}x^{\mu} = \left(\delta^{\nu}_{\mu}+\Delta\lambda^{\nu}_{\mu}\right)x^{\mu} \\
&= \left(\delta^{\nu}_{\mu}+\Delta\lambda^{0}{}_{1}\delta^{\nu}_{0}\delta^{1}_{\mu}+\Delta\lambda^{1}{}_{0}\delta^{\nu}_{1}\delta^{0}_{\mu}\right)x^{\mu} \qquad (19)
\end{aligned}$$

where the second equation comes from expanding out $\Delta\lambda^{\nu}_{\mu}$ in a sum of the non-zero terms. The $\Delta\lambda$ in these transformation equations are related to $\Delta\beta$ by

$$\begin{aligned}
\Delta\lambda^{0}{}_{1} &= g_{1\alpha}\Delta\lambda^{0\alpha} = g_{11}\Delta\lambda^{01} = -\Delta\lambda^{01} = -\Delta\beta \\
\Delta\lambda^{1}{}_{0} &= g_{0\alpha}\Delta\lambda^{1\alpha} = g_{00}\Delta\lambda^{10} = \Delta\lambda^{10} = -\Delta\beta \\
\Delta\lambda^{1}{}_{0} &= \Delta\lambda^{0}{}_{1} = -\Delta\beta \qquad (20)
\end{aligned}$$

Substituting equation 20 into equation 19 gives

$$(x')^{0} = x^{0} - \Delta\beta x^{1} \qquad (21)$$

$$(x')^{1} = x^{1} - \Delta\beta x^{0} \qquad (22)$$

$$(x')^{2} = x^{2} \qquad (23)$$

$$(x')^{3} = x^{3} \qquad (24)$$

which are just the infinitesimal Lorentz transformation for the primed frame moving along the positive x^1 direction. An obvious way of showing this is to sit at the origin of the primed frame. This imposes the requirement that $x^1/x^0 = \Delta\beta$, which is just the velocity of the primed frame relative to the unprimed frame. One final comment on the identification with the Lorentz transformations, if instead of selecting $\Delta\lambda^{10} \neq 0$, $\Delta\lambda^{20}$ or $\Delta\lambda^{30}$ are select not equal to zero, then the Lorentz transformation along the x^2 or x^3 directions respectively would have been arrived at.

Now select $\Delta\lambda^{12} = -\Delta\lambda^{21} = -\Delta\phi$ and all other $\Delta\lambda^{\mu\nu} = 0$. Then the transformation is given by

$$(x')^{\nu} = \left[\delta^{\nu}_{\mu} + \delta^{\nu}_{1}\delta^{2}_{\nu}\Delta\phi - \delta^{\nu}_{2}\delta^{1}_{\nu}\Delta\phi\right] x^{\mu} \tag{25}$$

where the $\Delta\lambda$ in these transformation equations are related to $\Delta\phi$ by

$$\begin{aligned}
\Delta\lambda^{1}{}_{2} &= g_{2\alpha}\Delta\lambda^{1\alpha} = g_{22}\Delta\lambda^{12} = -\Delta\lambda^{12} = -\Delta\phi \\
\Delta\lambda^{2}{}_{1} &= g_{1\alpha}\Delta\lambda^{2\alpha} = g_{11}\Delta\lambda^{21} = -\Delta\lambda^{21} = \Delta\phi \\
\Delta\lambda^{1}{}_{0} &= -\Delta\lambda^{0}{}_{1} = -\Delta\phi
\end{aligned} \tag{26}$$

This written out explicitly is

$$(x')^0 = x^0 \tag{27}$$
$$(x')^1 = x^1 + \Delta\phi x^2 \tag{28}$$
$$(x')^2 = x^2 - \Delta\phi x^1 \tag{29}$$
$$(x')^3 = x^3, \tag{30}$$

which amounts to a rotation about the x^3 axis. Likewise, if $\Delta\lambda^{13}$ or $\Delta\lambda^{23}$ are selected as the only non-zero parameters, the transformation arrived at would be a rotation about the x^2 or x^1 axis respectively.

