Physics 4213/5213
Lecture 2

1 Introduction

In order to have any understanding of particle physics, in particular how particles interact with each other, the special theory of relativity has to be combined with quantum mechanics. Relativity is a necessary ingredient for two reasons. First, most experiments are carried out with particles colliding at speeds close to those of light (relativistic velocities), and secondly relativity allows the conversion of energy into mass. The latter of these two being the most important, as there is no classical method to produce new particles. An important consequence of the mass-energy equivalence and the uncertainty principle, is that a particle cannot be localized to a distance less than:

$$\Delta x \approx \frac{\hbar}{\Delta p} \approx \frac{\hbar}{mc} \tag{1}$$

otherwise particle antiparticle pairs are created if the energy $E > 2mc^2$ (*note, the size constraint is equal to the Compton wavelength of the particle*). In fact, in relativistic quantum mechanics neither the spatial nor time components can be known with arbitrary precision since the energy and momentum are constrained by the mass of the particle and the speed of light (c).

This lecture will derive a relativistic version of the Schrödinger equation; a relativistic wave equation. It will also examine some of its properties.

2 Relativistic Notation

Before starting on the path toward developing the relativistic wave equation, a few words on relativistic notation are in order. In relativity all 4-vectors and their transformations are the most important quantities. The most fundamental 4-vector is the one that describes space and time, $x^\mu = (t, x, y, z)$. Its transformation properties are defined in terms of the invariant quantity $s^2 = x^\mu x_\mu = t^2 - x^2 - y^2 - z^2$. This introduces a second quantity $x_\mu = (t, -x, -y, -z)$—the 4-vector with the upper index is a contravariant vector, while that with the lower index is a covariant vector. To transform between the two types of vectors, one introduces the metric tensor:

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \Rightarrow \quad x^\mu = g_{\mu\nu} x^\nu \tag{2}$$

Transforming between two frames in relative motion, requires using the Lorentz transformation. The Lorentz transformation is given by:

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad x'^\mu = \Lambda^\mu_\nu x^\nu \tag{3}$$

Notice that the covariant vector transform differently from the contravariant vector $x'_\mu = \Lambda^\nu_\mu x_\nu$ where the two different transformations are defined by the invariance of $x^\mu x_\mu = x^\nu \Lambda^\mu_\nu \Lambda^\nu_\mu x_\nu$. This imposes the condition that $\Lambda^\mu_\nu \Lambda^\nu_\mu = \mathbf{1}$, that is they are inverse transformations of each other:

$$\Lambda^\nu_\mu = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad x'^\mu = \Lambda^\mu_\nu x^\nu \tag{4}$$

3 Relativistic Wave Equation

The development of a single particle relativistic wave equation follows the same line of reasoning as used to develop the Schrödinger equation. First the state of the system is defined in terms of a wave-function; $\psi(\mathbf{q})$ with $\mathbf{q}$ defining all the coordinates relavent to the system, spatial, temporal, spin, etc. The wave-function itself, contains all the information about the system. Further the magnitude squared $|\psi(\mathbf{q})|$ of the wave-function must be positive definite, since it is interpreted as a probability.

Next, every physical observable is represented by a linear Hermitian operator. The operators defining the energy and momentum are given by:

$$\left.\begin{array}{l} E \to i\frac{\partial}{\partial t} \\ p_i \to -i\frac{\partial}{\partial x_i} \end{array}\right\} \quad \Rightarrow \quad p^\mu \to i\partial^\mu \tag{5}$$

Finally, the time evolution of the wave-function is given by:

$$i\frac{\partial\psi}{\partial t} = \hat{H}\psi \tag{6}$$

where $\hat{H}$ is the Hamiltonian. Therefore, the problem of finding the relativistic wave equation, is that of finding the Hamiltonian.

The Hamiltonian for a free relativistic particle is simply the total energy of the particle. This is given by:

$$H = E = \sqrt{p^2 + m^2} \quad \Rightarrow \quad E^2 = p^2 + m^2 \tag{7}$$

where the square of the equation is used to avoid having square roots of operators. In principle, the square root can be expanded, but now derivatives to all orders will be introduced making the theory impossible to solve. But if two operators commute $[\hat{A}, \hat{B}] = 0$ and $\hat{A}\psi = \hat{B}\psi$ then $\hat{A}^2\psi = \hat{B}^2\psi$, since E and $\sqrt{p^2 + m^2}$ represent the energy, the two operators formed must commute. Using the prescription given above, the time evolution of the wave-function is given by:

$$(\partial^\mu \partial_\mu + m^2)\psi = 0 \tag{8}$$

The wave-function itself is found by solving the differential equation leading to:

$$\psi(x^\mu) = Ne^{i(\mathbf{p}\cdot\mathbf{r} - Et)} = Ne^{ip^\mu x_\mu} \tag{9}$$

where N is the normalization to be determined later. This solution is not complete, since substituting back into equation 8 leads to the conclusion that for each value of the momentum, there are two possible energies. One that is positive and a second that is negative. The solution must then be written as:

$$\psi(x^\mu) = N_\pm e^{i(\mathbf{p}\cdot\mathbf{r}\mp|E|t)} \tag{10}$$

The two solutions obviously need some interpretation.

