# Physics 4183 Electricity and Magnetism II
# Electrostatics and Ohm's Law

## 1 Introduction

The study of time independent electric fields is based on the set of integral and differential equations derived from Coulomb's law. In this lecture, we review the properties of time independent electric fields and the origin of the associated integral and differential equations. In addition, we will consider the case of non-static charges in conductors, Ohm's law, as an example of electrostatics.

### 1.1 The Static Electric Field

The starting point for discussing the electric field is Coulomb's law:

$$\vec{\mathbf{F}} = q\frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \tag{1}$$

where this assumes that $q$ and $Q$ are point charges separated by a distance $r$ and in the form written, the charge Q is assumed to be at the origin of the coordinate system. Typically we assume that the charge $Q$ is the source of the electric field acting on the charge $q$; note that Newton's third law allows us to interchange the roles of $q$ and $Q$. As an aside, this formula assumes that the charge is a constant, which it turns out to not be. It is known from quantum field theory and experiment that the charge measured depends on how close to the charge the measurement is made. When the charge of the electron is measured at a distance of $2.2 \times 10^{-16}$ cm the charge is 3.9% larger than at microscope distances, at a distance of $1.1 \times 10^{-14}$ cm, the charge is 1.5% larger, keep in mind that the size of the electron is $< 10^{-19}$ cm, while the size of the proton is $10^{-13}$ cm. At atomic distances ($10^{-8}$ cm), this effect is causes a shift in the energy levels of the hydrogen atom of $\approx 3 \times 10^{-5}$%. The reason for this change in charge is due to the vacuum acting like a dielectric material. The larger the applied field, the larger the polarization. If one observes the charge from a distance, the induced charge shields the bare charge. As one approaches the charge, one starts to see the full charge, as is the case when one is within a distance of a few atoms of the charge in a dielectric.

Let's define the electric field as the force acting at the location of the charge $q$ per unit of charge due to the charge $Q$ in the limit where $q$ approaches zero so as not to affect the field

$$\vec{\mathbf{E}} \equiv \lim_{q\to 0}\frac{\vec{\mathbf{F}}}{q} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}. \tag{2}$$

Equation 2 defines the electric field at a distance $r$ from the origin ($r = 0$) where a point charge, $Q$, is situated. To generalizes Eq. 2 to a point charge at an arbitrary location, $\vec{\mathbf{r}}'$, we replace $r \equiv |\vec{\mathbf{r}}|$ with $|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|$ and the unit vector, $\hat{\mathbf{r}}$, with $(\vec{\mathbf{r}} - \vec{\mathbf{r}}')/|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|$. The electric field becomes

$$\vec{\mathbf{E}} = \frac{Q}{4\pi\epsilon_0|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^2}\frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}. \tag{3}$$

From this expression, the electric field at the point $\vec{\mathbf{r}}$ due to a collection of charges at fixed locations is

$$\vec{\mathbf{E}} = \sum_i \frac{Q_i}{4\pi\epsilon_0|\vec{\mathbf{r}} - \vec{\mathbf{r}}'_i|^2}\frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}'_i}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'_i|} \tag{4}$$

under the assumption that the fields from each charges can be superimposed linearly. Superposition assumes that the field does not affect the charge. If we assume that the charges are placed close to each other, then the charge can be broken up into small elements such that the total charge is $Q = \sum_i \Delta Q_i \equiv \sum_i \Delta Q(\vec{\mathbf{r}}'_i)$. Finally, if the charge distribution is continuous then the charge elements become differentials $dq \equiv dq(\vec{\mathbf{r}}')$, and the sum becomes an integral $Q = \int dq = \int dq/dV\, dV = \int \rho(\vec{\mathbf{r}}')d\vec{\mathbf{r}}'^3$, with $\rho(\vec{\mathbf{r}}')$ the volume charge density[1]. The electric field from a differential element of charge is

$$d\vec{\mathbf{E}} = \frac{dq(\vec{\mathbf{r}}')}{4\pi\epsilon_0|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^2}\frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} = \frac{\rho(\vec{\mathbf{r}}')d\vec{\mathbf{r}}'^3}{4\pi\epsilon_0|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^2}\frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} \tag{5}$$

therefore the electric field at $\vec{\mathbf{r}}$ from a continuous volume charge density is

$$\vec{\mathbf{E}} = \iiint_{V'} \frac{\rho(\vec{\mathbf{r}}')}{4\pi\epsilon_0|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^2}\frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}d\vec{\mathbf{r}}'^3 \tag{6}$$

Note that a point charge can be treat as a charge density using a Dirac delta function

$$\rho(\vec{\mathbf{r}}') = Q\,\delta^3(\vec{\mathbf{r}}' - \vec{\mathbf{r}}'') \tag{7}$$

Equation 6 completely defines a static electric field, but in most situations, the charge density is not known. In fact, in most situations it is the potentials that are known or most easily determined. Therefore what we need is a differential equation to calculate the fields or potentials, and then apply boundary conditions.

To arrive at a set of differential equations, we start by examining the global properties of the electrostatic field. This is done by calculating two integrals, the first is the flux of electric field lines across a closed surface and the second is the work done by the field over a closed path. The flux of electric field lines through a closed surface is independent of the surface, since the field depends on $1/r^2$ as can be seen in the calculation of the differential flux

$$d\mathcal{F} = \frac{Q}{4\pi\epsilon_0 r^2}r^2 d\Omega = \frac{Q}{4\pi\epsilon_0}d\Omega \tag{8}$$

Therefore, the flux is independent of the shape and size of the surface, and its position relative to the charge. The total flux through any closed surface is

$$\mathcal{F} = \oiint \frac{Q}{4\pi\epsilon_0}d\Omega = \frac{Q}{\epsilon_0} \tag{9}$$

where $Q$ is the charge enclosed. By superposition, we can calculate the flux for each individual charge element and arrive at Gauss's law

$$\mathcal{F} = \oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \frac{Q_{\text{enclosed}}}{\epsilon_0}. \tag{10}$$

If the charge is outside the closed surface, then the net flux is zero, since the flux is independent of the distance to the surface and must enter and exit the closed surface.

[1] There are similar expressions for surface and linear charge densities.

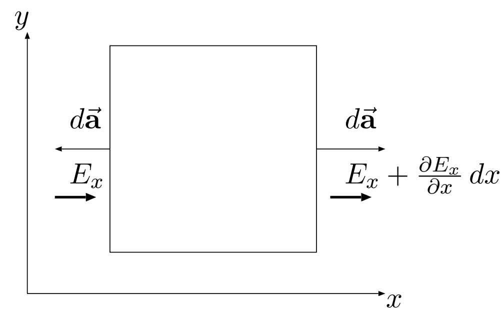


Figure 1: Slice of a differential cube through which the total flux is being calculated.

Gauss's law in integral form defines the general properties of the electric field. To arrive at the local properties, the differential form, we take the limit of equation Eq. 10 as the enclosed volume goes to zero

$$\lim_{V\to 0}\left\{\frac{1}{V}\oiint \vec{\mathbf{E}}\cdot d\vec{\mathbf{a}} = \frac{1}{V}\int_V \frac{\rho(\vec{\mathbf{r}}')}{\epsilon_0}dV\right\}. \tag{11}$$

The left hand side of the equation reduces to $\rho(\vec{\mathbf{r}}')$ after taking the limit, while the right hand side after integrating becomes which leads to

$$\lim_{V\to 0}\frac{1}{V}\left\{\left(E_x + \frac{\partial E_x}{\partial x}dx - E_x\right)dy\,dz + \left(E_y + \frac{\partial E_y}{\partial y}dy - E_y\right)dx\,dz + \left(E_z + \frac{\partial E_z}{\partial z} - E_z\right)dx\,dy\right\} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}, \tag{12}$$

where we use Fig. 1 to illustrate how the integral is performed. Taking Eqs. 11 and 12 together leads to the differential form of Gauss's law

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{E}} = \frac{\rho}{\epsilon_0}. \tag{13}$$

We next examine the work performed in moving a test charge around a closed loop. As before, we use the field of a point charge and then generalize the result using the principle of superposition. To calculate the work per unit charge to travel around a closed loop, we note that the electric field due to a point charge is radial. The line integral depends only on the radial displacement of the test charge. Therefore, the line integral around a closed loop is zero

$$\oint \vec{\mathbf{E}}\cdot d\vec{\boldsymbol{\ell}} = 0 \tag{14}$$

This result can be arrived at formally by noting that $\vec{\mathbf{E}}\cdot d\vec{\boldsymbol{\ell}}$ is a total differential of a scalar function

$$\vec{\mathbf{E}}\cdot d\vec{\boldsymbol{\ell}} = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}\cdot d\vec{\boldsymbol{\ell}} = -\vec{\boldsymbol{\nabla}}\frac{q}{4\pi\epsilon_0 r}\cdot d\vec{\boldsymbol{\ell}} = -d\left(\frac{q}{4\pi\epsilon_0 r}\right) \quad\Rightarrow\quad \int \vec{\mathbf{E}}\cdot d\vec{\boldsymbol{\ell}} = -\int d\Phi. \tag{15}$$

Since this is the integral of a total differential, the integral depends only on the value of $\Phi(\vec{\mathbf{r}})$ at the endpoints independent of path. Therefore the integral over a closed path is zero

$$\oint \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{\ell}} = -\oint d\Phi = 0. \tag{16}$$

To arrive at the differential relation, we calculate the integral in the limit of an infinitesimally small enclosed area per unit area

$$\lim_{V \to 0} \frac{1}{S} \left\{ \oint \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{\ell}} = \left( E_x - E_x - \frac{\partial E_x}{\partial y} dy \right) dx + \left( E_y + \frac{\partial E_y}{\partial x} - E_y dx \right) dy \right\}$$
$$= \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)_z \tag{17}$$

which is the circulation in the direction $\hat{\mathbf{z}}$; the direction is defined by the unit vector perpendicular to the surface bounded by the path ($\hat{\mathbf{z}}$ in this case) with the right hand rule giving the sign as shown in Fig. 2. If we take the other two independent surfaces, we arrive at similar expressions. The three expressions can be written in the following simple form

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = 0 \tag{18}$$

Since the curl of a gradient is zero, Eq. 18 implies that the electric field can be written as the gradient of a scalar function (potential)

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\nabla}}\Phi = 0 \quad \Rightarrow \quad \vec{\mathbf{E}} = -\vec{\boldsymbol{\nabla}}\Phi, \tag{19}$$

where the minus sign insure that the field lines point toward regions of low potential and the potential is defined up to a constant; in other words, only potential differences matter not the absolute value of the potential. Finally, we combine the result of Eq. 19 with the divergence of the electric field and arrive at the Poisson equation

$$\left.\begin{array}{r} \vec{\mathbf{E}} = -\vec{\boldsymbol{\nabla}}\Phi \\ \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = \dfrac{\rho}{\epsilon_0} \end{array}\right\} \quad \Rightarrow \quad \nabla^2 \Phi = -\frac{\rho}{\epsilon_0}; \tag{20}$$

recall that in a charge free region the right hand side is zero and the Poisson equation becomes the Laplace equation $\nabla^2\Phi = 0$. Note, the fact that the electric field is defined as the gradient of a potential is equivalent to Eq. 16, since $d\Phi = \vec{\boldsymbol{\nabla}}\Phi \cdot d\vec{\boldsymbol{\ell}}$.

## 1.2 Charge Conservation

Charge is know to be locally conserved. That is, not only is the total charge conserved, but it cannot suddenly disappear at one spot and reappear at a distant location. Any change in the charge must correspond to a flow out of some bounding surface, or a source or sink of charge must be with the bounding surface.

At present the best limits on charge conservation come from experiments looking at the decay of the electron through the process $e \to \gamma\nu_e$ where a lower limit is set on its lifetime of a 90% probability that it is $> 4.6 \times 10^{26}$ years. In addition, the best limit that the electron simply

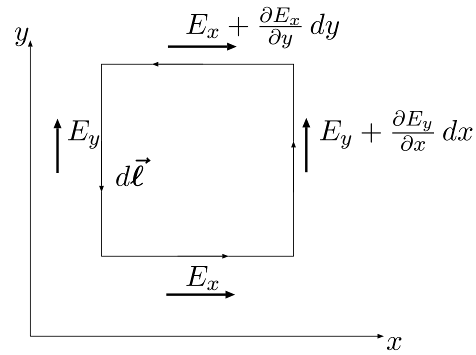


Figure 2: Differential path used to calculate the work done in moving a test charge (calculation of differential potential).

disappears is $> 6.4 \times 10^{24}$ also with a 90% probability. Keep in mind that the age of the universe is estimated at $\approx 12 \times 10^9$ years. Another limit comes from $n \to p\nu_e\bar{\nu}_e$

To arrive at a mathematical expression that expresses the conservation of charge, let's start by defining a charge current density as the rate of the total charge through a differential element of area

$$J = |\vec{\mathbf{J}}| = \frac{dq}{dtda} \tag{21}$$

with the total current through a surface being

$$I = \int_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{a}} \tag{22}$$

where the dot product states that the only component of the current density that contributes to the total current through the surface is that component that is normal to the surface. The tangential component of the current does not cross the surface.

To impose the condition of charge conservation, we take a volume $V$ bounded by a surface $S$ and assume a charge density $\rho$ is contained inside. If the charge contained within the volume varies with time, charge conservation requires that the charge flow through the surface

$$-\frac{d}{dt}\int_V \rho\, dV = \oiint \vec{\mathbf{J}} \cdot d\vec{\mathbf{a}} \tag{23}$$

where the minus sign is due to the current being positive if the charge flows out through the surface and the time derivative being negative for a decrease in charge within the volume. Next we apply the divergence theorem to the right hand side of this equation

$$-\frac{d}{dt}\int_V \rho\, dV = \oiint \vec{\mathbf{J}} \cdot d\vec{\mathbf{a}} = \int_V \vec{\mathbf{\nabla}} \cdot \vec{\mathbf{J}}\, dV \quad \Rightarrow \quad \int_V \left[\frac{\partial \rho}{\partial t} + \vec{\mathbf{\nabla}} \cdot \vec{\mathbf{J}}\right] dV = 0 \tag{24}$$

Since this equation has to be valid for any arbitrary volume, the integrand must be identically zero. Therefore, we arrive at the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\mathbf{\nabla}} \cdot \vec{\mathbf{J}} = 0 \tag{25}$$

This equation is interpreted as follows: The divergence gives the flux of the current density through an infinitesimal closed surface. The time derivative denotes the accumulation or depletion of charge within the closed surface. The fact that the sum of the two is zero implies that the charge is conserved, the change in the enclosed charge is equal to the charge flowing into or out of the volume. Finally, if the current density is independent of time (the case for static fields), then $\vec{\nabla} \cdot \vec{\mathbf{J}} = 0$.

### 1.3 Ohm's Law

As an application of the continuity equation

$$\vec{\nabla} \cdot \vec{\mathbf{J}} + \frac{\partial \rho}{\partial t} = 0 \tag{26}$$

and the electrostatic Maxwell equations

$$\vec{\nabla} \times \vec{\mathbf{E}} = 0 \qquad\qquad \vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon_0}, \tag{27}$$

we investigate the properties of Ohmic materials. These are material that satisfy a linear relation between current and applied field: $V = IR$ in familiar form and $\vec{\mathbf{J}} = \sigma \vec{\mathbf{E}}$ in a less familiar form. We will start by justifying this relation through a simple model and then move on to understand the properties of materials that obey this law.

Ohm's law states that the current density is proportional to the applied force per unit charge $\vec{\mathbf{J}} = \sigma \vec{\mathbf{f}}$, where $\vec{\mathbf{f}}$ is

$$\vec{\mathbf{f}} = \vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}. \tag{28}$$

But, since the velocity is typically small, the Lorentz force does not contribute and only the electric field matters, therefore Ohm's law is normally given as $\vec{\mathbf{J}} = \sigma \vec{\mathbf{E}}$ where $\sigma$ is the conductivity of the material given in units of $\Omega^{-1}$-m$^{-1}$). (The inverse of the conductivity, the resistivity $\rho$, is also used.) Since Ohm's law primarily applies to conductors, and we have considered conductors as materials with free charges, we would expect an electric field applied on the material to accelerate the charges. Therefore, the current density ($\vec{\mathbf{J}} = \rho \vec{\mathbf{v}}$) would be expected to increase linearly with time. We know from experiment that this is not the case. The current is constant for a given applied field, and varies linearly with the strength of the field. Therefore we need to understand the physics that leads to Ohm's law.

Consider the force of an electric field acting on a charge

$$m \frac{dv}{dt} = qE \tag{29}$$

where we treat the problem in one dimension and assume that we can generalize to three dimensions. Assume that the charge starts form rest and after a time $\tau$ the charge collides with an atom in the lattice coming to rest again. The average velocity is then

$$\langle v \rangle = \frac{q\tau}{2m} E \tag{30}$$

where the 1/2 comes from taking the average. The current density in this case is given by

$$J = n_q q \langle v \rangle = \frac{n_q q^2 \tau}{2m} E \quad \Rightarrow \quad \vec{\mathbf{J}} = \sigma \vec{\mathbf{E}} \quad \text{with} \quad \sigma = \frac{n_q q^2 \tau}{2m} \tag{31}$$

where $n_q$ is the number of free charges per unit volume. This, of course, is a linear approximation to the real problem. Yet even in this approximation, we can convert the conductivity into a tensor to allow for the electric field affecting non-parallel components of the current density

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \tag{32}$$

where the first index corresponds to the direction of the affected current, while the second corresponds to the direction of the applied electric field.

To get a feel for the numbers involved in the conduction of electrons, let's calculate the drift velocity in copper. Assume that we have a current of 1 A, and the radius of the cross-sectional area of the wire is 1 mm. The number per unit volume of conduction electrons is $n_q \approx 8.5 \times 10^{28}$ /m$^3$. Finally, The drift velocity is

$$\langle v \rangle = \frac{I/A}{n_q e} = \frac{1/(3.14 \times 10^{-6})}{(8.5 \times 10^{28})(1.6 \times 10^{-19})} \approx 23\ \mu\text{m/s} \tag{33}$$

which corresponds to 7 min/cm.

The conductivity of a few materials are given below to get a feel for the magnitude of the numbers

| Material | $\sigma$ $(\Omega\text{-m})^{-1}$ |
|---|---|
| Silver | $6.1 \times 10^7$ |
| Copper | $5.8 \times 10^7$ |
| Sea Water | $\approx 5$ |
| Silicon | $1.6 \times 10^{-3}$ |
| Pure Water | $2 \times 10^{-4}$ |
| Glass | $\approx 10^{-12}$ |

Let's consider a conductor of arbitrary shape with a conductivity $\sigma$, and a charge density $\rho$ placed within it. I would like to know what happens to the charge density as a function of time. To solve this problem, we note that the continuity equation defines how a charge density evolves in time

$$\vec{\nabla} \cdot \vec{\mathbf{J}} + \frac{\partial \rho}{\partial t} = 0 \tag{34}$$

where $\rho \equiv \rho_f$ corresponds to the free charge density. Since we know that the the current density depends linearly on the applied electric field, we can replace the divergence of the current density with the divergence of the electric field

$$\vec{\nabla} \cdot \vec{\mathbf{J}} = \sigma \vec{\nabla} \cdot \vec{\mathbf{E}} \tag{35}$$

Finally, we know that $\vec{\nabla} \cdot \vec{\mathbf{E}} = \rho_f/\epsilon$, where this expression comes from the use of $\vec{\mathbf{D}} = \epsilon \vec{\mathbf{E}}$ and $\vec{\nabla} \cdot \vec{\mathbf{D}} = \rho_f$, therefore

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon}\rho = 0 \quad \Rightarrow \quad \rho(t) = \rho(0) e^{-\sigma t/\epsilon} \tag{36}$$

For large times, the charge density goes to zero. But since we assume that the conductor is finite in size, the charge must end up on the surface where it stays forever.

Let's consider the situation of a steady current density (independent of time) and constant conductivity. In this case the continuity equation is given by

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{J}} = 0. \tag{37}$$

Replacing the current density through the use of Ohm's law, we get

$$\vec{\boldsymbol{\nabla}} \cdot (\sigma \vec{\mathbf{E}}) = 0 \quad \Rightarrow \quad \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = 0 \quad \text{if } \sigma \text{ is a constant.} \tag{38}$$

This states that there is no charge density inside the conductor (the divergence of $\vec{\mathbf{E}}$ is zero), which is equivalent to stating that Laplace's equation ($\nabla^2 \Phi = 0$) holds. The previous example stated that if a charge density is placed inside a conductor, it will flow to the surface. This example states that for a steady state system, there are no free charges inside the conductor, as has been argued before.