Returning to the derivation of S, start by writing

$$\Delta\lambda^{\nu}{}_{\mu} = \Delta\lambda\,(I_{\mathrm{n}})^{\nu}{}_{\mu} \tag{31}$$

where $(I_{\mathrm{n}})^{\nu}{}_{\mu}$ corresponds to a unit generalized rotation matrix about the axis n and the identification $\Delta\lambda = \Delta\beta$ is made; this is similar to a unit vector. For a boost along the x axis, the matrix is given by (see eq. 19):

$$(I_x)^{\nu}{}_{\mu} = -\left(\delta^{\nu}_{1}\delta^{0}_{\mu} + \delta^{\nu}_{0}\delta^{1}_{\mu}\right) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{32}$$

With this definition for $(I_{\mathrm{n}})^{\nu}{}_{\mu}$ the finite Lorentz transformation is given by

$$(x')^{\nu} = \lim_{N\to\infty} \left[\left(1 + \frac{\lambda}{N} I_x\right)^{N}\right]^{\nu}_{\ \mu} x^{\mu} = \left(e^{\lambda I_x}\right)^{\nu}_{\ \mu} x^{\mu} \tag{33}$$

To write the transformation in matrix form, start by expanding out the exponential in a Taylor series:

$$e^{\lambda I_x} = \sum_{i=0}^{\infty} \frac{(\lambda I_x)^i}{i!} \tag{34}$$

The matrix I_x has the property that $I_x^2 = \mathbf{1}$, allowing the odd and even powers to be separated:

$$e^{\lambda I_x} = \left(1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots\right)\mathbf{1} + \left(\frac{\lambda}{1!} + \frac{\lambda^3}{3!} + \cdots\right) I_x \tag{35}$$

where the first term in parenthesis is $\cosh\lambda$ and the second term is $\sinh\lambda$. Multiplying through by the matrices, and adding the two terms together yields:

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cosh\lambda & -\sinh\lambda & 0 & 0 \\ -\sinh\lambda & \cosh\lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{36}$$

with the parameter λ being identified as the rapidity and can be shown to be given by $\tanh\lambda = \beta$ with the β equal to the velocity of the primed frame.

So finally the form of S can be derived. This is given by applying an infinite number of infinitesimal transformations to the wave function

$$\psi'(x') = \mathrm{S}(\Lambda)\psi(x) = \lim_{N\to\infty}\left(1 - \frac{i}{4}\frac{\lambda}{N}\sigma_{\mu\nu}(I_{\mathrm{n}})^{\mu\nu}\right)^N \psi(x) = e^{\left(-i/4\lambda\sigma_{\mu\nu}I_n^{\mu\nu}\right)}\psi(x). \tag{37}$$

This expression gives both the Lorentz boosts along a given axis and rotations about a given axis. The first thing to look at is the boost along the x axis. Using the previously found value of I_x, the boost is given by

$$\psi'(x') = e^{-(i/2)\lambda\sigma_{01}}\psi(x) \tag{38}$$

For rotations about the x^3 axis, S can be arrived at using equation 25, which gives

$$\Delta\lambda^\nu{}_\mu = \Delta\phi(I_3)^\nu{}_\mu \tag{39}$$

where

$$(I_3)^\nu{}_\mu = \left(\delta_1^\nu\delta_\mu^2 - \delta_2^\nu\delta_\mu^1\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{40}$$

Therefore, the transformation corresponding to a rotation about the x^3 axis is given by

$$\psi'(x') = e^{(i/2)\phi\sigma^{12}}\psi(x) = e^{(i/2)\phi\Sigma^3}\psi(x), \tag{41}$$

with similar expressions for rotations about the other axis. Note that this transformation implies that a rotation by 4π is required to return the wave function to its original state. This implies that the wave function can only appear as a real physical object in a form that involves the wave function multiplied by its hermitian conjugate. Furthermore a four component object that transforms as $\psi'(x') = S(\lambda)\psi(x)$ is referred to as a spinor—typically a two component object is called a spinor and a four component object a bispinor.