Before attempting to understand the positive and negative solutions, the continuity equation for the probability density is derived. This is done along the same lines as for the Schrödinger equation. Take the Klein-Gordon equation and multiply from the left by the i times the complex conjugate of the wave-function. Then take the complex conjugate of the Klein-Gordon equation and multiply from the left by the wave function times i and subtract the two equations:

$$\left.\begin{array}{l} i\psi^*(\partial^\mu\partial_\mu + m^2)\psi = 0 \\ i\psi(\partial^\mu\partial_\mu + m^2)\psi^* = 0 \end{array}\right\} \quad \Rightarrow \quad \partial^\mu[i\psi^*\partial_\mu\psi - i\psi\partial_\mu\psi^*] = 0 \tag{11}$$

which has the form of the continuity equation $\partial^\mu j_\mu = 0$. The important term to examine in the current density, it the probability density j^0. This term is given by:

$$j^0 = i[\psi^*\partial^0\psi - \psi\partial^0\psi^*] \tag{12}$$

recall that ∂^0 is the time derivative. Since the Klein-Gordon equation is second order in the time derivative, the wave-function and its time derivative are independent of each other. This implies that j^0 can be either positive, negative, or even zero (when ψ is real) causing problems with it being interpreted as a probability density.

Before attempting to interpret the 4-current density, insert the plane wave (eq. 10) into the expression for the 4-current density (eq. 11). This leads to the following expression:

$$\rho = \pm 2|E||N|^2 \qquad \mathbf{j} = 2\mathbf{p}|N|^2 \tag{13}$$

There are two thing to notice in this expression, first the explicit $\pm$ sign, and secondly the dependence on the energy. The $\pm$ sign along with ρ being zero if the wave-function is real, leads to interpreting it as a charge density. In this case multiply it by the charge, take the $\pm$ to refer to two solutions with positive energy and the problem with negative energies and probabilities disappears.

Now for the energy dependence of the charge density. The continuity equation leads to charge conservation. Placing the charge in a box of finite size and not allowing any charge to flow out of the box ($\mathbf{j} = 0$), the charge in the box has to be the same independent of frame. Examine the total charge in two frames:

$$\begin{array}{ll} \text{Rest Frame:} & \displaystyle\int_V 2m|N|^2 d^3x \\ \text{Boosted Frame:} & \displaystyle\int_V 2E|N|^2(1/\gamma)d^3x = \int_V 2E|N|^2\left(\frac{m}{E}\right)d^3x = \int_V 2m|N|^2 d^3x \end{array} \tag{14}$$

therefore the energy term in the charge density, insures that the total charge is frame independent.

4 The Klein-Gordon Lagrangian

For completeness, and since it will make the physics more apparent, the Lagrangian for the Klein-Gordon equation is introduced. To do so, the wave-function is treated as a field, therefore instead of defining the Lagrangian, the Lagrangian density is used, where the two are related by:

$$L = \int_V \mathcal{L}\left(\psi_i, \partial^\mu\right) d^3x \quad \Rightarrow \quad \delta \int \mathcal{L}\left(\psi_i, \partial^\mu\right) d^4x = 0 \tag{15}$$

The second equation is the starting point for the determination of the equations of motion of the fields. This equation leads to the Euler-Lagrange equations for the fields:

$$\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial\psi_i \partial x^\mu)} - \frac{\partial \mathcal{L}}{\partial \psi_i} = 0 \tag{16}$$

The condition placed on the Lagrangian, is that it gives back the Klein-Gordon equations. There are two Lagrangian possible, one for the complex field (charged particles) and one for a real field (neutral particles):

$$\begin{aligned} &\text{Real Field:} \qquad && \mathcal{L} = \frac{1}{2}\left[(\partial^\mu\phi)(\partial_\mu\phi) - m^2\phi^2\right] \\ &\text{Complex Field:} \qquad && \mathcal{L} = \left[(\partial^\mu\psi)(\partial_\mu\psi^*) - m^2\psi\psi^*\right] \end{aligned} \tag{17}$$

Notice the term that is squared in the fields, this term is multiplied by the mass. In every theory that will be developed, any term in the Lagrangian that is proportional to the square of the fields, corresponds to a mass.