An interesting result that can be derived from Ohm's law, is that the field inside a current carrying conducting wire is uniform. Since we assume that the material surrounding our wire is non-conducting, therefore the normal component of the current density on the surface is zero $\vec{\mathbf{J}} \cdot \hat{\mathbf{n}} = 0$ as is the normal component of the electric field $\vec{\mathbf{E}} \cdot \hat{\mathbf{n}} = 0$. This defines the normal derivative of the potential on the surface $\partial V / \partial n = 0$. The potential on each end of the conducting wire is also specified, since we apply a potential difference $\Delta V$; call one end zero and the other $V_0$. Through the uniqueness theorem it can be shown that once the potential or its normal derivative has been specified on all surfaces, the potential can be uniquely given. For simplicity, assume that the wire is straight and aligned along the $z$-axis. The field is then given by solving the Laplace equation

$$\nabla^2 \phi = 0 \quad \Rightarrow \quad \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{39}$$

and matching the boundary conditions

$$\phi(z) = \frac{V_0}{L} z \quad \Rightarrow \quad \vec{\mathbf{E}} = -\frac{V_0}{L} \hat{\mathbf{z}} \tag{40}$$

where $L$ is the length of the wire and $\vec{\mathbf{E}}$ is seen to be uniform.

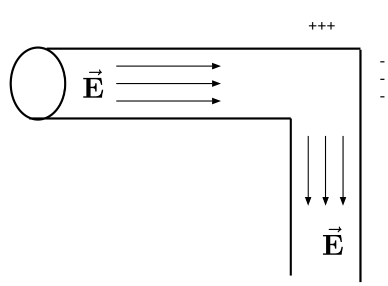


Figure 3: Depiction of what happens in order to keep the electric field uniform in a bent.

In this case considered above, we assumed that the wire is straight. What happens if there has a bent in it? For simplicity assume that the wire is bent by 90°. Since the field must be uniform, by Gauss's law the field induces a charge on the walls of the conductor (charges rearranged) to create the uniform field (see Fig. 3). The surface charge density induced is

$$q = \sigma_q S = \epsilon_0 E S = \epsilon_0 \frac{J}{\sigma} S = \frac{\epsilon_0}{\sigma} I \approx \frac{9 \times 10^{-12}}{6 \times 10^{7}} I \approx 1.5 \times 10^{-19} I \tag{41}$$

where the current in Amps and the charge in Coulombs; note the charge of a proton is $1.6 \times 10^{-19}$ C.

### 1.3.1 Example

Let's consider two concentric conducting cylinders of radii $a$ and $b$ with $a < b$, and a material of conductivity $\sigma$ separating them. The inner conductor is at a potential $V$ and the outer conductor is grounded. Calculate the total current flow between the inner and outer conductor over a region of length $L$.

Since the current density is uniform and the conductivity is assumed constant, the continuity equation is

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = 0 \quad \Rightarrow \quad \nabla^2 \Phi = 0 \tag{42}$$

Because of the symmetry of the system and the boundary conditions on the ends $\frac{\partial V}{\partial z} = 0$, the potential $\Phi$ is only a function of $r$. The Laplace equation for this system is written as follows

$$\begin{aligned} &\frac{1}{r}\frac{\partial}{\partial r}\left( r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \\ &\qquad \Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left( r \frac{\partial \Phi}{\partial r} \right) = 0 \quad \Rightarrow \quad \frac{d\Phi}{dr} = \frac{k}{r} \quad \Rightarrow \quad \Phi(r) = k \ln r / r_0 \end{aligned} \tag{43}$$

where $k$ and $r_0$ are constants of integration to be determined by the boundary conditions and because of the physics of the problem the potetnial depends only on $r$. The boundary conditions are

$$\left. \begin{aligned} \Phi(a) &= V = k \ln(a/r_0) \\ \Phi(b) &= 0 = k \ln(b/r_0) \end{aligned} \right\} \quad \Rightarrow \quad \Phi(r) = \frac{V}{\ln(a/b)} \ln(r/b) \tag{44}$$

In order to calculate the current, we use Ohm's law requiring us to calculate the electric field

$$E_r = -\frac{\partial \Phi}{\partial r} = -\frac{V}{r \ln(a/b)}. \tag{45}$$

The electric field can be used to calculate the current density and integrating the current density over a surface between the conductors gives the total current

$$\vec{\mathbf{J}} = \sigma \vec{\mathbf{E}} = \frac{V\sigma}{r \ln(b/a)} \quad \Rightarrow \quad I = \oiint \vec{\mathbf{J}} \cdot d\vec{\mathbf{a}} = \oiint \frac{V\sigma}{r \ln(b/a)} r d\theta dz = \frac{2\pi L \sigma}{\ln(b/a)} V \tag{46}$$

# Physics 4183 Electricity and Magnetism II
# EMF and Magnetic Fields

## 1 Introduction

In the mid 1800's, Faraday discovered, by accident, that when currents are time dependent they produce electric fields. At the time of this discovery, he was experimenting with magnetic fields to see if they could induce electric fields. He reasoned that if electric charges in motion generated magnetic fields, then maybe magnetic fields would generate electric field. His experiments concentrated on static magnetic fields and saw no effect, except when he switched on and off his apparatus. In these cases, he saw a small current induced in his apparatus, which flowed in opposite directions depending on whether the apparatus was being switch on or off. From these observations, he developed an apparatus that specifically looked for this effect.

In this lecture we will investigate the generation of electric fields by magnetic fields, which can occur in one of two ways. The first is a consequence of the Lorentz force ($\vec{\mathbf{F}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$), and the second is due to a changing magnetic flux through a closed circuit (Faraday's Law). We will also examine how Faraday's law and ties into the four static Maxwell equations. But before doing so, we will define the electromotive force (EMF).

### 1.1 Electromotive Force

Let's consider a typical circuit that consists of a source that cause opposite charges to pileup on opposite terminals. These charges generate an electric field through the conductors (wires) connecting the opposite terminals as shown in Fig. 1. Charges then flow between the opposite terminals through the conductor. When they reach the opposite terminal they must be transported across the source back to the appropriate terminals, so that the process can continue. Therefore, the force per unit charge to transport charge around the circuit is composed of the force move charges in the source $\vec{\mathbf{f}}_s$ and the electric field $\vec{\mathbf{E}}$

$$\vec{\mathbf{f}} = \vec{\mathbf{f}}_s + \vec{\mathbf{E}} \tag{1}$$

where $\vec{\mathbf{f}}$ is the net force necessary to overcome any resistance in the circuit.

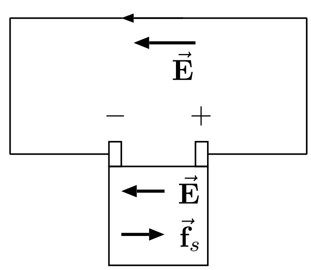


Figure 1: Electrical circuit.

Let's consider a line integral around the closed loop

$$\oint \vec{\mathbf{f}} \cdot d\vec{\boldsymbol{\ell}} = \oint (\vec{\mathbf{E}} + \vec{\mathbf{f}}_s) \cdot d\vec{\boldsymbol{\ell}} = \oint \vec{\mathbf{f}}_s \cdot d\vec{\boldsymbol{\ell}} = \mathcal{E} \tag{2}$$

where we use the fact that the line integral around a closed loop for a static electric field is zero[1] and $\mathcal{E}$ is the EMF or potential difference driving the current. Note that in most cases this quantity is the work per unit charge except for the subtle case of wires moving through static magnetic fields. Within an ideal source, there are no currents flowing (zero internal resistance), therefore $\vec{\mathbf{f}} = 0$ and

$$\vec{\mathbf{E}} + \vec{\mathbf{f}}_s = 0 \quad \Rightarrow \quad \vec{\mathbf{f}}_s = -\vec{\mathbf{E}} \tag{3}$$

Therefore, the potential difference between the terminals is

$$V = -\int_a^b \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{\ell}} = \int_a^b \vec{\mathbf{f}}_s \cdot d\vec{\boldsymbol{\ell}} = \oint \vec{\mathbf{f}}_s \cdot d\vec{\boldsymbol{\ell}} = \mathcal{E} \tag{4}$$

where the second to the last equality is due to the fact that $\vec{\mathbf{f}}_s = 0$ outside the source. Therefore, we have equated the EMF with the potential difference between the terminals.

### 1.2 Resistive EMF

If the conductor or part of it has a finite conductivity, then a resistive EMF is generated. Consider the conductor in Fig. 1 to have a conductivity $\sigma$. The line integral is then written as

$$\mathcal{E}_R = -\int_a^b \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{\ell}} = -\frac{1}{\sigma}\int_a^b \vec{\mathbf{J}} \cdot d\vec{\boldsymbol{\ell}} \tag{5}$$

assuming a constant cross sectional area $S$ and a length $\ell$, this equation integrates to

$$\mathcal{E}_R = -\left(\frac{\ell}{\sigma S}\right) I = -IR = -V \tag{6}$$

as would be expected. The result holds independent of geometry, with the form of the resistance changing to accommodate it.

### 1.3 Static Magnetic Fields

Before we begin the discussion of EMF's generated by magnetic fields, we quickly review the properties of static magnetic fields, taking the Biot-Savart law as the starting point for this review. The Biot-Savart law is

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\boldsymbol{\ell}}' \times (\hat{\mathbf{r}} - \hat{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^2} \tag{7}$$

where the primed coordinate corresponds to the location of a differential current element $Id\vec{\boldsymbol{\ell}}'$ and the unprimed variable is the location where the field is being calculated. The integration is therefore along the current; the current is assumed to be infinitesimally small in the transverse direction. In order to build a real current distribution, we can sum over current filaments. These can then be converted to a current density, which is the current transversing normal to a surface

$$I = \sum_i I_i = \iint_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{a}}' \tag{8}$$

[1] Figure 1 shows that the electric field outside the source points in the opposite direction to that inside the source.

This modifies the Biot-Savart law to

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi} \iiint_V \frac{\vec{\mathbf{J}} \times (\hat{\mathbf{r}} - \hat{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^2} dV' \tag{9}$$

In this case we will calculate the divergence and curl directly, since the calculation of surface and line integrals are difficult to arrive at in a general manner. In addition, calculating the derivatives directly allows us to see that they are applied to the field point not the source.

Let's start with the divergence

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}} = \frac{\mu_0}{4\pi} \iiint_V \vec{\boldsymbol{\nabla}} \cdot \frac{\vec{\mathbf{J}} \times (\hat{\mathbf{r}} - \hat{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^2} dV' \tag{10}$$

since the integral is over the source point and the divergence acts on the field, the divergence can be moved inside the integral. The vector product can be simplified using the following relation

$$\vec{\boldsymbol{\nabla}} \cdot (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \vec{\mathbf{B}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}) - \vec{\mathbf{A}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}}) \tag{11}$$

Applying this relation to Eq. 10, requires calculating the following two derivatives

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{J}}(\vec{\mathbf{r}}') \quad \text{and} \quad \vec{\boldsymbol{\nabla}} \times \frac{(\hat{\mathbf{r}} - \hat{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^2} \tag{12}$$

The first term ($\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{J}}(\vec{\mathbf{r}}')$) is zero, since the derivative acts only on the field point. The second term is also zero, since it is radial; recall that the curl of a radial field is zero, only if the field has a circulation is the curl non-zero. Therefore, the divergence of the Biot-Savart law is zero

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}} = 0 \tag{13}$$

Next we calculate the curl of the Biot-Savart law

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = \frac{\mu_0}{4\pi} \iiint_V \vec{\boldsymbol{\nabla}} \times \frac{\vec{\mathbf{J}} \times (\hat{\mathbf{r}} - \hat{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^2} dV' \tag{14}$$

again the derivative can be moved inside the integral, since the derivative and integral are over independent variables. To simplify the curl, we use the following relation

$$\vec{\boldsymbol{\nabla}} \times (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \vec{\mathbf{A}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}}) - \vec{\mathbf{B}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}) + (\vec{\mathbf{B}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{A}} - (\vec{\mathbf{A}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{B}} \tag{15}$$

Now make the substitutions $\vec{\mathbf{A}} = \vec{\mathbf{J}}$ and $\vec{\mathbf{B}} = (\hat{\mathbf{r}} - \hat{\mathbf{r}}')/|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^2$. Therefore, the second and third term of Eq. 15 will be zero, since the derivative acts on the field point not the source. The first term

$$\iiint_V \vec{\mathbf{J}}(\vec{\mathbf{r}}')\, \vec{\boldsymbol{\nabla}} \cdot \frac{(\vec{\mathbf{r}} - \vec{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3}\, dV' = 4\pi \vec{\mathbf{J}}(\vec{\mathbf{r}}) \tag{16}$$

where the current density is at the location of the field point. This is shown by noting

$$\vec{\boldsymbol{\nabla}} \cdot \frac{(\vec{\mathbf{r}} - \vec{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3} = 4\pi \delta^3(\vec{\mathbf{r}} - \vec{\mathbf{r}}') \tag{17}$$

The last term

$$(\vec{\mathbf{J}} \cdot \vec{\boldsymbol{\nabla}}) \frac{(\vec{\mathbf{r}} - \vec{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3} \tag{18}$$

is arrived at by using the following relation

$$(\vec{\mathbf{J}} \cdot \vec{\boldsymbol{\nabla}}) \frac{(\vec{\mathbf{r}} - \vec{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3} = -(\vec{\mathbf{J}} \cdot \vec{\boldsymbol{\nabla}}') \frac{(\vec{\mathbf{r}} - \vec{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3} \tag{19}$$

where $\vec{\boldsymbol{\nabla}}'$ represents the derivative with respect to $\vec{\mathbf{r}}'$. This relation holds for any function that is the difference between the source and field point $(\vec{\mathbf{r}} - \vec{\mathbf{r}}')$. Next we take the gradient of a single component of the $\vec{\mathbf{r}}/r^3$ and use the relation

$$\vec{\mathbf{J}} \cdot \vec{\boldsymbol{\nabla}}' \left( \left[ \frac{(\vec{\mathbf{r}} - \vec{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3} \right]_i \right) = \vec{\boldsymbol{\nabla}}' \cdot \left( \vec{\mathbf{J}} \left[ \frac{(\vec{\mathbf{r}} - \vec{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3} \right]_i \right) - \left[ \frac{(\vec{\mathbf{r}} - \vec{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3} \right]_i \left( \vec{\boldsymbol{\nabla}}' \cdot \vec{\mathbf{J}} \right) \tag{20}$$

The second term on the right hand side is zero, since the current density is time independent, therefore the continuity equation implies

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{J}} + \frac{\partial \rho}{\partial t} = 0 \quad \Rightarrow \quad \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{J}} = 0 \tag{21}$$

Integrating the first term on the right hand side over the volume and applying Gauss's law gives

$$\iiint\limits_V \vec{\boldsymbol{\nabla}}' \cdot \left( \vec{\mathbf{J}} \left[ \frac{(\vec{\mathbf{r}} - \vec{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3} \right]_i \right) dV' = \iint\limits_S \left( \vec{\mathbf{J}} \left[ \frac{(\vec{\mathbf{r}} - \vec{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3} \right]_i \right) \cdot d\vec{\mathbf{a}}' = 0 \tag{22}$$

since the volume integral is over the region containing the currents, but we can extend it beyond that region since it will contribute nothing to the integral, therefore the surface is selected outside the region containing the current density, and therefore the current density on the surface is zero making the surface integral zero. So finally, the curl of the magnetic field is

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}} \tag{23}$$

Therefore, the Biot-Savart law lead to the following two differential equations

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}} = 0 \qquad\qquad \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}} \tag{24}$$

Furthermore, since the divergence of a curl is zero, the magnetic field can be written in terms of the curl of a vector potential

$$\vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}} \tag{25}$$

## 1.4 Motional EMF

Let's consider what happens when a conductor is transported through a uniform magnetic field at constant velocity (see Fig. 2). Since a conductor contains charges that are free to move within its boundary, the Lorentz force applied on the charges will cause positive and negative charges to move to opposite ends of the conducting bar as shown in Fig. 2. One can then ask, what are the

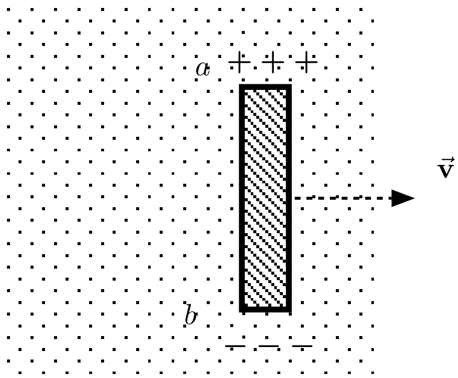


Figure 2: Sketch of a conducting bar moving through a magnetic field. The field is normal to the surface of the paper and points into the paper.

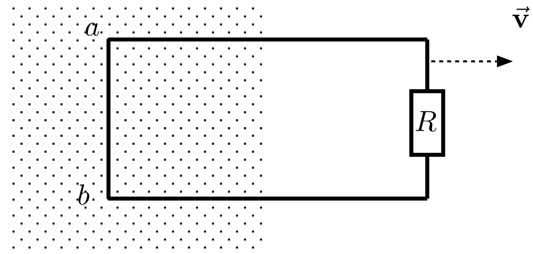


Figure 3: Sketch of a wire loop moving through a magnetic field. The field is normal to the surface of the paper and points into the paper.

forces acting on the charges? As in the discussion on EMFs, we will consider the force per unit charge, which is the Lorentz force separating the charges and the electric field generated by the charges pulling them together. The net force is zero, since the charges are bound to the ends of the conducting bar. Therefore, Newton's second law for this system gives

$$\vec{\mathbf{f}} = \vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}} = 0 \quad \Rightarrow \quad \vec{\mathbf{E}} = -\vec{\mathbf{v}} \times \vec{\mathbf{B}}, \tag{26}$$

where we see that the electric field and Lorentz force are equal in magnitude but opposite in direction. Given Eq. 26, the potential difference between the ends of the conducting bar is found to be

$$\int_a^b \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{\ell}} = -\int_a^b \left(\vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) \cdot d\vec{\boldsymbol{\ell}} = \oint \left(\vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) \cdot d\vec{\boldsymbol{\ell}} \equiv \mathcal{E} \tag{27}$$

where the last two integrals are equal since the Lorentz force only applies in the conductor, therefore whatever path closes the loop is irrelevant since the it doesn't contribute to the integral. Note that we used the definition of the EMF in the last step. If we close the loop as shown in Fig. 3, charges would flow through the resistor $R$. with the generated EMF still between points $a$ and $b$. So this setup, can be used as a generator to power and electrical device.

An obvious question one might ask is what does the work to move the charges, since magnetic fields cannot do work on electrical charges because the Lorentz force always acts perpendicular the direction of motion. For the system that we have setup, the EMF is given by $\mathcal{E} = vBh$ where $h \equiv |b-a|$. The only force other than the Lorentz force acting on the charge is that required to pull the wire, see Fig. 4. Therefore, the work must be done by this force. Let's see if this is the case. From Fig. 4 we see that the velocity of the charge is the vector sum of $\vec{\mathbf{v}}$, the velocity of the wire, $\vec{\mathbf{u}}$ the velocity due to the Lorentz force $\vec{\mathbf{w}} = \vec{\mathbf{v}} + \vec{\mathbf{u}}$. The Lorentz force has two components, one is due to the charge being pulled with a velocity $\vec{\mathbf{v}}$ which points upward and has magnitude $f_v = vB$ and the second component is due to the Lorentz force acting on the charge moving upwards along the wire with velocity $\vec{\mathbf{u}}$, which is $f_u = uB$. Therefore, the work done in transporting a charge from $a$

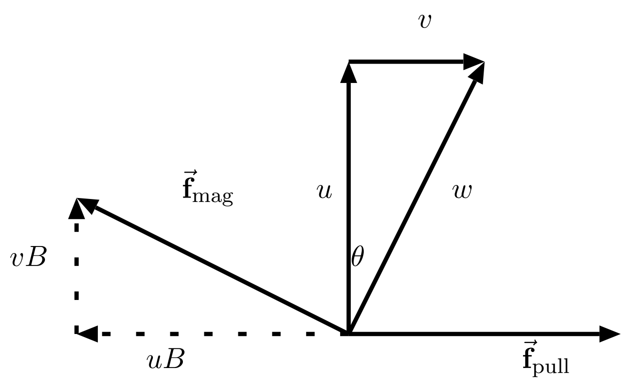


Figure 4: Diagram showing the forces acting on a charge and the velocities of the charge

to $b$ is

$$\int_a^b \vec{\mathbf{f}}_{\text{pull}} \cdot d\vec{\boldsymbol{\ell}} = uB\left(\frac{h}{\cos\theta}\right)\sin\theta = vBh = \mathcal{E} \tag{28}$$

The work per unit charge done in pulling the wire with velocity $\vec{\mathbf{v}}$ is equal to the EMF as expected.