For the transformation to be valid, it must be unitary in order to preserve the normalization of the wave function and conserve the probability. The transformation corresponding to rotations is clearly unitary

$$\mathrm{S_R} = e^{-(i/4)\sigma^{ij}\lambda_{ij}} \Longrightarrow S_{\mathrm{R}}^{\dagger} = S_{\mathrm{R}}^{-1}, \qquad i,j = 1,2,3 \tag{42}$$

where the Hermiticity of σ^{ij} was used. The boost (Lorentz) transformations on the other hand are not unitary

$$\mathrm{S_L} = e^{-(i/2)\lambda\sigma^{01}} = e^{(\lambda/2)\alpha_1} \Longrightarrow \mathrm{S_L^{\dagger}} \neq \mathrm{S_L^{-1}} \tag{43}$$

where the following was used to arrive at the result

$$\sigma_{01} = \frac{i}{2}\left(\gamma_0\gamma 1 - \gamma 1\gamma 0\right) = \frac{i}{2}\left(\beta\beta\alpha_1 - \beta\alpha_1\beta\right) = \frac{i}{2}\left(\alpha_1 + \alpha_1\right) = i\alpha_1 \tag{44}$$

but it can be shown that

$$\mathrm{S_L^{-1}} = \gamma_0 \mathrm{S_L^{\dagger}} \gamma_0. \tag{45}$$

In fact the general transform (both rotations and boosts) are satisfied by the relation given in equation 45

$$\mathrm{S}^{-1} = \gamma_0 \mathrm{S}^{\dagger} \gamma_0. \tag{46}$$

3 The Four-Current Density

As a reminder, the probability and probability current density were found in the first lecture to be

$$\rho = \psi^{\dagger}\psi, \qquad \vec{j} = \psi^{\dagger}\vec{\alpha}\psi \tag{47}$$

which can be written in the more convenient form

$$j^{\mu} = \psi^{\dagger}\gamma^{0}\gamma^{\mu}\psi \tag{48}$$

using the relations $\beta^2 = 1$, $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha^i$. This form of the 4-current density leads to the continuity equation $\partial_\mu j^\mu = 0$ which appears to be in a covariant form. To show that it is in fact covariant j^μ has to be shown to be a 4-vector.

To show that j^μ is a 4-vector, it has to be shown to transform like a 4-vector $j'^{\nu} = \Lambda^{\nu}_{\mu} j^{\mu}$. Writing the 4-current density in terms of the wave function gives

$$j'^{\mu} = {\psi'}^{\dagger}\gamma^{0}\gamma^{\mu}\psi' \tag{49}$$

Next transform the wave function to the unprimed frame

$$\begin{aligned} j'^{\mu} &= \psi^{\dagger}\mathrm{S}^{\dagger}\gamma^{0}\gamma^{\mu}\mathrm{S}\psi' && (50)\\ &= \psi^{\dagger}\gamma^{0}\mathrm{S}^{-1}\gamma^{\mu}\mathrm{S}\psi' && (51)\\ &= \Lambda^{\mu}{}_{\nu}\psi^{\dagger}\gamma^{0}\gamma^{\nu}\psi' && (52)\end{aligned}$$

where equation 12 is used to get to the last step. As a point of notation $\bar{\psi} \equiv \psi^{\dagger}\gamma^{0}$ making the 4-current density $j^{\mu} = \bar{\psi}\gamma^{\mu}\psi$.

4 Problems

1. Starting with equation 37, derive the transformation equation of ψ to a frame moving in some arbitrary direction with velocity $\beta = \tanh\lambda$. *To do this, you first need to determine what the value of I_n is. This specifies the direction of the velocity. The next step is to expand the exponential. The final step is to convert the hyperbolic functions into energy and momenta.*

   The transformation is given by:

$$\sqrt{\frac{E+m}{2m}}\begin{pmatrix} 1 & 0 & \frac{p_z}{E+m} & \frac{p_x-ip_y}{E+m} \\ 0 & 1 & \frac{p_x+ip_y}{E+m} & \frac{-p_z}{E+m} \\ \frac{p_z}{E+m} & \frac{p_x-ip_y}{E+m} & 1 & 0 \\ \frac{p_x+ip_y}{E+m} & \frac{-p_z}{E+m} & 0 & 1 \end{pmatrix} \tag{53}$$

References

[1] Relativistic Quantum Mechanics *Greiner* Chapter 1 and Chapter 2.

[2] Quarks and Leptons *Halzen and Martin* Chapter 3 and Chapter 5.