An interesting means of calculating the EMF for cases where we have closed loops of wires is by calculating the rate of change of the magnetic flux through the loop; this will not work for the bar discussed initially since there is no closed path. Let's examine this for the loop shown in Fig. 3. The magnetic flux is given by

$$\Phi_B = \iint \vec{\mathbf{B}} \cdot d\vec{\mathbf{a}} = Bhx \tag{29}$$

where $x$ represents the length of wire parallel to the velocity $\vec{\mathbf{v}}$. The rate of change of the flux is

$$\frac{d\Phi_B}{dt} = Bh\frac{dx}{dt} = -Bhv \tag{30}$$

where the minus comes from the fact that $\frac{dx}{dt} < 0$. Therefore, the EMF is given by

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \tag{31}$$

for the system in Fig. 3. It would be interesting to see if this can be generalized.

We now show that the flux rule

$$\mathcal{E} = \oint \left(\vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) \cdot d\vec{\boldsymbol{\ell}} = -\frac{d\Phi_B}{dt} \tag{32}$$

holds in general. Consider the surface $\mathcal{S}$ bounded by the solid loop in Fig. 5. The loop moves in time $dt$ to the new position that is indicated by the dashed loop. The change in flux over the time $dt$ is

$$d\Phi_B = \Phi_B(t+dt) - \Phi_B(t) = \int_{\text{ribbon}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{a}} \tag{33}$$

where the differential area is $d\vec{\mathbf{a}} = \vec{\mathbf{v}}\,dt \times d\vec{\boldsymbol{\ell}}$ making the change in flux

$$\frac{d\Phi_B}{dt} = \oint \vec{\mathbf{B}} \cdot (\vec{\mathbf{v}} \times d\vec{\boldsymbol{\ell}}). \tag{34}$$

As before, we note that the velocity of the charge is composed of two components, the velocity at which the wire is moved ($\vec{\mathbf{v}}$) and the velocity along the wire ($\vec{\mathbf{u}}$) caused by the Lorentz force $\vec{\mathbf{w}} = \vec{\mathbf{v}} + \vec{\mathbf{u}}$. Since $\vec{\mathbf{u}}$ is parallel to $d\vec{\boldsymbol{\ell}}$, the cross product $\vec{\mathbf{v}} \times d\vec{\boldsymbol{\ell}} = \vec{\mathbf{w}} \times d\vec{\boldsymbol{\ell}}$ and the change in flux is

$$\frac{d\Phi_B}{dt} = \oint \vec{\mathbf{B}} \cdot (\vec{\mathbf{w}} \times d\vec{\boldsymbol{\ell}}). \tag{35}$$

The triple product in the integral can be transformed as follows

$$\vec{\mathbf{B}} \cdot (\vec{\mathbf{w}} \times d\vec{\boldsymbol{\ell}}) = -\vec{\mathbf{w}} \times \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}}, \tag{36}$$

which gives the Lorentz force per unit charge acting on the charge. Using this transformation, the flux equation becomes

$$\frac{d\Phi_B}{dt} = -\oint \vec{\mathbf{w}} \times \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = -\oint \vec{\mathbf{f}}_{\text{mag}} \cdot d\vec{\boldsymbol{\ell}} = -\mathcal{E} \quad \Rightarrow \quad \mathcal{E} = -\frac{d\Phi_B}{dt}. \tag{37}$$

Note, this equation applies when a clear path for the current to flow is given, there are cases where it does not apply and one has to go back to the Lorentz force to solve them.

As a final comment, keep in mind that directions are given by the right hand rule. The positive direction of the loop is given by the direction that the fingers on the right hand curl and the positive direction of the enclosed surface area is the direction of the thumb. Therefore, all possible sign ambiguities in Eq. 37 are accounted for.

### 1.5 Faraday's Law

As has already been stated, Faraday saw a current generated in a closed circuit whenever there was a change in the magnetic fields in the region of the circuit. After further experimentation he showed that an EMF was generated around any closed loop that was proportional to the change in the magnetic flux

$$\mathcal{E} = \oint \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{\ell}} = -\frac{d\Phi_m}{dt} \tag{38}$$

where $\Phi_m$ is the magnetic flux bounded by the integration path; note, that even though we have the same expression as for the motional emf, the physics is fundamentally different. In addition, the circulation is no longer zero, which implies that the curl is also not zero and the generated electric field is not conserved.

To show that the curl is not zero, let's follow the procedure we used before. Take the limit of the integral so that the path of integration is infinitesimally small

$$\lim_{S \to 0} \left[ \oint \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{\ell}} = -\int_S \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{a}} \right] \quad \Rightarrow \quad \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{39}$$

where we followed the procedure used earlier, or apply Stokes's theorem directly to arrive at the second expression. This modifies the Maxwell equations for time dependent magnetic fields.

### 1.5.1 Example

Let's consider the following simple example. A infinitely long solenoid of radius $a$ with a time varying magnetic field $B(t)$ is concentric to a loop of wire with resistance $R$ and radius $b$; $b > a$. Determine the current in the loop of wire.

Initially we ignore the loop of wire. Recall that for an infinitely long solenoid, the magnetic field is zero outside the solenoid. Nonetheless, Faraday's law tells us that a changing magnetic flux will produce an electric field

$$\oint \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{\ell}} = -\frac{d}{dt}\left(\oiint \vec{\mathbf{B}} \cdot d\vec{\mathbf{a}}\right). \tag{40}$$

The magnetic field inside the solenoid will be uniform and parallel to the axis of the solenoid, therefore the generated electric field will be azimuthal

$$E_\phi(2\pi r) = -\pi a^2 \frac{\partial B_z(t)}{\partial t} \quad \Rightarrow \quad E_\phi = -\frac{a^2}{2r}\frac{\partial B_z(t)}{\partial t} \qquad r > a. \tag{41}$$

This electric field will cause a current to flow in the wire loop. The current flowing in the loop is

$$\mathcal{E} = \oint E_\phi d\ell = IR \quad \Rightarrow \quad \mathcal{E} = -\pi a^2 \frac{\partial B_z(t)}{\partial t} = IR \quad \Rightarrow \quad I = -\frac{\pi a^2}{R}\frac{\partial B_z(t)}{\partial t} \tag{42}$$

Let's examine the result. First notice that even though the magnetic field is zero at the position of the loop, an electric field is generated there. Secondly we see that the current that is generated produces a magnetic field that oppose the change in the field in the solenoid. An increasing field along the positive axis, produces an increasing field along the negative axis. The represent an inertial term, the magnetic field does not want to change. A final comment. This electric field generated is only valid for small distance from the time dependent magnetic field. As we will see later, a time dependent electric field will generate a time dependent magnetic field and the fields generated propagate at the speed of light. These effects have yet to be included.

## 1.6 Magnetic Field Moving

We have considered the cases of conductors moving through magnetic fields and time dependent magnetic fields. The first case is an application of the Lorentz force, even though under many circumstances the flux rule can be applied. The second case is Faraday's law, which is a direct application of the flux rule. The one remaining case that has not be treated, is the case of a magnet field moving while the conductor is held fixed. One could of course argue that this is the same as the case of a conductor moving through a magnetic field as viewed by a different observer. This is a application of the Galilean principle of relativity, which states that observers moving uniformly relative to each other observe the same physics. Therefore, one can solve the problem from a reference frame that is at rest relative to the magnetic field.

Another way of looking at the problem is through direct application of the flux rule. If the field is uniform some section of the closed loop must be outside the field. Since the flux is changing, an emf is induced.

A final comment on this issue is to revisit Eq. 39. Recall that the total time derivative is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{\mathbf{v}} \cdot \vec{\boldsymbol{\nabla}} \tag{43}$$

where the $\vec{\mathbf{v}}$ is the velocity of the observer relative to the magnetic field; the derivative asks how the magnetic field changes for the observer. In this case, the flux rule becomes

$$\oint \vec{\mathbf{E}}' \cdot d\vec{\boldsymbol{\ell}} = \oiint \left[ \frac{\partial \vec{\mathbf{B}}}{\partial t} + \vec{\mathbf{v}} \cdot \vec{\boldsymbol{\nabla}} \vec{\mathbf{B}} \right] \cdot d\vec{\mathbf{a}}. \tag{44}$$

To simplify this relation, we use the identity

$$\vec{\boldsymbol{\nabla}} \times (\vec{\mathbf{B}} \times \vec{\mathbf{v}}) = (\vec{\mathbf{v}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{B}} - (\vec{\mathbf{B}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{v}} + \vec{\mathbf{B}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{v}}) - \vec{\mathbf{v}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}}) \quad \Rightarrow \quad \vec{\mathbf{v}} \cdot \vec{\boldsymbol{\nabla}} \vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times (\vec{\mathbf{B}} \times \vec{\mathbf{v}}) \tag{45}$$

where we use the $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}} = 0$ and the derivatives of the velocity are also zero. Using this identity, Eq. 44 becomes

$$\oint \vec{\mathbf{E}}' \cdot d\vec{\boldsymbol{\ell}} = \oint \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{a}} + \oint \vec{\mathbf{v}} \times \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} \quad \Rightarrow \quad \mathcal{E} = \oiint \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{a}} + \oint \vec{\mathbf{v}} \times \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} \tag{46}$$

we see that the emf has two components, one for the time variation of the magnetic field and one due to the relative motion. Note, this expression is only valid for small velocities relative to the speed of light.

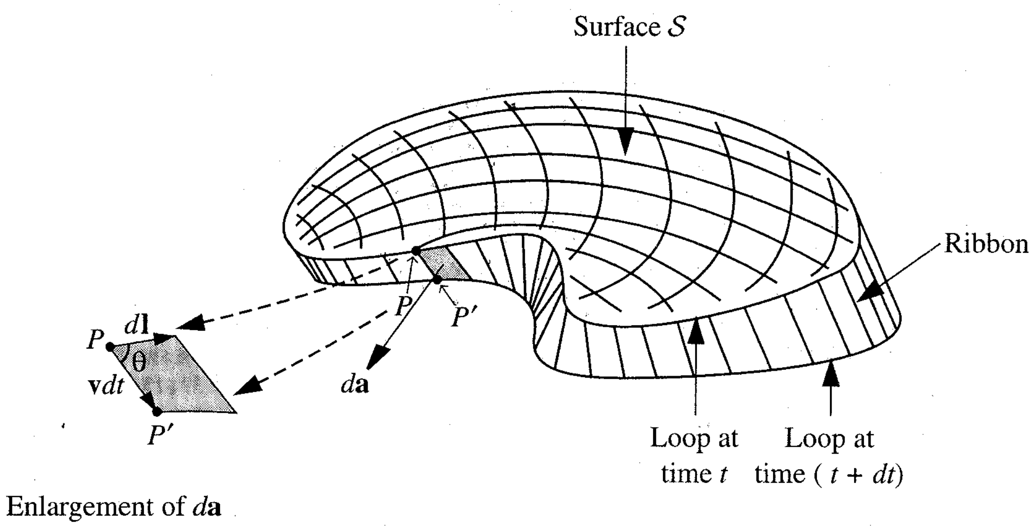


Figure 5: Figure shows a loop that moves through a magnetic field. The bounding surface is composed of the loop at time $t$ and $t + dt$ plus the surface connecting the initial and final positions.

# Physics 4183 Electricity and Magnetism II
# Inductance and Energy in the Magnetic Field

## 1 Introduction

In this lecture, we will define self and mutual inductance. Then use these to arrive at an expression for the energy in a magnetic field.

### 1.1 Inductance

In many electronic applications, one wants to know the effect of one circuit element on other elements. In the case of Faraday's law, any time dependent current will induce a current in any nearby circuit elements. To study this effect, we consider two loops of wire (see Fig. **??**). Assuming that loop 1 has a current $I_1$, the magnetic field generated is given by the Biot-Savart law as

$$\vec{\mathbf{B}}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\vec{\boldsymbol{\ell}}_1 \times \hat{\mathbf{r}}}{r^2} \tag{1}$$

therefore, so is the flux through loop 2

$$\Phi_{B_2} = \iint \vec{\mathbf{B}}_1 \cdot d\vec{\mathbf{a}} \quad \Rightarrow \quad \Phi_{B_2} = M_{21} I_1 \tag{2}$$

where $M_{21}$ is the constant of proportionality known as the mutual inductance of the two loops.

Using the vector potential, one can arrive at an interesting relation between the two loops. The flux through loop 2 can be rewritten as

$$\Phi_{B_2} = \iint \vec{\mathbf{B}}_1 \cdot d\vec{\mathbf{a}} = \iint (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}_1) \cdot d\vec{\mathbf{a}} = \oint \vec{\mathbf{A}}_1 \cdot d\vec{\boldsymbol{\ell}}. \tag{3}$$

The vector potential is given by

$$\vec{\mathbf{A}}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\vec{\boldsymbol{\ell}}_1}{r}. \tag{4}$$

combining the two equations together leads to

$$\Phi_{B2} = \frac{\mu_0 I_1}{4\pi} \oint \left( \oint \frac{d\vec{\boldsymbol{\ell}}_1}{r} \right) \cdot d\vec{\boldsymbol{\ell}}_2 \quad \Rightarrow \quad M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{\boldsymbol{\ell}}_1 \cdot d\vec{\boldsymbol{\ell}}_2}{r}. \tag{5}$$

Notice that changing the subscript doesn't change the integral. Therefore, the mutual inductance is dependent on the geometry and separation of the two loops and not which has the current flowing in it; $M \equiv M_{21} = M_{12}$.

Suppose we vary the current in loop 1. This will give a time dependent magnetic flux through loop two. Using Faraday's law, we calculate the induced emf in loop 2

$$\mathcal{E}_2 = -\frac{d\Phi_{B2}}{dt} = -M \frac{dI_1}{dt}. \tag{6}$$

Notice the minus sign tells us that the induced current will generate a magnetic field that will try to oppose the change.

In addition, the magnetic field generated by a loop of wire will surround its own flux

$$\Phi_B = LI \tag{7}$$

where $L$ is the self-inductance and $I$ the current in the loop. Therefore, there will also be a self inductance. This will generate an emf in the circuit that will attempt to oppose the change in current

$$\mathcal{E} = -L\frac{dI}{dt}. \tag{8}$$

From this equation, it can be seen that the inductance acts like an inertial term trying to oppose change to the magnetic field.

### 1.2 Energy in the Magnetic Field

As we have seen, in order for currents to flow energy is required since a change in the current causes a change in the magnetic field, which causes a back emf. Once the system reaches steady state, no additional energy is required to overcome the back emf; note, energy is required to overcome any resistance in the circuit.

To determine the amount of energy required for charges to flow, we start with the energy applied on a charge $q$, which is $W = q\mathcal{E}$. The work to overcome the back emf would multiply the right hand side by a minus sign and finally, the work per unit time acting on all charges changes $q$ to $I$ leading to

$$\frac{dW}{dt} = -\mathcal{E}I = LI\frac{dI}{dt} \tag{9}$$

where we use Eq. 8 for the last equality. This quickly leads to the expression for the total energy required to setup a steady current $I$

$$W = \frac{1}{2}LI^2 \tag{10}$$

notice this depends only on the final current, not on how long it takes to build up the current.

We would like to express the work in terms of the magnetic fields that are generated. This allows this expression to have a wider range of use. Recall that $\Phi_B = LI$, therefore

$$\Phi_B = \iint \vec{\mathbf{B}} \cdot d\vec{\mathbf{a}} = \iint (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}) \cdot d\vec{\mathbf{a}} = \oint \vec{\mathbf{A}} \cdot d\vec{\boldsymbol{\ell}} = LI. \tag{11}$$

The work done to produce the steady state current can now be expressed in terms of the current density and the vector potential

$$W = \frac{1}{2}I \oint \vec{\mathbf{A}} \cdot d\vec{\boldsymbol{\ell}} = \frac{1}{2} \oint (\vec{\mathbf{A}} \cdot \vec{\mathbf{I}})d\ell = \frac{1}{2} \iiint (\vec{\mathbf{A}} \cdot \vec{\mathbf{J}})dV \tag{12}$$

which is a more general expression, but still not in its simplest form. Let's introduce the magnetic field into this expression through the use of Ampere's law

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = \mu_0\vec{\mathbf{J}} \quad \Rightarrow \quad W = \frac{1}{2\mu_0} \iiint \vec{\mathbf{A}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}})dV. \tag{13}$$

To simplify this expression, we use the vector identity

$$\vec{\boldsymbol{\nabla}} \cdot (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \vec{\mathbf{B}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}) - \vec{\mathbf{A}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}}) \quad \Rightarrow \quad \vec{\mathbf{A}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}}) = \vec{\mathbf{B}} \cdot \vec{\mathbf{B}} - \vec{\boldsymbol{\nabla}} \cdot (\vec{\mathbf{A}} \times \vec{\mathbf{B}}), \tag{14}$$

where we used $\vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}$ in the last step. The work done in building up the magnetic field is

$$W = \frac{1}{2\mu_0}\left[\iiint B^2\,dV - \iiint \vec{\boldsymbol{\nabla}} \cdot (\vec{\mathbf{A}} \times \vec{\mathbf{B}})\,dV\right] = \frac{1}{2\mu_0}\left[\iiint B^2\,dV - \iint (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \cdot d\vec{\mathbf{a}}\right]. \tag{15}$$

If we integrate over all space, the surface integral goes to zero for any physically realistic field. Therefore, the work done to build up the field is

$$W = \frac{1}{2\mu_0}\iiint B^2\,dV. \tag{16}$$

Since Eq. 12 and 16 both represent the energy required to build the magnetic fields, there energy can be thought of as being stored either be in the current density (Eq. 12) or the magnetic field (Eq. 16).

# Physics 4183 Electricity and Magnetism II
# The Maxwell Equations

## 1 Introduction

So far we have arrived at a set of 5 differential equations that describe electric and magnetic fields in vacuum under the assumption that the Coulomb and the Biot-Savart laws are valid and statify the superposition principle

$$\vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$
$$\vec{\nabla} \cdot \vec{\mathbf{B}} = 0 \qquad \vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}}$$
$$\vec{\nabla} \cdot \vec{\mathbf{J}} + \frac{\partial \rho}{\partial t} = 0 \tag{1}$$

Several points can be made about these equations:

- The last equation only describes the conservation of charge and says nothing directly about the fields;
- There are no magnetic charges;
- Magnetic fields are due to charged currents (moving electric charges);
- Time dependent magnetic fields can generate electric fields.
- The description of time dependent electric fields does not seem to appear in these equations.

In additio, we add to these equations the force law

$$\vec{\mathbf{F}} = q(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}). \tag{2}$$

In this lecture we will complete the description of electric and magnetic fields by adding a description of the time dependent electric field. We will see that we can arrive at these descriptions through several different paths.

### 1.1 Time Dependent Currents

Let's consider the continuity equation for the case of time dependent currents (charge densities)

$$\vec{\nabla} \cdot \vec{\mathbf{J}} + \frac{\partial \rho}{\partial t} = 0 \tag{3}$$

Notice that the charge density is given by

$$\rho = \epsilon_0 \vec{\nabla} \cdot \vec{\mathbf{E}} \tag{4}$$

which implies

$$\vec{\nabla} \cdot \left( \vec{\mathbf{J}} + \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \right) = 0 \tag{5}$$

This tell us that the time dependent electric field behaves like a current, we in fact define

$$\vec{\mathbf{J}}_D = \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \tag{6}$$

as the displacement current.

Next, let's consider what happens to the Maxwell equations if we take the divergence of the two curl equations. First recall that the divergence of a curl is zero. Therefore, the divergence of the curl of $\vec{\mathbf{E}}$ (Faraday's Law) is

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\mathbf{E}}) = -\frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{\mathbf{B}}\right) = 0 \tag{7}$$

both sides are zero, so we don't learn any thing new. Now let's apply the divergence to the curl of $\vec{\mathbf{B}}$ (Ampere's Law)

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\mathbf{B}}) = \mu_0 \vec{\nabla} \cdot \vec{\mathbf{J}} \tag{8}$$

in this case the left hand side is zero, but the right hand side is not zero in general. In fact Eq. 5 tells us that the right hand side is incomplete, lacking the displacement current. Therefore, it makes sense to modify Ampere's Law by adding the displacement current to the right hand side

$$\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}} + \mu_0 \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}, \tag{9}$$

which introduces time dependent electric fields into the Maxwell equations. In addition, it incorporates the continuity equation into the four Maxwell equations, therefore instead of needing 5 equations to describe the fields and charge conservation, we need only four.

## 1.2 The Displacement Current

Let's see what we can learn about the displacement current and whether it makes sense to add it to the Maxwell equations. First of all, the net flux of the displacement current and the conduction current ($\vec{\mathbf{J}}$) through any closed surface is zero

$$\vec{\nabla} \cdot (\vec{\mathbf{J}} + \vec{\mathbf{J}}_D) = 0 \quad \Rightarrow \quad \oint\!\!\!\oint (\vec{\mathbf{J}} + \vec{\mathbf{J}}_D) \cdot d\vec{\mathbf{a}} = 0 \tag{10}$$

where we have used Gauss's theorem.

Let's consider a current density composed of both a conduction and displacement current. The current density crosses a surface $S_1$, which is bounded by a closed path $\ell$ as shown in Fig. 1. The current flowing through surfaces $S_2$ and $S_3$ must be the same in order for the divergence of the current density to be zero. In fact, any surface bounded by the path $\ell$ must have the same current flowing through it

$$I + I_D = \oint\!\!\!\oint (\vec{\mathbf{J}} + \vec{\mathbf{J}}_D) \cdot d\vec{\mathbf{a}} \tag{11}$$

Next, let's apply this to a capacitor that is being charged (see Fig. 2). Assume a steady current flowing though a wire. By Ampere's law we can calculate the magnetic field by integrating over a circular path about the wire

$$\oint_\ell \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 \int_S (\vec{\mathbf{J}} + \vec{\mathbf{J}}_D) \cdot d\vec{\mathbf{a}} \tag{12}$$

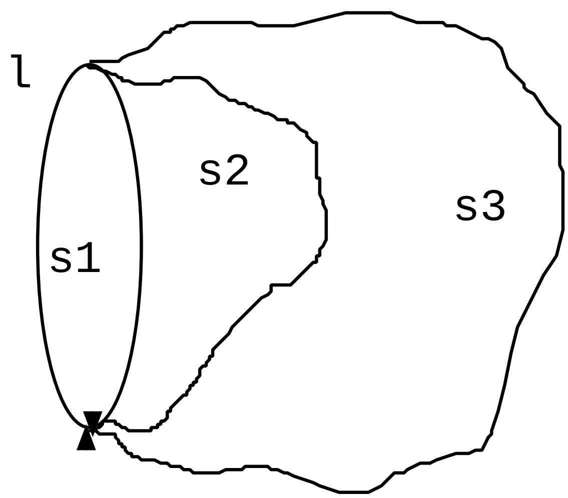


Figure 1: The figure depicts various surfaces that are bounded by the path $\ell$.

The question is, what surface do we take to calculate the current on the right hand side? Let's take surface $S_1$. In this case $\vec{\mathbf{J}}_D = 0$, since the electric field inside the wire is constant if the current is constant. In this case Eq. 12 gives us

$$\oint \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 I \tag{13}$$

Now consider surface $S_2$. In this case $\vec{\mathbf{J}} = 0$ since no currents are flowing through the capacitor, which is assume to be infinite resistance between the plates and the plates are much larger than their separation. But charges are accumulating on the plates, therefore the electric field between the plates is time dependent. The electric field between the plates is given by

$$\vec{\mathbf{E}} = \frac{q}{A\epsilon_0}\hat{\mathbf{n}} \tag{14}$$

where $A$ is the surface area of the plates and $q$ is the accumulated charge on the plates. If the charge is increasing then we have

$$\frac{\partial \vec{\mathbf{E}}}{\partial t} = \frac{1}{A\epsilon_0}\frac{dq}{dt}\hat{\mathbf{n}} = \frac{1}{A\epsilon_0} I\hat{\mathbf{n}} \tag{15}$$

If we integrate over $S_2$, we get

$$I_D = \epsilon_0 \int_{S_2} \frac{\partial \vec{\mathbf{E}}}{\partial t} \cdot da = I \tag{16}$$

note the only the portion of $S_2$ that has field going through it contributes to the integration, which in this case corresponds to $A$. Therefore, the two surfaces give the same value for the line integral of $\vec{\mathbf{B}}$.

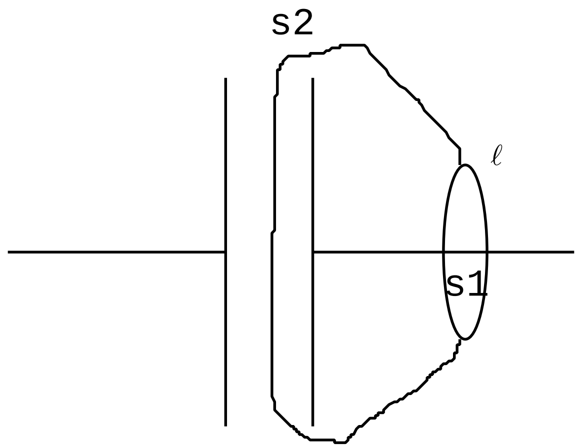


Figure 2: This shows two possible surfaces through which the flux of the current density can be calculated using the path $\ell$ in Ampere's Law.

### 1.3 The Maxwell Equations

We now have the 4 Maxwell equations. These describe the electric and magnetic fields whether static or time dependent. The four equations are

$$\vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{17}$$
$$\vec{\nabla} \cdot \vec{\mathbf{B}} = 0 \qquad \vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}} + \mu_0 \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

The final thing we would like to do is write the Maxwell equations using only the free charges and currents, those sources that we have the ability to control. This implies that we use the $\vec{\mathbf{D}}$ and $\vec{\mathbf{H}}$ fields instead of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$. Let's start by recalling that the bound charge and current densities are

$$\rho_b = -\vec{\nabla} \cdot \vec{\mathbf{P}} \qquad \vec{\mathbf{J}}_b = \vec{\nabla} \times \vec{\mathbf{M}} \tag{18}$$

Before proceeding, let's ask what happens if the polarization is a function of time. To investigate this, take a cylindrical surface and ask what happens on the ends. If the polarization is increasing then the surface charge density increases, if the polarization decreases then the surface charge density decreases. In other words, charge is flowing through the surfaces, therefore we can define a polarization current density as

$$dI = \frac{\partial \sigma_b}{\partial t} \hat{\mathbf{n}} \cdot d\vec{\mathbf{a}} = \frac{\partial P}{\partial t} da_{\perp} \quad \Rightarrow \quad \vec{\mathbf{J}}_p = \frac{\partial P}{\partial t} \tag{19}$$

We now have all the components to rewrite the Maxwell equations. First let's write the charge and current densities in terms of free and bound charge and current densities

$$\rho = \rho_f + \rho_b = \rho_f - \vec{\nabla} \cdot \vec{\mathbf{P}} \tag{20}$$
$$\vec{\mathbf{J}} = \vec{\mathbf{J}}_f + \vec{\mathbf{J}}_b + \vec{\mathbf{J}}_p = \vec{\mathbf{J}}_f + \vec{\nabla} \times \vec{\mathbf{M}} + \frac{\partial \vec{\mathbf{P}}}{\partial t}$$

Start with the divergence of $\vec{\mathbf{E}}$

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = \frac{\rho_f}{\epsilon_0} + \frac{1}{\epsilon_0}\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{P}} \quad \Rightarrow \quad \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{D}} = \rho_f \quad \text{where} \quad \vec{\mathbf{D}} = \epsilon_0 \vec{\mathbf{E}} + \vec{\mathbf{P}} \tag{21}$$

Next we take the curl of $\vec{\mathbf{B}}$

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}}_f + \mu_0 \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{M}} + \epsilon_0 \mu_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} + \mu_0 \frac{\partial \vec{\mathbf{P}}}{\partial t} \quad \Rightarrow \quad \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_f + \frac{\partial \vec{\mathbf{D}}}{\partial t} \tag{22}$$

where

$$\vec{\mathbf{H}} = \frac{1}{\mu_0}\vec{\mathbf{B}} - \vec{\mathbf{M}}. \tag{23}$$

Therefore, an alternate form of the Maxwell equations is

$$\begin{aligned} \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{D}} &= \rho_f & \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} &= -\frac{\partial \vec{\mathbf{B}}}{\partial t} \\ \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}} &= 0 & \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{H}} &= \vec{\mathbf{J}}_f + \frac{\partial \vec{\mathbf{D}}}{\partial t} \end{aligned} \tag{24}$$

where these are used to calculate the fields inside of materials.

# Physics 4183 Electricity and Magnetism II
# Inferring The Maxwell Equations

## 1 Introduction

In this lecture, we would like to show that the 4 Maxwell equations can be inferred starting from Coulomb's law and its assumptions, plus the experimental fact that light is an electromagnetic phenomena. It should be pointed out that this is not a proof of the Maxwell equations, this will show that they can be arrived at in an alternate manner.

### 1.1 The Maxwell Equations

Let's assume that Coulomb's law is given and it is assumed to apply only to static situations. This leads to the following 4 equations

$$\vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon_0} \qquad \frac{\partial \rho}{\partial t} = 0 \tag{1}$$
$$\vec{\nabla} \times \vec{\mathbf{E}} = 0 \qquad \frac{\partial \vec{\mathbf{E}}}{\partial t} = 0$$

Assume that the charges are moving at a constant velocity $\vec{\mathbf{v}}$ relative to a observer. The velocity is assumed to be much smaller than some critical velocity ($v \ll c$), which we will identify later with the speed of light. We will assume that the physics remains the same in both frames. In addition, we will assume that all partial time derivatives are transformed to total time derivatives

$$\frac{\partial}{\partial t} \rightarrow \frac{d}{dt} \tag{2}$$

The total time derivative is given by

$$\frac{d}{dt} = \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z} + \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \vec{\mathbf{v}} \cdot \vec{\nabla} \tag{3}$$

therefore in the rest frame of the charge we get back the original results.

We start by transforming the equation for the charge density

$$\frac{\partial \rho}{\partial t} = 0 \quad \Rightarrow \quad \vec{\mathbf{v}} \cdot \vec{\nabla}\rho + \frac{\partial \rho}{\partial t} = 0 \quad \Rightarrow \quad \vec{\nabla} \cdot (\rho \vec{\mathbf{v}}) + \frac{\partial \rho}{\partial t} = 0 \tag{4}$$

where we have used the fact the $\vec{\mathbf{v}}$ is a constant, and the vector identity

$$\vec{\nabla} \cdot (\vec{\mathbf{v}}\rho) = \rho \vec{\nabla} \cdot \vec{\mathbf{v}} + \vec{\mathbf{v}} \cdot \vec{\nabla}\rho \tag{5}$$

which can be derived by applying the chain rule of differentiation. The quantity $\rho\vec{\mathbf{v}}$ is the current density, therefore Eq. 5 becomes

$$\vec{\nabla} \cdot \vec{\mathbf{J}} + \frac{\partial \rho}{\partial t} = 0 \tag{6}$$

which is the continuity equation. Therefore, from a simple transformation we have arrived at charge conservation.

Let's next take the equation for a time independent electric field and transform it

$$\frac{\partial \vec{\mathbf{E}}}{\partial t} = 0 \quad \Rightarrow \quad \frac{d\vec{\mathbf{E}}}{dt} = \frac{\partial \vec{\mathbf{E}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{E}} = 0 \tag{7}$$

The second term of the second equation can be rewritten using the following vector identity

$$\vec{\boldsymbol{\nabla}} \times (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = (\vec{\mathbf{B}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{A}} - (\vec{\mathbf{A}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{B}} + \vec{\mathbf{A}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}}) - \vec{\mathbf{B}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}) \tag{8}$$

where we identify the variables as $\vec{\mathbf{E}} \equiv \vec{\mathbf{A}}$ and $\vec{\mathbf{v}} \equiv \vec{\mathbf{B}}$, and we keep in mind that $\vec{\mathbf{v}}$ is a constant, we get

$$\frac{\partial \vec{\mathbf{E}}}{\partial t} + \vec{\mathbf{v}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}}) - \vec{\boldsymbol{\nabla}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{E}}) = 0 \tag{9}$$

The divergence of $\vec{\mathbf{E}}$ can be replaced by the charge density and the quantity $\vec{\mathbf{v}} \times \vec{\mathbf{E}}$ represents a new field that arises from the motion of the charges. We call this new field $k\vec{\mathbf{B}} = \vec{\mathbf{v}} \times \vec{\mathbf{E}}$ the magnetic (induction) field, where $k$ is a constant that fixes the units between the two fields. Inserting these quantities into Eq. 9 gives us

$$\frac{\partial \vec{\mathbf{E}}}{\partial t} + \vec{\mathbf{v}}\frac{\rho}{\epsilon_0} - k\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = 0 \quad \Rightarrow \quad \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}} + \epsilon_0 \mu_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \tag{10}$$

where we have identified the constant $k = (\mu_0 \epsilon_0)^{-1}$ in order to arrive at the same form of this equation as our earlier considerations. Notice that we have arrived at Ampere's law and Maxwell's modification by transforming from the rest frame of the charge distribution.

Given that we have both the divergence and curl of the electric field and only the curl of the magnetic field, it is natural to next ask what is the divergence of $\vec{\mathbf{B}}$. Taking the divergence of $\vec{\mathbf{B}}$ we get

$$k\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{E}}) \quad \Rightarrow \quad k\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}} = \vec{\mathbf{E}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}}) - \vec{\mathbf{v}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}}) \tag{11}$$

Since the velocity is constant $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}} = 0$ and from Eq. 1 we have $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = 0$, therefore we get

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}} = 0 \tag{12}$$

The one remaining quantity that we need to determine is the affect of a time dependent magnetic field and its relation to the electric field. In the rest frame of the charge distribution, the magnetic field is independent of time

$$\frac{\partial \vec{\mathbf{B}}}{\partial t} = 0 \tag{13}$$

Therefore when we transform to the moving frame, the total derivative is zero

$$\frac{d\vec{\mathbf{B}}}{dt} = \frac{\partial \vec{\mathbf{B}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{B}} = 0 \quad \Rightarrow \quad \frac{\partial \vec{\mathbf{B}}}{\partial t} = \vec{\boldsymbol{\nabla}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{B}}) \tag{14}$$

where the vector identity given in Eq. 8 is used to arrive at the second equation.

We aren't finished yet, we need to determine what $\vec{\mathbf{v}} \times \vec{\mathbf{B}}$ is; we would like to remove the velocity from the equation. To do so, we use the experimental fact that light is an electromagnetic oscillation. To include this fact into the equations we are inferring, we note that the electric field must have the form

$$\vec{\mathbf{E}} = \vec{\mathbf{f}}(z \pm ct) \tag{15}$$

for a wave propagating in the $\pm z$ direction with a velocity $c$. A wave of this form must satisfy the following second order partial differential equation

$$\frac{\partial^2 \vec{\mathbf{E}}}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \quad \Rightarrow \quad \nabla^2 \vec{\mathbf{E}} = \frac{1}{c^2}\frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \tag{16}$$

where $c$ is the hase velocity of the wave (in this case the speed of light), and the second equation generalizes the first for an arbitrary direction of propagation; note we are assuming that the wave is outside the charge distribution that produces the field, otherwise this equation will need to be modified. The left hand side of the wave equation can be rewritten in terms of a curl using the following vector identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{E}}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{\mathbf{E}}) - \nabla^2 \vec{\mathbf{E}} \quad \Rightarrow \quad \nabla^2 \vec{\mathbf{E}} = -\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{E}}) \tag{17}$$

since $\vec{\nabla} \cdot \vec{\mathbf{E}} = 0$ outside the charge distribution. The second partial time derivative of the electric field can be written in terms of the magnetic field as follows

$$\frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0}\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{\mathbf{B}}\right) = \frac{1}{\mu_0 \epsilon_0}\vec{\nabla} \times \left[\vec{\nabla} \times (\vec{\mathbf{v}} \times \vec{\mathbf{B}})\right] \tag{18}$$

where the first equality uses Eq. 10 with $\vec{\mathbf{J}} = 0$, since we are assuming that we are outside the charge distribution, and the second equality uses Eq. 14. Combining Eq. 17 with Eq. 18 leads to

$$\frac{1}{\mu_0 \epsilon_0 c^2}\vec{\nabla} \times \left[\vec{\nabla} \times (\vec{\mathbf{v}} \times \vec{\mathbf{B}})\right] = -\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{E}}) \tag{19}$$

therefore $\mu_0 \epsilon_0 = 1/c^2$ where $c$ is the speed of light and $\vec{\mathbf{E}} = -\vec{\mathbf{v}} \times \vec{\mathbf{B}}$; note one can add a gradient to both sides, but that does not add anything to the final result, so we ignore it. This transforms Eq. 14 to

$$\vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{20}$$

which is Faraday's law. Note that one might argue that the derivation of $\vec{\nabla} \cdot \vec{\mathbf{B}}$ might be wrong since we assumed the curl of $\vec{\mathbf{E}}$ to be zero. But under the assumption that we made, the velocity $v \ll c$ the error that we made is only of order $v^2/c^2$, that is we kept terms up to order $v/c$

$$\vec{\nabla} \cdot (c\vec{\mathbf{B}}) = \frac{\vec{\mathbf{v}}}{c} \cdot \left[\vec{\nabla} \times \vec{\mathbf{E}}\right] = \frac{\vec{\mathbf{v}}}{c^2} \cdot \frac{\partial}{\partial t}\left(\vec{\mathbf{v}} \times \vec{\mathbf{E}}\right) \tag{21}$$

Consider a typical velocity of 100 km/h $\approx$ 30 m/s, this gives $v/c = 10^{-7}$ and $(v/c)^2 = 10^{-14}$ as you can see this is a very small effect. To improve on our approximation, we have to use special relativity, this will discuss later. None-the-less, we have arrived at the four Maxwell equations which are

$$\begin{aligned} \vec{\nabla} \cdot \vec{\mathbf{E}} &= \frac{\rho}{\epsilon_0} & \qquad \vec{\nabla} \times \vec{\mathbf{E}} &= -\frac{\partial \vec{\mathbf{B}}}{\partial t} \\ \vec{\nabla} \cdot \vec{\mathbf{B}} &= 0 & \qquad \vec{\nabla} \times \vec{\mathbf{B}} &= \mu_0 \vec{\mathbf{J}} + \mu_0 \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \end{aligned} \tag{22}$$

At this point we remove the condition $v \ll c$, and compare to experiment to see how well we agree.

# Physics 4183 Electricity and Magnetism II
# Electromagnetic Waves in Matter

## 1 Introduction

We will discuss the propagation of electromagnetic waves in linear media, and the propagation of waves between two or more different media. The material in this section will have application to the field of optics, and shielding.

### 1.1 Electromagnetic Waves in Matter

The standard form for the Maxwell equations in a media is

$$\vec{\nabla}\cdot\vec{\mathbf{D}} = \rho_f \qquad \vec{\nabla}\times\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{\mathbf{B}} = 0 \qquad \vec{\nabla}\times\vec{\mathbf{H}} = \vec{\mathbf{J}}_f + \frac{\partial\vec{\mathbf{D}}}{\partial t} \tag{1}$$

Now consider a linear homogeneous isotropic media of permittivity $\epsilon$, permeability $\mu$ and conductivity $\sigma$, the Maxwell equations become

$$\vec{\nabla}\cdot\vec{\mathbf{E}} = \frac{\rho_f}{\epsilon} \qquad \vec{\nabla}\times\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{\mathbf{B}} = 0 \qquad \vec{\nabla}\times\vec{\mathbf{B}} = \mu\sigma\vec{\mathbf{E}} + \mu\epsilon\frac{\partial\vec{\mathbf{E}}}{\partial t} \tag{2}$$

where we have used the relations $\vec{\mathbf{J}}_f = \sigma\vec{\mathbf{E}}$, $\vec{\mathbf{D}} = \epsilon\vec{\mathbf{E}}$, and $\vec{\mathbf{B}} = \mu\vec{\mathbf{H}}$.

If we take the divergence of the curl of $\vec{\mathbf{H}}$ equation, then we get the previously found continuity for a linear homogeneous isotropic material

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{\mathbf{B}} = \mu\sigma\vec{\mathbf{E}} + \mu\epsilon\frac{\partial\vec{\mathbf{E}}}{\partial t}\right) \quad\Rightarrow\quad \frac{\partial\rho_f}{\partial t} + \frac{\sigma}{\epsilon}\rho_f = 0 \quad\Rightarrow\quad \rho_f(\vec{\mathbf{r}},t) = \rho_f(\vec{\mathbf{r}},0)e^{-t/\tau} \tag{3}$$

where $\tau = \epsilon/\sigma$ is the time required for $\approx 63\%$ of charge density to move to the surface. For silver the time constant is $\approx 1.4\times10^{-19}$ sec, while for graphite it is $\approx 1.2\times10^{-16}$ sec, where we have taken $\epsilon = \epsilon_0$.

Now let's consider a harmonic electromagnetic wave traveling through our media. The fields are given by

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},t) = \vec{\mathbf{E}}_0 e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t)} \qquad \vec{\mathbf{B}}(\vec{\mathbf{r}},t) = \vec{\mathbf{B}}_0 e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t)} \tag{4}$$

As in our earlier discussion, we can find relations between the electric and magnetic fields, and the wave vector. Start by inserting the electric and magnetic fields into the divergence equations, and assume that there are no free charges

$$\left.\begin{array}{l}\vec{\nabla}\cdot\vec{\mathbf{E}} = 0\\ \vec{\nabla}\cdot\vec{\mathbf{B}} = 0\end{array}\right\} \quad\Rightarrow\quad \left\{\begin{array}{l}\vec{\mathbf{k}}\cdot\vec{\mathbf{E}}_0 = 0\\ \vec{\mathbf{k}}\cdot\vec{\mathbf{B}}_0 = 0\end{array}\right. \tag{5}$$

These relations tell us that the fields are perpendicular to the direction of propagation. Next we use the curl equations. These lead to

$$\left.\begin{array}{l} \vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \\ \vec{\nabla} \times \vec{\mathbf{B}} = \mu\sigma\vec{\mathbf{E}} + \mu\epsilon\frac{\partial \vec{\mathbf{E}}}{\partial t} \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} \vec{\mathbf{k}} \times \vec{\mathbf{E}}_0 = \omega\vec{\mathbf{B}}_0 \\ -(1/\omega)\vec{\mathbf{k}} \times \vec{\mathbf{B}}_0 = \vec{\mathbf{E}}_0\,(i\mu\sigma/\omega + \mu\epsilon) \end{array}\right. \tag{6}$$

which imply that the electric and magnetic fields are perpendicular to each other. Next we use the fact that the electric and magnetic fields are perpendicular to each other and to the direction of propagation to arrive at the following relation

$$\vec{\mathbf{k}} \times (\vec{\mathbf{k}} \times \vec{\mathbf{B}}_0) = -k^2\vec{\mathbf{B}}_0 \tag{7}$$

Taking the cross cross product of the lower of Eqs. 6 with $\vec{\mathbf{k}}$ using the upper of these equations to substitute for $\vec{\mathbf{k}} \times \vec{\mathbf{E}}$ and using Eq. 7 leads to the dispersion relation

$$\frac{k^2}{\omega^2} = \epsilon\mu + i\frac{\mu\sigma}{\omega} \tag{8}$$

for the case of $\sigma = 0$ (dielectric), the relation gives the inverse speed of light in the media

$$\frac{k}{\omega} = \sqrt{\mu\epsilon} = \frac{1}{v_p} \quad \text{or in vacuum} \quad \frac{k}{\omega} = \sqrt{\mu_0\epsilon_0} = \frac{1}{c} \tag{9}$$

so finally, as in elementary physics courses, we define the index of refraction

$$n \equiv \frac{kc}{\omega} = c\sqrt{\epsilon\mu + i\frac{\mu\sigma}{\omega}} = \sqrt{\frac{\epsilon\mu + i\mu\sigma/\omega}{\mu_0\epsilon_0}} \tag{10}$$

which is now a complex number.

Let's examine $n$ in various limits. First take the case of a very poor conductor or very high frequency. The imaginary term is small, therefore the index of refraction is real and is the ratio of the speed of light in vacuum and the speed of light in the medium

$$n \approx \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} = \frac{c}{v_p} \tag{11}$$

If in addition the material is non-magnetic, $\mu = \mu_0$, then $n$ is the dielectric constant (this holds for all the materials we will treat)

$$n \approx \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{\epsilon_r} \tag{12}$$

On the other hand, if the material is a very good conductor, then we rewrite $n$ as follows

$$n = \sqrt{\frac{\epsilon\mu + i\mu\sigma/\omega}{\mu_0\epsilon_0}} = \sqrt{\frac{|\epsilon\mu + i\mu\sigma/\omega|e^{i\phi}}{\mu_0\epsilon_0}} \quad \text{where} \quad \phi = \tan^{-1}\left(\frac{\sigma}{\epsilon\omega}\right) \tag{13}$$

so in the limit of very large $\sigma$ or very small frequency, $n$ is given by

$$n \approx \sqrt{\frac{\mu\sigma/\omega}{\mu_0\epsilon_0}}\, e^{i\pi/4} = (1+i)\sqrt{\frac{\mu\sigma}{2\omega\mu_0\epsilon_0}} \tag{14}$$

Using the index of refraction, we can write the wave equation such that it shows the complex nature of the wave vector. Start by writing $n$ in a form that shows it to be complex

$$n \equiv \mathrm{Re}(n) + i\,\mathrm{Im}(n) \tag{15}$$

Now recall that the wave equation for the electric field is

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}, t) = \vec{\mathbf{E}}_0 e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega t)} \tag{16}$$

Make the substitution $\omega = kc/n$

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}, t) = \vec{\mathbf{E}}_0 e^{i\omega(\vec{\mathbf{k}}/\omega\cdot\vec{\mathbf{r}} - t)} = \vec{\mathbf{E}}_0 e^{i\omega[(n/c)\hat{\mathbf{k}}\cdot\vec{\mathbf{r}} - t]} = \vec{\mathbf{E}}_0\, e^{-\left[(\mathrm{Im}\, n/c)\,\omega\hat{\mathbf{k}}\cdot\vec{\mathbf{r}}\right]}\, e^{i\omega\left[(\mathrm{Re}\, n/c)\hat{\mathbf{k}}\cdot\vec{\mathbf{r}} - t\right]} \tag{17}$$

This corresponds to a wave that is attenuate by the piece corresponding to the imaginary part of $n$, and a propagation piece corresponding to the real part. The phase velocity can easily be derive, and is found to be

$$v_p \equiv \frac{dz}{dt} = \frac{c}{\mathrm{Re}\, n} \tag{18}$$

assuming that the wave travels along the $z$-axis. The attenuation piece gives the depth that a wave will travel into the material. The skin depth is defined as

$$\delta = \frac{c}{(\mathrm{Im}\, n)\omega} = \sqrt{\frac{2}{\sigma\mu\omega}} \tag{19}$$

where Eq. 14 is used in the last step.

As examples of skin depth for different materials, silver has a conductivity of $\sigma = 3 \times 10^7$ $(\Omega\mathrm{m})^{-1}$ at microwave frequencies $\nu = 3 \times 10^9$ Hz, leading to a skin depth of $\delta \approx 5 \times 10^{-6}$ cm. At the other extreme, sea water has a conductivity of $\sigma = 4.3$ $(\Omega\mathrm{m})^{-1}$ at a frequency of 75 Hz, leading to a skin depth of 28 m, at $10^6$ Hz (FM frequencies), the skin depth is 2.5 cm. In both cases $\mu$ is approximately equal to $\mu_0$.

## 1.2 Propagation of EM Waves in an Ionized Gas

Let's consider an ionized gas of very low density, such that the ions never interact with each other. This situation is very different from a conductor where the electrons have a large number of collisions with atoms that make up the crystal lattice. In the case of an ionized gas, we can make the approximation $\epsilon = \epsilon_0$ and $\mu = \mu_0$. An electromagnetic wave will interact with the gas through its electric and magnetic fields. The equation of motion for the ions is given by the Lorentz force as follows

$$\vec{\mathbf{F}} = q\left[\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}\right] \tag{20}$$

where $q$ is the charge of the individual ion or electron that make up the gas. We assume that the wave propagates along the $z$-axis, and that the electric field is along the $x$-axis. In this case the magnetic field is along the $y$ axis. The equation of motion for the the ions and electrons is

$$\vec{\mathbf{F}} = q\left[E_x\hat{\mathbf{x}} + B_y\left(v_x\hat{\mathbf{z}} - v_z\hat{\mathbf{x}}\right)\right] \quad \Rightarrow \quad \begin{cases} \frac{d^2x}{dt^2} = \frac{qE}{m}\left(1 - \frac{1}{c}\frac{dz}{dt}\right) \\ \frac{d^2y}{dt^2} = 0 \\ \frac{d^2z}{dt^2} = \frac{qE}{mc}\frac{dx}{dt} \end{cases} \tag{21}$$

where the relation $E/B = c$ was used, we define $E \equiv E_x$, and $m$ is the mass of the charged particle (ion or electron).

To solve the set of differential equations, we start by assuming that $|dz/dt|/c \ll 1$. The equation of motion along the direction of the electric field becomes

$$\frac{d^2x}{dt^2} = \frac{qE}{m} \quad \Rightarrow \quad x(t) = -\frac{q}{m\omega^2}E_0 \cos\omega t \quad \Rightarrow \quad \frac{dx}{dt} = \frac{q}{m\omega}E_0 \sin\omega t \tag{22}$$

where $E_x(t) = E_0 \cos\omega t$. In addition to the oscillatory motion in the $x$ direction, the same occurs along the direction of propagation of the wave due to the magnetic field

$$\frac{d^2z}{dt^2} = \frac{qE}{mc}\frac{dx}{dt} \quad \Rightarrow \quad z(t) = -\frac{q^2E_0^2}{8\omega^3m^2c}\sin 2\omega t \tag{23}$$

From this expression we can show that the approximation $c \gg v_z$ is valid. The $z$ component of the velocity is

$$\frac{1}{c}\left(\frac{dz}{dt}\right)_{\max} = \frac{q^2}{4m_e^2c^2}\frac{E_0^2}{\omega^2} \approx \frac{2\times 10^3}{f^2}E_0^2 = 1.6\times 10^6 \frac{\left\langle \vec{\mathbf{S}} \right\rangle \cdot \hat{\mathbf{n}}}{f^2} \tag{24}$$

where the time average Poynting vector is

$$\left\langle \vec{\mathbf{S}} \right\rangle \cdot \hat{\mathbf{n}} = \frac{1}{2}\mathrm{Re}\left(\vec{\mathbf{E}} \times \vec{\mathbf{H}}^*\right) \cdot \hat{\mathbf{n}} = \frac{1}{2}\,c\epsilon_0 E_0^2 \quad \Rightarrow \quad E_0^2 = \frac{2}{c\epsilon_0}\left\langle \vec{\mathbf{S}} \right\rangle \cdot \hat{\mathbf{n}} \tag{25}$$

As an example, consider a laser with a power density of $10^{16}$ watts/m$^2$ and a frequency $f \approx 10^{15}$ Hz (ultraviolet), the ratio $v_z/c \approx 10^{-8}$.

To calculate the conductivity of the gas, we use Ohm's law

$$J = \sum_i n_i q_i \left(\frac{dx}{dt}\right)_i \tag{26}$$

where $n_i$ is the density of charged particles of type $i$. Substituting from Eq. 22, the current density is[1]

$$J = \sigma E_0 e^{i\omega t} = -i\sum_j \frac{n_j q_j^2}{m_j\omega}E_0 e^{i\omega t} \quad \Rightarrow \quad \sigma = -\frac{i}{\omega}\sum_j \frac{n_j q_j^2}{m_j} \approx -i\frac{n_e q_e^2}{m_e\omega} = -i\epsilon_0\frac{\omega_p^2}{\omega} \tag{27}$$

where $e^{i\omega t}$ is used to be consistent with Eq. 22,, the final approximation is due to the large difference in electron and ion masses (the electron mass is approximately $5\times 10^{-5}$ times lighter than the proton, therefore has much less inertia than any ion, and will therefore dominate the current), and the plasma frequency ($\omega_p$) is defined as

$$\omega_p^2 = \frac{n_e q_e^2}{\epsilon_0 m_e} \quad \Rightarrow \quad f_p \approx 8.98\sqrt{n_e}\ \text{Hz}. \tag{28}$$

As an example, in the ionosphere $n_e \approx 10^{11}$ electrons/meter$^3$, which leads to $f_p \approx 3$ MHz.

[1]Note that we transform from trigonometric functions to complex exponentials. This simplifies the step of showing that the conductivity must be imaginary.

In addition to the conduction current, we need to add the displacement current to understand the properties of the gas. The displacement current is given by

$$J_D = \frac{\partial D}{\partial t} = \epsilon_0 \frac{\partial E}{\partial t} = i\omega\epsilon_0 E \tag{29}$$

The total current is therefore

$$J_t = i\omega\epsilon_0 E - i\frac{n_e q_e^2}{m_e \omega} E = i\omega\epsilon_0 E \left(1 - \frac{\omega_p^2}{\omega^2}\right) \tag{30}$$

Looking at the form of $J_t$, we conclude that for $\omega = \omega_p$, Ampere's law implies $B = 0$; no current, therefore no magnetic field. Notice also that the electric and magnetic fields are now related by

$$\omega\vec{\mathbf{B}} = \vec{\mathbf{k}} \times \vec{\mathbf{E}} \quad \Rightarrow \quad \frac{B_0}{E_0} = \frac{k}{\omega} \tag{31}$$

where the index of refraction can be derived from Faraday's law

$$i\vec{\mathbf{k}} \times \vec{\mathbf{B}} \,=\, \mu_0 \vec{\mathbf{J}}_t \quad \Rightarrow \quad \frac{k}{\omega}\frac{B_0}{E_0} \,=\, \frac{k^2}{\omega^2} \,=\, \epsilon_0\mu_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right) \quad \Rightarrow \quad \frac{k}{\omega} \,=\, \frac{n}{c} \,=\, \frac{1}{c}\sqrt{\left(1 - \frac{\omega_p^2}{\omega^2}\right)}. \tag{32}$$

Notice, for large values of the frequency, we get back the original relation $E_0/B_0 = c$. But for $\omega < \omega_p$, the $B$-field is larger than it would be in vacuum. In addition, a phase shift of $\pi/2$ is introduced between the electric and magnetic fields, because $B_0/E_0$ is imaginary.

Equations 31 and 10 tells us that the the index of refraction is given by

$$n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \tag{33}$$

If $\omega_p < \omega$, then the wave propagates through the gas unimpeded. If $\omega_p > \omega$, then the wave will be attenuated and not propagate since $n$ is imaginary (see Eq. 17).

### 1.3 Propagation of Light in a Crystal

The propagation of light (electromagnetic waves) in a crystal is more complicated than in a conducting media. The primary reason is that crystals are generally anisotropic; the susceptibility, dielectric constant, index of refraction, depend on the direction in which the wave propogates. A simple model can be built on the following ideas:

1. The electrons are bound to the atoms by a spring type force;
2. The spring constant depends on the direction; typically corresponding to 3 orthogonal directions.
3. The electric field from the wave generates the only force acting on the electron.

Recall that we are characterizing all materials using the index of refraction. The general form of the index of refraction is given by

$$n = \sqrt{\frac{\epsilon\mu + i\mu\sigma/\omega}{\mu_0\epsilon_0}}, \tag{34}$$

which for a crystal there are no currents, therefore the conductivity is zero, and it is non-magnetic $\mu = \mu_0$. Therefore, the index of refraction is given by

$$n = \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{\epsilon_r} \tag{35}$$

To characterize the system, we write the electric field in terms of the polarization; note, since a crystal has no currents flowing in general, the electric field polarizes the material

$$\vec{\boldsymbol{P}} = \epsilon_0 \overleftrightarrow{\boldsymbol{\chi}}\vec{\mathbf{E}} \quad \text{and} \quad \overleftrightarrow{\boldsymbol{\epsilon}} = \epsilon_0(\overleftrightarrow{\mathbf{I}} + \overleftrightarrow{\boldsymbol{\chi}}) \tag{36}$$

where $\overleftrightarrow{\boldsymbol{\chi}}$ is the susceptibility tensor, and $\overleftrightarrow{\boldsymbol{\epsilon}}$ is the dielectric tensor. In matrix form, this equation takes the following form

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \epsilon_0 \begin{pmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{37}$$

An addition point to note is that the relation between the electric and displacement fields is related by a tensor

$$\vec{\mathbf{D}} = \epsilon_0(\overleftrightarrow{\mathbf{I}} - \overleftrightarrow{\boldsymbol{\chi}})\vec{\mathbf{E}} = \overleftrightarrow{\boldsymbol{\epsilon}}\vec{\mathbf{E}} \tag{38}$$

therefore, the electric field and displacement fields do not in general point along the same direction.

For an ordinary non-absorbing crystal, the susceptibility tensor is symmetric and therefore the tensor can be diagonalized

$$\begin{pmatrix} \chi_{11} & 0 & 0 \\ 0 & \chi_{22} & 0 \\ 0 & 0 & \chi_{33} \end{pmatrix} \tag{39}$$

where the three $\chi$'s are known as the principle susceptibilities and the principle dielectric constants are derived using Eq. 36. At this point, the index of refraction is given by

$$\begin{aligned} n_1 &= \sqrt{1 + \chi_{11}} \\ n_2 &= \sqrt{1 + \chi_{22}} \\ n_3 &= \sqrt{1 + \chi_{33}} \end{aligned} \tag{40}$$

### 1.4 Faraday Rotation

If a beam of polarized light traverses a linear homogeneous isotropic dielectric that has been placed in a magnetic field, the plane of polarization rotates. This phenomena was first discovered by Faraday in 1845.

In order to explain this phenomena, we will proceed as we did for the dilute ionized gas, by calculating the index of refraction for the dielectric. Again we apply Newton's second law to the charges in the dielectric, but in this case we are not looking for the current density since the charges are bound, we are calculating the polarization. Recall that the polarization is related to the electric field and the permittivity by

$$\vec{\boldsymbol{P}} = \epsilon_0 \overleftrightarrow{\boldsymbol{\chi}} \vec{\mathbf{E}} \quad \text{and} \quad \overleftrightarrow{\boldsymbol{\epsilon}} = \epsilon_0 (\overleftrightarrow{\mathbf{I}} + \overleftrightarrow{\boldsymbol{\chi}}) \tag{41}$$

where we allow for the possibility of the material not being isotropic after the magnetic field is applied.

We start by writing down the equation of motion for a bound charge in an oscillating electric field of the wave and an externally applied static magnetic field

$$m \frac{d^2 \vec{\mathbf{r}}}{dt^2} = -m\omega_0^2 \vec{\mathbf{r}} - e\vec{\mathbf{E}} - e \left( \frac{d\vec{\mathbf{r}}}{dt} \right) \times \vec{\mathbf{B}} \tag{42}$$

where we have assumed that the electrons are the main component of the motion, and a restoring force is included to account for the electrons being bound. Assuming the usual harmonic time dependence $e^{-i\omega t}$, the equation of motion becomes

$$-m\omega^2 \vec{\mathbf{r}} + m\omega_0^2 \vec{\mathbf{r}} = -e\vec{\mathbf{E}} + i\omega e \vec{\mathbf{r}} \times \vec{\mathbf{B}} \quad \Rightarrow \quad (-m\omega^2 + m\omega_0^2)\vec{\boldsymbol{P}} = n_e e^2 \vec{\mathbf{E}} + i\omega e \vec{\boldsymbol{P}} \times \vec{\mathbf{B}} \tag{43}$$

where the second equation is obtained from the first by multiplying both sides by $-n_e e$, where $n_e$ is the number of electrons per unit volume, and then using the definition of the polarization $\vec{\boldsymbol{P}} = -n_e e \vec{\mathbf{r}}$. If we assume that the magnetic field is parallel to the direction of propagation of the wave and we take that direction to be along the $z$-axis, Eq. 43 can be used to solve for the elements of the “effective” susceptibility tensor. The most straight forward method to solve for the elements of the susceptibility tensor is to write Eq. 43 in matrix form, and and the proceed to solve for the elements of $\vec{\boldsymbol{P}}$, which can then be expressed in terms of $\overleftrightarrow{\mathbf{T}}\chi$. Following this procedure, the “effective” susceptibility tensor is

$$\overleftrightarrow{\boldsymbol{\chi}} = \begin{pmatrix} \chi_{11} & i\chi_{12} & 0 \\ -i\chi_{12} & \chi_{11} & 0 \\ 0 & 0 & \chi_{33} \end{pmatrix} \tag{44}$$

where

$$\chi_{11} = \frac{n_e e^2}{m\epsilon_0} \left[ \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 - \omega^2 \omega_c^2} \right] \tag{45}$$

$$\chi_{12} = \frac{n_e e^2}{m\epsilon_0} \left[ \frac{\omega \omega_c}{(\omega_0^2 - \omega^2)^2 - \omega^2 \omega_c^2} \right] \tag{46}$$

$$\chi_{33} = \frac{n_e e^2}{m\epsilon_0} \left[ \frac{1}{\omega_0^2 - \omega^2} \right] \tag{47}$$

with

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \qquad \omega_c = \frac{eB}{m} \tag{48}$$

We have come up with some rather complicated expressions, but what do they mean. First of all, it should be obvious that the basis set that we have used to describe the fields (unit vectors along the Cartesian axis) is probably not the best choice, since this leads to off-diagonal terms in the susceptibility tensor. Therefore, we diagonalize the tensor; that is find the eigenvalues and eigenvectors. The eigenvalues are

$$\begin{pmatrix} \chi_{11} & i\chi_{12} & 0 \\ -i\chi_{12} & \chi_{11} & 0 \\ 0 & 0 & \chi_{33} \end{pmatrix} \vec{\mathbf{E}} = \lambda \vec{\mathbf{E}} \quad \Rightarrow \quad \lambda_i = \left\{ \begin{array}{l} \chi_{11} - \chi_{12} \\ \chi_{11} + \chi_{12} \\ \chi_{33} \end{array} \right\} \quad \Rightarrow \quad \left\{ \begin{array}{l} n_r = \sqrt{1 + \chi_{11} - \chi_{12}} \\ n_l = \sqrt{1 + \chi_{11} + \chi_{12}} \\ n_z = \sqrt{1 + \chi_{33}} \end{array} \right. \tag{49}$$

and the eigenvectors are

$$E_r = E_x - iE_y, \quad E_l = E_x + iE_y, \quad E_z \tag{50}$$

where $E_r$ corresponds to a wave whose electric field vector rotates in a clockwise direction (right circular polarization), $E_l$ corresponds to counter-clockwise rotation (left circular polarization). Notice that the material has two indices of refraction, one for right circular polarization, the other for left circular polarization. The waves associated with these, assuming propagation along the $z$-axis, are

$$E_r \propto (\hat{\mathbf{x}} - i\hat{\mathbf{y}}) e^{i\omega(n_r/cz - t)} \qquad E_l \propto (\hat{\mathbf{x}} + i\hat{\mathbf{y}}) e^{i\omega(n_l/cz - t)} \tag{51}$$

A linearly polarized wave can be written as

$$\vec{\mathbf{E}} = \frac{1}{2}(E_r + E_l) = \frac{1}{2} E_0 e^{i\omega/c(n_r + n_l)z/2} \left[ (\hat{\mathbf{x}} - i\hat{\mathbf{y}}) e^{i\omega/c(n_r - n_l)z/2} + (\hat{\mathbf{x}} + i\hat{\mathbf{y}}) e^{-i\omega/c(n_r - n_l)z/2} \right] \tag{52}$$

where we have an overall propagation term and and terms corresponding to the amount the direction of the electric field vector rotates about the propagation direction. Therefore, from these equations the amount of rotation is seen to be the difference in the phases over some distance $\ell$. This is given by

$$\Delta\theta = \omega/c(n_r - n_l)\ell/2 \approx \frac{\pi n_e e^3}{\lambda m_e^2 \epsilon_0} \left[ \frac{\omega B}{(\omega_0^2 - \omega^2)^2} \right] \ell \tag{53}$$

where the approximation $\omega\omega_c \ll |\omega_0^2 - \omega^2|$ is used. Notice that this expression has the form $\Delta\theta = VB\ell$ where $V$ corresponds to a constant that depends on the material. For glass this constant is $\approx 3.5^\circ/T/m$.

# Physics 4183 Electricity and Magnetism II
# Reflection and Transmission of E& M Waves

## 1 Introduction

Having given a brief of introduction to electromagnetic waves propagating in different media, will now discuss what happens at an interface between two different media. We will start by discussing the boundary condition, then move on to solve the problem of waves crossing the boundary at normal incidence, and finish up with waves at oblique angles.

### 1.1 Boundary Conditions

When electrostatics and magnetostatics were discussed, we always used boundary conditions to determine how to match up the fields between two different media. For electromagnetic waves we need to add the time dependent terms in Maxwell's equations. To determine the boundary conditions, it is easier to use the integral form of the equations which are

$$\begin{aligned} &\oint_S \vec{\mathbf{D}} \cdot d\vec{\mathbf{a}} = Q_f && \oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{a}} = 0 \\ &\oint \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{\ell}} = -\oint_S \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{a}} && \oint \vec{\mathbf{H}} \cdot d\vec{\boldsymbol{\ell}} = I_f + \oint \frac{\partial \vec{\mathbf{D}}}{\partial t} \cdot d\vec{\mathbf{a}} \end{aligned} \tag{1}$$

The top two equations are the same as in the static case, therefore they lead to the same boundary conditions

$$\begin{aligned} &D_\perp^1 - D_\perp^2 = \sigma_f \quad \Rightarrow \quad \epsilon_1 E_\perp^1 - \epsilon_2 E_\perp^2 = \sigma_f \\ &B_\perp^1 - B_\perp^2 = 0 \end{aligned} \tag{2}$$

where in the first equation we apply the assumption that the media is linear and isotropic. Next we use the lower two of Eqs. 1. To determine the boundary conditions, we take a loop about the interface with the length of the sides perpendicular to the interface taken in the limit where they go to zero. In this case the flux of the $\vec{\mathbf{D}}$ and $\vec{\mathbf{B}}$ fields goes to zero, and one is left with the component of the of the $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$ fields parallel to the surface and the surface current density. The boundary conditions derived from these two equations are

$$\begin{aligned} &\vec{\mathbf{E}}_\parallel^1 - \vec{\mathbf{E}}_\parallel^2 = 0 \\ &H_\parallel^1 - H_\parallel^2 = \vec{\mathbf{K}}_f \times \hat{\mathbf{n}} \quad \Rightarrow \quad \frac{1}{\mu_1}\vec{\mathbf{B}}_\parallel^1 - \frac{1}{\mu_2}\vec{\mathbf{B}}_\parallel^2 = \vec{\mathbf{K}}_f \times \hat{\mathbf{n}} \end{aligned} \tag{3}$$

where we have used the linearity of the $\vec{\mathbf{H}}$ field to get the last expression. Finally, if we assume that the materials are free of any free charges or currents, as is the normal case, then the boundary conditions are

$$\begin{aligned} &\epsilon_1 E_\perp^1 - \epsilon_2 E_\perp^2 = 0 && \vec{\mathbf{E}}_\parallel^1 - \vec{\mathbf{E}}_\parallel^2 = 0 \\ &B_\perp^1 - B_\perp^2 = 0 && \frac{1}{\mu_1}\vec{\mathbf{B}}_\parallel^1 - \frac{1}{\mu_2}\vec{\mathbf{B}}_\parallel^2 = 0 \end{aligned} \tag{4}$$

### 1.2 Reflection and Transmission at Normal Incidence

Let's begin by calculating the intensity of the reflected and transmitted waves at the boundary of two dielectrics. The dielectric constants are $\epsilon_1$ and $\epsilon_2$ In addition, the material is assumed to be non-magnetic $\mu_1 = \mu_2 = \mu_0$. We assume that the incident wave propagates along the $z$-axis, and the electric field is along the $x$ axis. Since the Poynting vector must be in the $z$ direction, the magnetic field must be in the $y$ direction. The incident fields are

$$\vec{\mathbf{E}}_i(z,t) = \hat{\mathbf{x}} E_i^0 e^{i(k_1 z - \omega t)} \qquad \vec{\mathbf{B}}_i(z,t) = \hat{\mathbf{y}} \frac{n_1}{c} E_i^0 e^{i(k_1 z - \omega t)} \tag{5}$$

where the relation between the electric and magnetic fields comes from the Maxwell equation

$$\vec{\nabla} \times \vec{\mathbf{H}} = \frac{\partial \vec{\mathbf{D}}}{\partial t} \quad \Rightarrow \quad B = \mu\epsilon \frac{\omega}{k} E \quad \Rightarrow \quad B = \frac{1}{v_p} E \quad \Rightarrow \quad B = \frac{n}{c} E \tag{6}$$

Next we write down the reflected wave

$$\vec{\mathbf{E}}_r(z,t) = \hat{\mathbf{x}} E_r^0 e^{i(-k_1 z - \omega t)} \qquad \vec{\mathbf{B}}_r(z,t) = -\hat{\mathbf{y}} \frac{n_1}{c} E_r^0 e^{i(-k_1 z - \omega t)} \tag{7}$$

where the minus for the magnetic field is required to keep the direction of the Poynting vector the same as $\vec{\mathbf{k}}$. Finally, we write down the transmitted wave

$$\vec{\mathbf{E}}_t(z,t) = \hat{\mathbf{x}} E_t^0 e^{i(k_2 z - \omega t)} \qquad \vec{\mathbf{B}}_t(z,t) = \hat{\mathbf{y}} \frac{n_2}{c} E_t^0 e^{i(k_2 z - \omega t)} \tag{8}$$

We now have all the elements to calculate the amplitudes of the transmitted and reflected waves. For simplicity, we place the interface at $z = 0$ without loss of generality. The fields are parallel to the interface, therefore we only need to consider the boundary conditions associated with the parallel components

$$\left.\begin{array}{l} \vec{\mathbf{E}}_\parallel^1 - \vec{\mathbf{E}}_\parallel^2 = 0 \\ \vec{\mathbf{B}}_\parallel^1 - \vec{\mathbf{B}}_\parallel^2 = 0 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} E_i^0 + E_r^0 = E_t^0 \\ n_1(E_i^0 - E_r^0) = n_2 E_t^0 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} E_r^0 = \frac{n_1 - n_2}{n_1 + n_2} E_i^0 \\ E_t^0 = \frac{2n_1}{n_1 + n_2} E_i^0 \end{array}\right. \tag{9}$$

Even though knowing the transmitted and reflected amplitudes is important, they are not the quantities that are measured. Of more importance, is the time averaged intensity, which is given by

$$\vec{\mathbf{S}}_{\text{avg}} = \frac{1}{2} \text{Re}\left(\vec{\mathbf{E}} \times \vec{\mathbf{H}}^*\right) = \frac{1}{2}\, nc\epsilon_0 E_0^2\, \hat{\mathbf{k}} \quad \Rightarrow \quad |\vec{\mathbf{S}}_{\text{avg}}| = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}} n E^2 \tag{10}$$

where $Z_0 \equiv \sqrt{\mu_0/\epsilon_0} = 377\ \Omega$ is the impedance of vacuum. The ratio of reflected to incident intensity is

$$\left.\begin{array}{l} I_r = \frac{1}{2} Z_0^{-1} n_1 \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 (E_i^0)^2 \\ I_i = \frac{1}{2} Z_0^{-1} n_1 (E_i^0)^2 \end{array}\right\} \quad \Rightarrow \quad R = \frac{I_r}{I_i} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \tag{11}$$

while the ratio for transmitted to incident intensity is

$$I_t = \frac{1}{2} Z_0^{-1} n_2 \left[\frac{2n_1}{n_1 + n_2} E_i^0\right]^2 \quad \Rightarrow \quad T = \frac{I_t}{I_i} = \left(\frac{4 n_1 n_2}{(n_1 + n_2)^2}\right) \tag{12}$$

The two expressions can be combined to show that the energy flow is conserved $R + T = 1$.

### 1.3 Reflection and Refraction at Oblique Angles

Before we calculate the reflected and transmitted intensities for a wave incident at an oblique angle on a boundary, we deduce a few well known laws of optics. Let's consider the electric field of the incident, reflected and transmitted waves at a boundary. The normal and tangential fields must be continuous across the boundary for all times and at all points (we assume a material with no free charges)

$$\vec{\mathbf{E}}_i e^{i(\vec{\mathbf{k}}_i\cdot\vec{\mathbf{r}}-\omega t)} + \vec{\mathbf{E}}_r e^{i(\vec{\mathbf{k}}_r\cdot\vec{\mathbf{r}}-\omega t)} = \vec{\mathbf{E}}_t e^{i(\vec{\mathbf{k}}_t\cdot\vec{\mathbf{r}}-\omega' t)} \tag{13}$$

Let's first select a fixed point on the boundary; with no loss of generality let's select that point to be $\vec{\mathbf{r}} = 0$. In order for the boundary conditions to be satisfied for all times, the phase must be the same on both sides of the boundary, which implies $\omega = \omega'$. Let's take $t = 0$, and require that the boundary condition be independent of position on the boundary. In this case, the phase implies

$$\vec{\mathbf{k}}_i \cdot \vec{\mathbf{r}} = \vec{\mathbf{k}}_r \cdot \vec{\mathbf{r}} = \vec{\mathbf{k}}_t \cdot \vec{\mathbf{r}} \tag{14}$$

where we place the origin on the interface, the interface is situated along one of the axis (see Fig. 1), we take this to be $z = 0$. This condition implies that

$$xk_{ix} + yk_{iy} = xk_{rx} + yk_{ry} = xk_{tx} + yk_{ty} \tag{15}$$

for any value of the coordinates, which are orthogonal and therefore independent of each other. This implies that

$$k_{ix} = k_{rx} = k_{tx} \qquad k_{iy} = k_{ry} = k_{ty} \tag{16}$$

which implies that the 3 wavevectors form a plane, which we can take to be the $x$-$z$ plane.

A second condition on the wavevectors can be derived by using the first equality in Eq. 14 and the fact that both wavevectors are in the same media ($|\vec{\mathbf{k}}_i| = |\vec{\mathbf{k}}_r|$). This implies that the incident and reflected angles are equal ($\theta_i = \theta_r$). A third condition can be derived using the second equality in Eq. 14. This leads to

$$|\vec{\mathbf{k}}_i| \cos(\pi/2 - \theta_i) = |\vec{\mathbf{k}}_t| \cos(\pi/2 - \theta_t) \quad \Rightarrow \quad n_1 \sin\theta_i = n_2 \sin\theta_t \tag{17}$$

where we used the relation $n = ck/\omega$. This expression is known as Snell's law.

Next we derive expressions between the amplitudes of the incident, transmitted and reflected waves; to simplify the algebra, we will take the amplitude of the incident wave to be 1. There are two cases that are normally considered with all others being combinations of these. The first of these has the electric field parallel to the interface. This case is called the transverse electric (TE) case; the electric field is transverse to the plane of the wavevectors. The second case has the magnetic field parallel to the interface, which is called the transverse magnetic (TM) case (the electric field is in the plane of the wavevectors).

We will first discuss the transverse electric case. At the boundary, the fields parallel to the interface are related by

$$\begin{aligned} &1 + E_r = E_t \\ &H_i \cos\theta_i - H_r \cos\theta_i = H_t \cos\theta_t \quad \Rightarrow \quad \frac{k_1}{\mu_1}\left[\cos\theta_i - E_r \cos\theta_i\right] = \frac{k_2}{\mu_2} E_t \cos\theta_t \end{aligned} \tag{18}$$

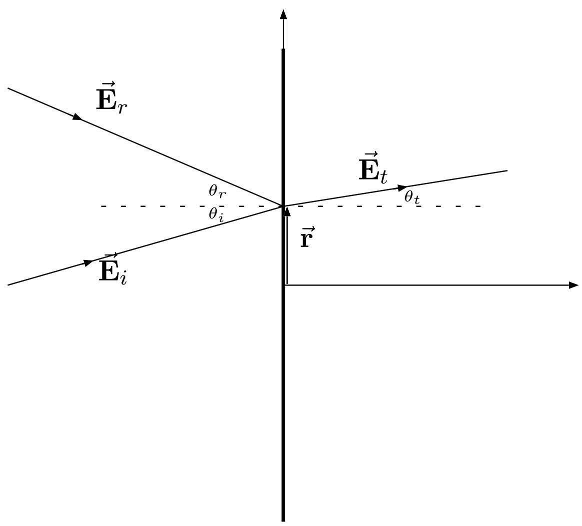


Figure 1: This figure defines the quantities needed for discussing the problem of reflection and transmission at oblique angles.

where we use

$$H = \frac{B}{\mu} = \frac{E}{\mu v} = \frac{kE}{\mu\omega} \tag{19}$$

and the fact that $\omega$ is the same on both sides of the interface. We solve for $E_r$ first, by substituting for $E_t$ from the first equation into the second equation. Then we substitute $k_j = n_j\omega/c$, to get

$$E_r = \frac{\cos\theta_i - \beta\cos\theta_t}{\cos\theta_i + \beta\cos\theta_t} \quad \text{where} \quad \beta = \frac{\mu_1 n_2}{\mu_2 n_1} \tag{20}$$

with the transmitted amplitude calculated by substituting this expression back into first of Eq. 18

$$E_t = \frac{2\cos\theta_i}{\cos\theta_i + \beta\cos\theta_t} \tag{21}$$

For a typical dielectric $\mu_1 = \mu_2 = \mu_0$. In addition, using Snell's law these equation can be written in terms of the incident angle and the ratio of indices of refraction

$$\begin{aligned} E_r &= \frac{\cos\theta_i - \sqrt{n^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}} \\ E_t &= \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}} \end{aligned} \tag{22}$$

where $n = n_2/n_1$.

We now repeat the procedure for the TM case. The boundary conditions are

$$\begin{aligned} &H_i - H_r = H_t && \Rightarrow \quad \frac{k_1}{\mu_1}\left[1 - E_r\right] = \frac{k_2}{\mu_2}E_t \\ &\cos\theta_i + E_r\cos\theta_i = E_t\cos\theta_t && \end{aligned} \tag{23}$$

The algebra is a little more complicated, with the resulting equations being

$$E_r = \frac{-\beta\cos\theta_i + \cos\theta_t}{\beta\cos\theta_i + \cos\theta_t} \tag{24}$$
$$E_t = \frac{2\cos\theta_i}{\beta\cos\theta_i + \cos\theta_t}.$$

After setting $\mu_1 = \mu_2$, and apply Snell's law, we arrive at

$$E_r = \frac{-n^2\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}}{n^2\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}} \tag{25}$$
$$E_t = \frac{2n\cos\theta_i}{n^2\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}}$$

In real applications, the amplitude is difficult to measure since it has to be extracted through a time measurement. The most useful quantity is the time averaged Poynting vector crossing or being reflected by the surface. This quantity is the intensity or energy per unit area normal to the surface

$$I = \left\langle \vec{\mathbf{S}} \right\rangle \cdot \hat{\mathbf{n}}_s = \frac{1}{2}\vec{\mathbf{E}} \times \vec{\mathbf{H}}^* \cos\theta = \frac{1}{2} Z_0^{-1} n |E_0|^2 \cos\theta \tag{26}$$

where $\hat{\mathbf{n}}_s$ corresponds to a unit vector normal to the surface. Since we can calculate everything we need from the reflected intensity, we will only write down this quantity

$$I_r^p = \frac{I_r^p}{I_i^p} = \left| \frac{-n^2\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}}{n^2\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}} \right|^2 \tag{27}$$

$$I_r^s = \frac{I_r^s}{I_i^s} = \left| \frac{\cos\theta_i - \sqrt{n^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}} \right|^2 \tag{28}$$

where $I^s$ corresponds to the TE case ($\vec{\mathbf{E}}$ perpendicular to the plane of the wave vectors)[1], and $I^p$ corresponds to the TM case ($\vec{\mathbf{E}}$ parallel to the plane of the wave vectors)[2].

Let's consider the case where $n_1 < n_2$. Figure 2 show the reflected intensity for $n_1 = 1$ and $n_2 = 1.5$. As can be seen, the TM wave has a smaller reflectance than the TE wave, and has a zero. The zero occurs at the angle

$$\begin{aligned}
&-n^2\cos\theta_i + \sqrt{n^2 - \sin\theta_i} = 0 \quad \Rightarrow \quad n^4\cos^2\theta_i = n^2 - \sin^2\theta_i \\
&n^4\cos^2\theta_i = n^2 - 1 + \cos^2\theta_i \quad \Rightarrow \quad (n^4 - 1)\cos^2\theta_i = n^2 + 1 \\
&(n^2 - 1)(n^2 + 1)\cos^2\theta_i = (n^2 - 1) \quad \Rightarrow \quad (n^2 + 1)\cos^2\theta_i = 1 \\
&\cos\theta_i = \frac{1}{\sqrt{1 + n^2}} \quad \Rightarrow \quad \sin\theta_i = \frac{n}{\sqrt{1 + n^2}} \\
&\tan\theta_B \equiv \tan\theta_i = n
\end{aligned} \tag{29}$$

[1] $s$ stands for senkrecht, the German word for perpendicular.
[2] $p$ stands for parallel.

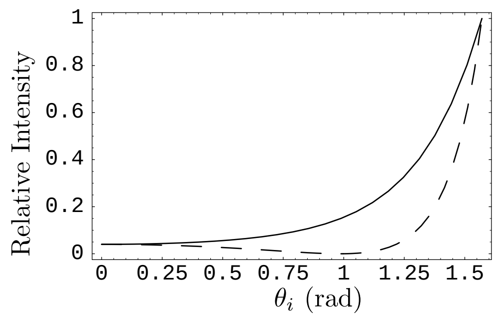


Figure 2: Intensity of the reflected wave as a function of incident angle for $n_2 > n_1$. The solid curve corresponds to the TE wave, while the dashed curve is the TM wave.

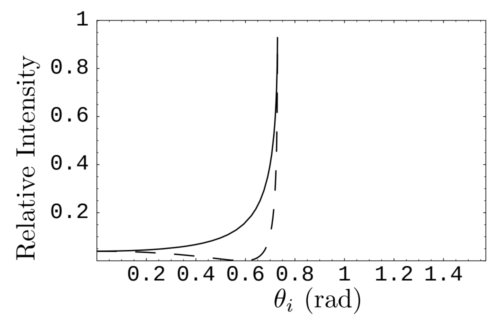


Figure 3: Intensity of the reflected wave as a function of incident angle for $n_1 > n_2$. The solid curve corresponds to the TE wave, while the dashed curve is the TM wave.

where $\theta_B$ is the Brewster angle. In addition, Snell's law at the Brewster angle gives

$$\sin\theta_i = n\sin\theta_t \quad \Rightarrow \quad \frac{n}{\sqrt{1+n^2}} = n\sin\theta_t \tag{30}$$

$$\sin\theta_t = \frac{1}{\sqrt{1+n^2}} = \cos\theta_i = \cos\theta_r \quad \Rightarrow \quad \theta_t = \frac{\pi}{2} - \theta_r$$

therefore there is a 90° angle between the reflected and transmitted wave. Since the electric field is perpendicular to the propagation direction and for TM waves it is in the plane of the wavevectors, the electric field of the transmitted wave is parallel to the reflected wave; see Fig 4. The standard argument is that the TM wave is not reflected because the charges in media, which radiate to form the reflected and transmitted waves, don't radiate in the direction of the electric field as we will show later.

Now let's consider the case where $n_1 > n_2$ (see Fig. 3). Here again we see that the TM mode has a lower reflectance than the TE mode. Again we find that at the Brewster angle the reflectance goes to zero. The major difference being that the reflectance is equal to 1 at angles less than 90°. This occurs when $\sin\theta_i = n$, referred to as the critical angle, which according to Snell's law corresponds to the transmitted angle being 90°

$$n_1\sin\theta_i = n_2\sin\theta_t \quad \Rightarrow \quad \sin\theta_t = \frac{n_1}{n_2}\sin\theta_i \tag{31}$$

$$\sin\theta_t = \frac{1}{n}\sin\theta_i = 1 \quad \Rightarrow \quad \theta_t = \frac{\pi}{2} \tag{32}$$

Beyond the critical angle, the transmission angle is imaginary

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-\frac{\sin^2\theta_i}{n^2}} = \pm i\sqrt{\frac{\sin^2\theta_i}{n^2} - 1} \tag{33}$$

Based on this, you might guess that the electric field is zero in the other media (media 2). This of course is not obvious, since we have selected the transmitted and reflected intensities normal to the surface. Therefore, there may still be an electric field on both sides of the surface. In fact to satisfy the boundary conditions based on the phase relations, the field cannot be zero $\vec{\mathbf{k}}_i\cdot\vec{\mathbf{r}} = \vec{\mathbf{k}}_r\cdot\vec{\mathbf{r}} = \vec{\mathbf{k}}_t\cdot\vec{\mathbf{r}}$.

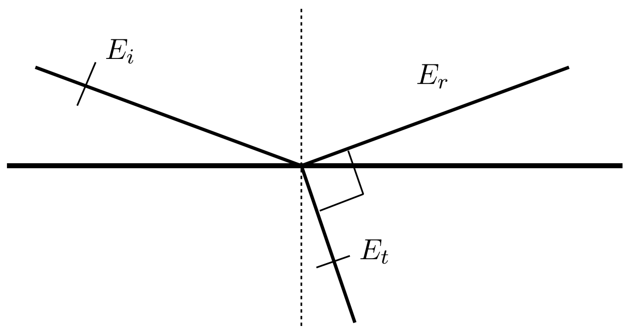


Figure 4: The geometry at which the Brewster angle occurs.

To determine the analytic form of the transmitted wave, we substitute the transmitted wave vector into the wave representing the wave in media 2

$$\vec{\mathbf{k}}_t = k_t(\hat{\mathbf{x}}\sin\theta_t + \hat{\mathbf{z}}\cos\theta_t) \quad \Rightarrow \quad \vec{\mathbf{E}}_t e^{i(\vec{\mathbf{k}}_t\cdot\vec{\mathbf{r}}-\omega t)} \tag{34}$$

Next use Snell's law to rewrite the angles in terms of the incident angles

$$\vec{\mathbf{E}}_t = \vec{\mathbf{E}}_t^0 e^{i(\omega n_1/c)(x\sin\theta_i \pm iz\sqrt{\sin^2\theta_i - n^2} - c/n_1 t)} = \vec{\mathbf{E}}_t^0 e^{-|z|(n_1\omega/c)\sqrt{\sin^2\theta_i - n^2}}\, e^{i(\omega/c)(xn_1\sin\theta_i - ct)}. \tag{35}$$

Equation 35 shows that the trnasmitted wave propagates parallel to the interface and the amplitude of the electric field decays as a function of distance from the interface.

Finally, we consider the phase shift on total internal reflection. The difference in phase shift between the TE and TM mode determines the direction of the electric field vector as the wave propagates through a material. For example, a linearly polarized wave can become circularly polarized if the difference in phase shifts between the two modes is $\pi/2$.

To arrive at the phase shift for either mode, we recall that a wave that is totally reflected, the magnitude of the amplitude is equal to one. Nonetheless, there can be a phase shift. The amplitude of the reflected wave for both modes can be written in the following form

$$E_r = e^{i\delta} = \frac{ae^{-i\alpha}}{ae^{i\alpha}} \tag{36}$$

where $a$ is the magnitude of the numerator and denominator and $\alpha$ is the phase (note that the numerator and denominator are complex conjugates of each other, which holds for both TE and TM waves). For the TE wave the denominator is

$$ae^{i\alpha} = \cos\theta_i + i\sqrt{\sin^2\theta_i - n^2} \tag{37}$$

in addition, since $2\alpha = \delta_s$ then $\tan\delta_s/2 = \tan\alpha$ where

$$\tan\alpha = \tan\left(\frac{\delta_s}{2}\right) = \frac{\sqrt{\sin^2\theta_i - n^2}}{\cos\theta_i} \tag{38}$$

with a similar expression for the TM wave

$$\tan\left(\frac{\delta_p}{2}\right) = \frac{\sqrt{\sin^2\theta_i - n^2}}{n^2\cos\theta_i}. \tag{39}$$

The relative phase shift between the two polarizations is

$$\tan\left(\frac{\delta_p - \delta_s}{2}\right) = \frac{\cos\theta_i\sqrt{\sin^2\theta_i - n^2}}{\sin^2\theta_i} \tag{40}$$

where the relation

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} \tag{41}$$

was used.

### 1.4 Reflection and Transmission In Absorptive Media

As a final example of reflection and transmission of waves at a boundary, we will consider the case of waves incident from a non-absorptive dielectric media onto an absorptive media. We previously showed that the general form of the index of refraction is complex as is the general form of the wavevector for linear media. Of course this will hold for any media. We therefore write the complex index of refraction and wavevectors as

$$\vec{\mathcal{K}} = \vec{\mathbf{k}} + i\vec{\boldsymbol{\alpha}} \qquad\qquad \mathcal{N} = n + i\kappa \tag{42}$$

To determine the reflection and transmission coefficients we apply boundary conditions between the incident, transmitted, and reflected waves. The first condition that we get out of this is that the phases of the three waves must be equal in order for the boundary conditions on the amplitudes to be valid for all times and all points on the boundary. The incident and reflected waves give the following condition

$$\vec{\mathbf{k}}_i \cdot \vec{\mathbf{r}} = \vec{\mathbf{k}}_r \cdot \vec{\mathbf{r}} \quad\Rightarrow\quad \theta_i = \theta_r \tag{43}$$

as we found for dielectric media. The condition for incident and transmitted gives

$$\vec{\mathbf{k}}_i \cdot \vec{\mathbf{r}} = \vec{\mathcal{K}} \cdot \vec{\mathbf{r}} \quad\Rightarrow\quad \begin{cases} \vec{\mathbf{k}}_i \cdot \vec{\mathbf{r}} = \vec{\mathbf{k}}_r \cdot \vec{\mathbf{r}} \\ \vec{\boldsymbol{\alpha}} \cdot \vec{\mathbf{r}} = 0 \end{cases} \tag{44}$$

The top equation on the right is Snell's law, while the second condition states that the attenuation of the wave occurs in a direction perpendicular to the surface (see Fig. **??**. Note that the transmitted wave is non-homogeneous in that the amplitude varies along a point of constant phase.

As stated above, the top equation on the right is Snell's law

$$k_i \sin\theta_i = k_t \sin\theta_t \tag{45}$$

If $k_t$ were a constant, then the transmitted angle could be calculated, but for non-homogeneous waves this is not the case. To derive the relation between the transmitted wavevector and the other variables, we start with the wave equation

$$\nabla^2\vec{\mathbf{E}} = \frac{\mathcal{N}^2}{c^2}\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2} \tag{46}$$

which for a plane wave leads to

$$\vec{\mathcal{K}} \cdot \vec{\mathcal{K}} = \frac{\mathcal{N}^2\omega^2}{c^2} = \mathcal{N}^2 k_0 \quad \text{where} \quad k_0 = \frac{\omega}{c} \tag{47}$$

Next we expand this expression out in terms of real and imaginary parts

$$(\vec{\mathbf{k}} + i\vec{\boldsymbol{\alpha}}) \cdot (\vec{\mathbf{k}} + i\vec{\boldsymbol{\alpha}}) = (n + i\kappa)^2 k_0^2 \quad \Rightarrow \quad \begin{cases} k^2 - \alpha^2 = (n^2 - \kappa^2) k_0^2 \\ \vec{\mathbf{k}} \cdot \vec{\boldsymbol{\alpha}} = k\alpha \cos\theta_t = n\kappa k_0^2 \end{cases} \tag{48}$$

After some algebraic manipulation, these expressions can be written in the following form

$$k \cos\theta_t + i\alpha = k_0 \sqrt{\mathcal{N}^2 - \sin^2\theta_i} \tag{49}$$

Now we express the complex index of refraction in a purely formal manner

$$\mathcal{N} = \frac{\sin\theta_i}{\sin\tilde{\theta}_t} \tag{50}$$

where $\tilde{\theta}_t$ is a complex angle with no simple physical interpretation and for simplicity we have assumed $n_1 = 1$. This equation can be rewritten as follows

$$\cos\tilde{\theta}_t = \sqrt{1 - \frac{\sin^2\theta_i}{\mathcal{N}^2}} \quad \Rightarrow \quad \mathcal{N}\cos\tilde{\theta}_t = \sqrt{\mathcal{N}^2 - \sin\theta_i} \tag{51}$$

Finally combine this equation with Eq. 49, to arrive at the fairly simple relation between $k$ and $\mathcal{N}$

$$\mathcal{N} = \frac{k_t \cos\theta_t + i\alpha}{k_0 \cos\tilde{\theta}_t} \tag{52}$$

Now we can start to calculate the transmission and reflection coefficients. First the relation between the magnetic and electric fields is given as before

$$\vec{\mathbf{H}} = \frac{1}{\mu_0\omega} \vec{\mathcal{K}} \times \vec{\mathbf{E}} \quad \Rightarrow \quad \begin{cases} H = \dfrac{1}{\mu_0\omega} E_0 k_0 \\ H = \dfrac{1}{\mu_0\omega}(k_t \cos\theta_t + i\alpha) E_r \end{cases} \tag{53}$$

where the first equation holds for both the incident and reflected waves, and the second holds only for the transmitted wave; note that the direction of the electric field in the absorptive media is parallel to the interface. Next we apply the boundary conditions. As before we use the component of the fields parallel to the surface. For the TE mode, the boundary conditions are

$$\begin{aligned} &1 + E_r = E_t \\ &H \cos\theta_i - H_r \cos\theta_r = H \quad \Rightarrow \quad k_0 \cos\theta_i - k_0 E_r \cos\theta_i = (k \cos\theta_t + i\alpha) E_t = \mathcal{N} k_0 E_t \cos\tilde{\theta}_t \end{aligned} \tag{54}$$

where we take the electric field of the incident wave to be 1. At this point we can solve for the transmitted and reflected fields, but as before only the reflected field is necessary

$$E_r = \frac{\cos\theta_i - \mathcal{N}\cos\tilde{\theta}_t}{\cos\theta + \mathcal{N}\cos\tilde{\theta}_t} \tag{55}$$

which is similar in form to the case for dielectrics. The TM case leads to

$$E_r = \frac{-\mathcal{N}\cos\theta_i + \cos\tilde{\theta}_t}{\mathcal{N}\cos\theta_i + \cos\tilde{\theta}_t} \tag{56}$$

To calculate the reflected intensity, we use the previously derived expression intensity and take the ratio of the reflected to incident

$$I = \left\langle \vec{\mathbf{S}} \right\rangle = \frac{1}{2}(\vec{\mathbf{E}} \times \vec{\mathbf{H}}^*) \quad \Rightarrow \quad \frac{I_r}{I_i} = |E_r|^2 \tag{57}$$

# Physics 4183 Electricity and Magnetism II
# Waveguides

## 1 Introduction

An important application of electrodynamics in bounded regions is that of guiding signals from location to location through the use of wave guides. Waveguides can be as simple as a wire or trace on a printed circuit board, or more complex, such as coaxial cables, rectangular waveguides, and fiber optic. All of these rely on the same principles requiring that the fields match boundary conditions on the various surface.

The discussion in this set of lectures will concentrate on rectangular waveguides. We will start the discussion by giving a physical description of how a wave travels through the guide by using the case a pair of conducting parallel plates. From this point we will move on to formally solve the problem by applying the boundary conditions required by the Maxwell equations.

### 1.1 Propagation Between Parallel Conducting Plates

As a simple physical problem, let's consider an electromagnetic wave confined between two parallel conducting planes separated by a distance $b$ (see Fig. 1). As for any other case where we have waves incident on an interface, the problem can be broken up into two case, the TE case where the electric field is perpendicular to the plane of the wave vectors, and the TM case where the magnetic field is perpendicular to the plane of the wave vectors.

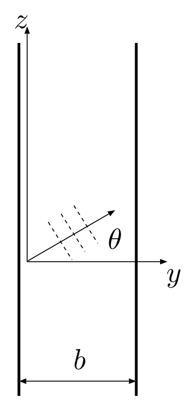


Figure 1: The figure shows the wave along with its direction confined between two parallel conducting plates. The parallel dashed lines correspond to the plane of constant phase, which is also the plane the fields are in.

To determine the conditions that the conducting plates impose on the wave, let's take a wave that is incident on the plates making an angle $\theta$ with respect to the $y$-axis and propagates in the $y$-$z$ plane. From our previous discussion, and assuming the plates are perfect conductors ($\sigma = \infty$)

the amplitude of the reflected waves is

$$E_r^s = \frac{n_1 \cos\theta_i - \sqrt{\mathcal{N}^2 - n_1^2 \sin\theta_i}}{n_1 \cos\theta_i + \sqrt{\mathcal{N}^2 - n_1^2 \sin\theta_i}} = -E_i^s \qquad E_r^p = \frac{-\mathcal{N}^2 \cos\theta_i + \sqrt{\mathcal{N}^2 - n_1^2 \sin\theta_i}}{\mathcal{N}^2 \cos\theta_i + \sqrt{\mathcal{N}^2 - n_1^2 \sin\theta_i}} = -E_i^p \tag{1}$$

where $\mathcal{N}$ is the complex index of refraction of media 2 (the conducting walls), and we have used the approximation for a good conductor

$$\mathcal{N} \approx (1+i)\sqrt{\frac{\mu\sigma}{2\omega\mu_0\epsilon_0}} \tag{2}$$

Notice that in both cases the amplitude ($E_0$) is inverted, this allows for the boundary conditions to be satisfied at the surface. The component of the electric field parallel to the surface cancel (incident + reflected). This is required since the parallel component of the field must be continuous across the interface and it must be zero inside a conductor. The components of the electric field normal to the surface for the incident and reflected waves add, since a charge is expected to be induced on the surface. The remaining discussion will focus only on the TE case, the TM case is similar. The wave-vector for the incident wave is

$$\vec{\mathbf{k}}_i = k_0(\hat{\mathbf{y}}\cos\theta + \hat{\mathbf{z}}\sin\theta) \tag{3}$$

and for the reflected wave it is

$$\vec{\mathbf{k}}_r = k_0(-\hat{\mathbf{y}}\cos\theta + \hat{\mathbf{z}}\sin\theta) \tag{4}$$

The electric field associated with the TE wave between the plates is

$$\vec{\mathbf{E}} = \hat{\mathbf{x}}E_0\left[e^{ik_0(y\cos\theta + z\sin\theta - \omega t)} - e^{ik_0(-y\cos\theta + z\sin\theta - \omega t)}\right] \tag{5}$$

At $y = 0$ the electric field has to vanish since it is parallel to the surface, inside a conductor there is no field, and the boundary conditions require it to be continuous (as stated above). This condition is automatically satisfied, since we used the calculated reflection amplitude, which already had this condition imposed. Next, the field is required to be zero at $y = b$ for the same reason as at $y = 0$. First we rewrite Eq. 5 as follows

$$\vec{\mathbf{E}} = \hat{\mathbf{x}}E_0\left[e^{ik_0 y\cos\theta} - e^{-ik_0 y\cos\theta}\right]e^{i(k_0 z\sin\theta - \omega t)} = \hat{\mathbf{x}}E_0\left[2i\sin(k_0 y\cos\theta)\right]e^{i(k_0 z\sin\theta - \omega t)} \tag{6}$$

From this expression, the condition that the field be zero at $y = b$, is easily seen to be

$$\sin(k_0 b\cos\theta) = 0 \quad \Rightarrow \quad k_0 b\cos\theta = n\pi \quad \text{with} \quad n = 1,\, 2,\, 3,\, \ldots \tag{7}$$

where we ignore $n = 0$, since it leads to a trivial solution; $\vec{\mathbf{E}} = 0$ for all times and all positions. Notice that we have set up a standing wave in the $y$ direction, but the wave continues to propagate along the $z$-axis.

To understand what the boundary condition imposes on the wave as it propagates between the plates, we write the wave-vector as a two wavelengths: One associated with the standing wave, and one with the propagating wave. The wavelength associated with the $y$-direction (standing wave) is

$$k_c = k_0\cos\theta \quad \Rightarrow \quad \frac{2\pi}{\lambda_c} = \frac{2\pi}{\lambda_0}\cos\theta \quad \Rightarrow \quad \lambda_c = \frac{2\pi}{k_0\cos\theta} = \frac{\lambda_0}{\cos\theta} \quad \text{where} \quad \lambda_0 = \frac{2\pi}{k_0} \tag{8}$$

and $\lambda_0$ is the wavelength in vacuum. The wavelength associated with the $z$-direction (propagation) is

$$k_g = k_0 \sin\theta \quad \Rightarrow \quad \lambda_g = \frac{2\pi}{k_0 \sin\theta} = \frac{\lambda_0}{\sin\theta} \tag{9}$$

Notice that the three wavelengths are related by

$$k_0^2 = k_c^2 + k_g^2 \quad \Rightarrow \quad \frac{1}{\lambda_c^2} + \frac{1}{\lambda_g^2} = \frac{1}{\lambda_0^2} \quad \Rightarrow \quad \frac{1}{\lambda_0^2} - \frac{1}{\lambda_c^2} = \frac{1}{\lambda_g^2} \tag{10}$$

The value of $\lambda_c$ is fixed by the boundary conditions, and the value of the incident angle. If we take $n = 1$, this gives $\lambda_c = 2b/n = 2b$, next increase $\lambda_0$ such that $\lambda_g$ is negative, the wave no longer propagates, but is extinguished since $\lambda_g$ is imaginary

$$\vec{\mathbf{E}} = \hat{\mathbf{x}} E_0 \left[\sin(2\pi y/\lambda_c)\right] e^{-2\pi z/\lambda_g} e^{-i\omega t} \quad \text{where} \quad E_0 \to 2iE_0 \quad \text{and} \quad \lambda_0 > \lambda_c \tag{11}$$

Another interesting phenomena occurs in the waveguide. The phase velocity is greater than $c$. This can be seen by noticing that the phase in the $z$ direction has to advance in the same time a longer distance than in the $\hat{\mathbf{k}}$ direction. The phase velocity along the waveguide is

$$v_p = \frac{\lambda_0}{\sin\theta} \frac{1}{\Delta t} = \frac{c}{\sin\theta} \tag{12}$$

The group velocity is given by $d\omega/dk_g$. Using Eq. 10, the group velocity is found to be

$$ck_0 = \omega = c\sqrt{k_g^2 - k_c^2} \quad \Rightarrow \quad v_g = \frac{d\omega}{dk_g} = c\frac{k_g}{k_0} = c \sin\theta \tag{13}$$

notice that the $v_g v_p = c^2$. The group velocity is the velocity at which energy (signals) propagate, therefore being consistent with special relativity.

## 1.2 Waves in Hollow Conductors

We now consider the general case of an electromagnetic wave bounded by conducting surfaces on all sides; the surfaces are assumed to be perfect conductors and the bounded media vacuum. The only condition that we will impose is that the bounding surface have a uniform cross section. We start with the boundary conditions on the surface of the conductor

$$\vec{\mathbf{E}}_{\parallel} = 0 \qquad \vec{\mathbf{B}}_{\perp} = 0 \tag{14}$$

$$\vec{\mathbf{E}}_{\perp} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \qquad \vec{\mathbf{B}}_{\parallel} = \mu_0 \vec{\mathbf{K}} \times \hat{\mathbf{n}} \tag{15}$$

The boundary conditions on the first line determine the shape of the field. The second set determine the induced currents and charges of the boundaries once the fields are known.

To determine the wave solutions that propagate through the guide, we align the $z$-axis of our coordinate system along the axis of the guide. Since the wave is assumed to propagate in the $z$-direction, we assume that the fields associated with the wave have the form

$$\vec{\mathbf{E}} = \vec{\boldsymbol{\mathcal{E}}}(x, y)\, e^{i(k_g z - \omega t)} \qquad \vec{\mathbf{B}} = \vec{\boldsymbol{\mathcal{B}}}(x, y)\, e^{i(k_g z - \omega t)} \tag{16}$$

where I have used the notation $k_g$ from the last section. With this assumed form of the field, the Maxwell equations between the conductors in Cartesian coordinates take on the following forms: The divergence of the electric field

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{E}_x}{\partial x} + \frac{\partial \mathcal{E}_y}{\partial y} + ik_g \mathcal{E}_z = 0 \tag{17}$$

The divergence of the magnetic field

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}} = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{B}_x}{\partial x} + \frac{\partial \mathcal{B}_y}{\partial y} + ik_g \mathcal{B}_z = 0 \tag{18}$$

The curl of the electric field

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \quad \Rightarrow \quad \begin{cases} \dfrac{\partial \mathcal{E}_z}{\partial y} - ik_g \mathcal{E}_y = i\omega \mathcal{B}_x \\ ik_g \mathcal{E}_x - \dfrac{\partial \mathcal{E}_z}{\partial x} = i\omega \mathcal{B}_y \\ \dfrac{\partial \mathcal{E}_y}{\partial x} - \dfrac{\partial \mathcal{E}_x}{\partial y} = i\omega \mathcal{B}_z \end{cases} \tag{19}$$

The curl of the magnetic field

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = \frac{1}{c^2}\frac{\partial \vec{\mathbf{E}}}{\partial t} \quad \Rightarrow \quad \begin{cases} \dfrac{\partial \mathcal{B}_z}{\partial y} - ik_g \mathcal{B}_y = -\dfrac{i\omega}{c^2} \mathcal{E}_x \\ ik_g \mathcal{B}_x - \dfrac{\partial \mathcal{B}_z}{\partial x} = -\dfrac{i\omega}{c^2} \mathcal{E}_y \\ \dfrac{\partial \mathcal{B}_y}{\partial x} - \dfrac{\partial \mathcal{B}_x}{\partial y} = -\dfrac{i\omega}{c^2} \mathcal{E}_z \end{cases} \tag{20}$$

To determine the fields in the waveguide, take the second of the curl of $\vec{\mathbf{E}}$ equations and the first of the curl of $\vec{\mathbf{B}}$ equations, and solve for $\mathcal{E}_x$ and $\mathcal{B}_y$ in terms of the $z$ component of the fields

$$\mathcal{E}_x = \frac{i}{k_c^2}\left(k_g \frac{\partial \mathcal{E}_z}{\partial x} + \omega \frac{\partial \mathcal{B}_z}{\partial y}\right) \tag{21}$$

$$\mathcal{B}_y = \frac{i}{k_c^2}\left(k_g \frac{\partial \mathcal{B}_z}{\partial y} + \frac{\omega}{c^2} \frac{\partial \mathcal{E}_z}{\partial x}\right) \tag{22}$$

Likewise, we can solve for $\mathcal{E}_y$ and $\mathcal{B}_x$ in terms of the $z$ component of the fields using the first of the curl of $\vec{\mathbf{E}}$ equations and the second of the curl of $\vec{\mathbf{B}}$ equations

$$\mathcal{E}_y = \frac{i}{k_c^2}\left(k_g \frac{\partial \mathcal{E}_z}{\partial y} - \omega \frac{\partial \mathcal{B}_z}{\partial x}\right) \tag{23}$$

$$\mathcal{B}_x = \frac{i}{k_c^2}\left(k_g \frac{\partial \mathcal{B}_z}{\partial x} - \frac{\omega}{c^2} \frac{\partial \mathcal{E}_z}{\partial y}\right) \tag{24}$$

where $k_c^2 = \omega^2/c^2 - k_g^2 = k_0^2 - k_g^2$. This set of equations give the transverse fields in terms of the longitudinal fields. We would still like a set of equations that contain only the longitudinal fields. These can be found by substituting $\mathcal{E}_x$ and $\mathcal{E}_y$ into Eq 17, and $\mathcal{B}_x$ and $\mathcal{B}_y$ into Eq. 18

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right)\mathcal{E}_z = 0 \qquad\qquad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right)\mathcal{B}_z = 0 \tag{25}$$

or in a more compact notion, which is more general

$$\left(\nabla_T^2 + k_c^2\right)\mathcal{E}_z = 0 \qquad \left(\nabla_T^2 + k_c^2\right)\mathcal{B}_z = 0 \tag{26}$$

where $\nabla_T^2$ is the transverse Laplacian operator, therefore we can work in any coordinate system we choose.

Let's now ask what type of waves can propagate down the guide. First we consider the case where the fields are transverse (TEM). In this case $\mathcal{E}_z = \mathcal{B}_z = 0$. The equations for the divergence of the fields in this case become

$$\frac{\partial \mathcal{E}_x}{\partial x} + \frac{\partial \mathcal{E}_y}{\partial y} = 0 \qquad \frac{\partial \mathcal{B}_x}{\partial x} + \frac{\partial \mathcal{B}_y}{\partial y} = 0 \tag{27}$$

and the third of the two curl equations become

$$\frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y} = 0 \qquad \frac{\partial \mathcal{B}_y}{\partial x} - \frac{\partial \mathcal{B}_x}{\partial y} = 0 \tag{28}$$

The curl equations imply that each of the transverse fields can be written as potentials

$$\mathcal{E}_x = -\frac{\partial \Phi_E}{\partial x} \qquad \mathcal{E}_y = -\frac{\partial \Phi_E}{\partial y} \qquad \mathcal{B}_x = -\frac{\partial \Phi_B}{\partial x} \qquad \mathcal{B}_y = -\frac{\partial \Phi_B}{\partial y} \tag{29}$$

Since the bounding surface is a conductor, it is at a constant electric potential. Therefore, the electric field along the surface is zero. Equation 27 states that the electric field is constant inside the region bounded by the conductors. Therefore, the electric field must be zero everywhere. Using the curl equations for the electric and magnetic fields, the magnetic field is likewise zero. Therefore, we must conclude that no TEM wave can propagate through a hollow waveguide; we have assumed that there is only one continuous conducting surface, so only in this case does this hold.

Let's now consider the other two possible modes, TE where $\mathcal{E}_z = 0$, and TM where $\mathcal{B}_z = 0$. For the TE mode, Eqs. 21-24 become

$$\begin{aligned} \mathcal{E}_x &= c\frac{ik_0}{k_c^2}\frac{\partial \mathcal{B}_z}{\partial y} \qquad & \mathcal{E}_y &= -c\frac{ik_0}{k_c^2}\frac{\partial \mathcal{B}_z}{\partial x} \\ \mathcal{B}_x &= \frac{ik_g}{k_c^2}\frac{\partial \mathcal{B}_z}{\partial x} \qquad & \mathcal{B}_y &= \frac{ik_g}{k_c^2}\frac{\partial \mathcal{B}_z}{\partial y} \end{aligned} \tag{30}$$

The lower set of equations lead to the following relation between the transverse gradient of the longitudinal magnetic field and the transverse magnetic field components

$$\vec{\boldsymbol{\nabla}}_T \mathcal{B}_z = -\frac{ik_c^2}{k_g}\vec{\boldsymbol{\mathcal{B}}}_t \tag{31}$$

Combining this equation with the upper equations of Eq. 30, leads to

$$\vec{\boldsymbol{\mathcal{E}}}_t = -\frac{ck_0}{k_g}(\hat{\mathbf{z}} \times \vec{\boldsymbol{\mathcal{B}}}_t) \tag{32}$$

A similar set of relations can be derived for the TM case, where $\mathcal{B}_z = 0$

$$\vec{\boldsymbol{\mathcal{B}}}_t = \frac{k_0}{k_g c}(\hat{\mathbf{z}} \times \vec{\boldsymbol{\mathcal{E}}}_t) \qquad \vec{\boldsymbol{\nabla}}_T \mathcal{E}_z = -\frac{ik_c^2}{k_g}\vec{\boldsymbol{\mathcal{E}}}_t \tag{33}$$

These two sets of equations (Eqs. 31-33), along with the differential equations

$$(\nabla_T^2 + k_c^2)\mathcal{B}_z = 0 \quad \text{TE} \tag{34}$$

$$(\nabla_T^2 + k_c^2)\mathcal{E}_z = 0 \quad \text{TM} \tag{35}$$

and the boundary conditions

$$\begin{aligned} \vec{\mathbf{E}}_{\|}|_S = \hat{\mathbf{n}} \times \vec{\mathbf{E}}|_S = 0 \\ \vec{\mathbf{B}}_{\perp}|_S = \hat{\mathbf{n}} \cdot \vec{\mathbf{B}}|_S = 0 \end{aligned} \tag{36}$$

give the fields in the waveguide. It should be clear from these equations that the TE fields are completely defined in terms of $\mathcal{B}_z$, while the TM fields are completely defined in terms of $\mathcal{E}_z$. To see what these boundary conditions mean, let's define a local coordinate system on the inner wall of the waveguide as follows

- $z$ is parallel to the axis of the waveguide;
- $y$ is normal to the surface;
- $x$ is parallel to the surface, and normal to the $y$-$z$ plane ($\hat{\mathbf{y}} \times \hat{\mathbf{z}}$).

For the TE mode ($\mathcal{E}_z = 0$, Eq. 36 imply that $\mathcal{E}_x = 0$ and $\mathcal{B}_y = 0$. But Eq. 30 says that these two components are proportional to $\frac{\partial \mathcal{B}_z}{\partial y}$, which leads to

$$\left.\frac{\partial \mathcal{B}_z}{\partial n}\right|_S \equiv \left.\frac{\partial \mathcal{B}_z}{\partial y}\right|_S = 0 \tag{37}$$

For the TM case ($\mathcal{B}_z = 0$), the boundary conditions (Eq. 36) require that $\mathcal{E}_x|_S = 0$, $\mathcal{E}_z|_S = 0$, and $\mathcal{B}_y = 0$. We find from Eq. 21 that

$$\frac{\partial \mathcal{E}_z}{\partial x} = 0 \tag{38}$$

which is automatically implied, since $\mathcal{E}_z = 0$ and the derivative is along the surface. The only condition for the TM mode is

$$\mathcal{E}_z|_S = 0 \tag{39}$$

### 1.3 Rectangular Waveguide

The most common shape of a hollow waveguide is rectangular, which we will consider in this section. We assume that the waveguide has dimensions $a \times b$ with $a > b$, and that the walls are perfect conductors forming an equipotential surface. The most common waveguides are used for transporting TE waves, therefore we start with the TE case. The longitudinal ($z$) component of the magnetic field, from which all the other components can be calculated, is found by solving the differential equation

$$(\nabla_T^2 + k_c^2)\mathcal{B}_z = 0 \tag{40}$$

Since the there are no cross terms involving the coordinates, the equation can be solved by separation of variables where we assume a solution of the form $\mathcal{B}_z(x, y) = X(x)Y(y)$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) X(x)Y(y) = 0 \quad \Rightarrow \quad \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = -k_c^2 \tag{41}$$

Since the two terms on the right hand side depend on a single variable, the only way that the equality can be satisfied is if the two terms are independently equal to a constant

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = -\alpha^2 \qquad \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = -\beta^2 \quad \text{where} \quad k_c^2 = \alpha^2 + \beta^2 \tag{42}$$

The two differential equations are trivial to solve, the solutions being

$$X(x) = A\sin\alpha x + B\cos\alpha x \qquad Y(y) = C\sin\beta y + D\cos\beta y \tag{43}$$

We now need to apply the boundary conditions. These are given by Eq. 36. Along the surfaces at $x = 0$ and $x = a$, the boundary conditions are

$$\mathcal{E}_y = 0 \qquad \mathcal{B}_x = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{B}_z}{\partial x} = 0 \tag{44}$$

where the equality comes from Eqs. 30. Along the surfaces at $y = 0$ and $y = b$, the boundary conditions are

$$\mathcal{E}_x = 0 \qquad B_y = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{B}_z}{\partial y} = 0 \tag{45}$$

Starting with the surface at $x = 0$, we get

$$\frac{\partial \mathcal{B}_z}{\partial x} = A\alpha\cos\alpha x - B\alpha\sin\alpha x \quad \Rightarrow \quad \left.\frac{\partial \mathcal{B}_z}{\partial x}\right|_{x=0} = 0 = A\alpha \quad \Rightarrow \quad A = 0 \tag{46}$$

where we impose the condition on $A$ not $\alpha$ to avoid getting a trivial solution. The boundary conditions at $x = a$ are

$$\frac{\partial \mathcal{B}_z}{\partial x} = -B\alpha\sin\alpha x \quad \Rightarrow \quad \left.\frac{\partial \mathcal{B}_z}{\partial x}\right|_{x=0} = 0 = -B\alpha\sin\alpha a \quad \Rightarrow \quad \alpha = \frac{n\pi}{a} \tag{47}$$

again we select the condition such that we don't get a trivial solution. Therefore, the solution for $X(x)$ is

$$X(x) \propto \cos\frac{n\pi}{a}x \tag{48}$$

We can carry through the same procedure on the surfaces at $y = 0$ and $y = b$ to get

$$Y(y) \propto \cos\frac{m\pi}{b}y \tag{49}$$

Therefore the $z$ component of the magnetic field is

$$\mathcal{B}_z = B_0\cos\frac{n\pi}{a}x\,\cos\frac{m\pi}{b}y \quad \text{with} \quad k_c^2 = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right) \tag{50}$$

with the longest allowed wavelength being $\lambda = 2a$, which corresponds to $n = 1$, $m = 0$ (lowest frequency $\omega = \pi c/a$).

To calculate the remaining components of the fields, we use Eqs. 30. The components for the $\mathrm{TE}_{10}$ mode are

$$\begin{aligned} \mathcal{E}_x &= 0 & \mathcal{B}_x &= -i\left[\frac{k_g a}{\pi}B_0\sin\frac{\pi}{a}x\right]\,e^{i(k_g z-\omega t)} \\ \mathcal{E}_y &= ic\left[\frac{k_0 a}{\pi}B_0\sin\frac{\pi}{a}x\right]\,e^{i(k_g z-\omega t)} & \mathcal{B}_y &= 0 \\ \mathcal{E}_z &= 0 & \mathcal{B}_z &= B_0\left[\cos\frac{\pi}{a}x\right]\,e^{i(k_g z-\omega t)} \end{aligned} \tag{51}$$

with the actual fields being the real part of these equations.

# Physics 4183 Electricity and Magnetism II
# Introduction to Radiation

## 1 Introduction

In the following set of lectures, we will discuss the generation of electromagnetic waves. So far we have discussed the properties of these waves as they propagate through different media, but we have not discussed how the waves were generated.

Since the discussion of radiation is most easily done using potentials, we will start with a discussion of the use of potentials in electrodynamics. From here, we will discuss the general problem of radiation from charge distributions and point particles. We will end the discussion by considering multipole radiation.

### 1.1 Radiation—A Simple Picture

Before we start the formal discussion of radiation, we will build a simple model to explain this phenomena. Select a coordinate system such that a particle of charge $q$ sits at the origin at time $t = 0$. We will assume that at that instant of time its velocity is zero, but a constant force is applied, which causes it to accelerate with acceleration $a$ along the $z$-axis. The force is applied for a time interval $\Delta t$, such that it is traveling at a velocity $v = a\Delta t$ and has traveled a distance $z_0 = \frac{1}{2}a(\Delta t)^2$ (see Fig. 1). The particle continues traveling at velocity $v$ for $t > \Delta t$.

Lets examine the field after a time $t \gg \Delta t$, when the particle is at position $z_1$. For radial distance from the origin greater than $ct$, the field lines point back to the origin, since the information that the particle has accelerated travels at the speed of light $c$; the information has not reached this distance yet. For radial distance less than $c(t - \Delta t)$, the field lines point to the current position of the uniformly moving particle. Since the field lines can only terminate on charges, and the only charge in the system is generating the field, the two sets of field lines must connect in the shell of thickness $c\Delta t$ (the kink in Fig. 1).

We are now in a position to calculate the fields in the transition region. From the geometry in Fig. 1, we see that the ratio of polar too radial field is

$$\frac{E_\theta}{E_r} \approx \frac{vt\sin\theta}{c\Delta t} = a\left(\frac{r}{c^2}\right)\sin\theta \tag{1}$$

where we have used $a = v/\Delta t$ and $r = ct$. As time increases, the ratio increases, that is the polar component of the field increase more rapidly than the radial component of the field. The radial field must always obey Gauss's law, the radial flux through a closed shell about the charge must be a constant equal to $q/\epsilon_0$. Therefore, the radial field is

$$E_r = \frac{q}{4\pi\epsilon_0 r^2} \tag{2}$$

The polar field is therefore

$$E_\theta = \frac{aq}{4\pi\epsilon_0 c^2 r}\sin\theta \tag{3}$$

Notice that along the $z$-axis ($\theta = 0$ and $\theta = \pi$) the polar component of the field is zero, therefore no radiation in these direction. The maximum polar component occurs at 90° to the acceleration.

Another important point to note, is that the polar field falls off as $1/r$, not $1/r^2$. We will see below that this is a very important point.

Next we investigate the radiated power. To do so, we first need to calculate the magnetic field. Notice that the wave is propagating radially outward from the origin (also from the current position of the charge). Therefore, the wave vector is $\vec{\mathbf{k}} = k\hat{\mathbf{r}}$, and the magnetic field associated with the pulse is

$$\omega\vec{\mathbf{B}} = \vec{\mathbf{k}} \times \vec{\mathbf{E}} \quad \Rightarrow \quad \vec{\mathbf{B}} = \frac{1}{c}\,\hat{\mathbf{r}} \times \vec{\mathbf{E}} \quad \Rightarrow \quad B_\phi = \frac{aq}{4\pi\epsilon_0 c^3}\frac{\sin\theta}{r} \tag{4}$$

notice that we are assuming that the pulse is sufficiently far from the source, such that the wave approximates a plane wave. The infinitesimal power carried radially outward at a radius $r = ct$ is

$$dP = \vec{\mathbf{S}} \cdot \hat{\mathbf{r}}\, r^2 d\Omega \tag{5}$$

The radial intensity is

$$\vec{\mathbf{S}} \cdot \hat{\mathbf{r}} = \left(\frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}}\right) \cdot \hat{\mathbf{r}} = \frac{1}{\mu_0}(\hat{\mathbf{r}} \times \vec{\mathbf{E}}) \cdot \vec{\mathbf{B}} = \epsilon_0 c E_\theta^2 = \frac{a^2 q^2}{(4\pi)^2 \epsilon_0 c^3 r^2}\sin^2\theta \tag{6}$$

notice the $1/r^2$ dependence. Now integrate over the surface at $r = ct$ and multiply by $\Delta t$ to get the total energy

$$\Delta U = P\Delta t = \left[\frac{a^2 q^2}{(4\pi)^2\epsilon_0 c^3}\, 2\pi \int_0^\pi \sin^3\theta\, d\theta\right] \Delta t = \frac{q^2}{4\pi\epsilon_0}\left(\frac{2a^2}{3c^3}\right)\Delta t \tag{7}$$

which is referred to as the Larmor formula. Notice the following items, if the field in the pulse has a $1/r^2$ dependence like the static Coulomb field, then the energy goes to zero at infinity implying no radiation; the energy radiated depends on the square of the acceleration and the duration of the acceleration.

As a simple example, consider a 1 C charge accelerated at 100 times the acceleration of gravity for 100 s. The total radiated energy is $10^{-7}$ J (or $10^{-9}$ W). Therefore, to radiate significant amounts of energy, requires large charges and/or large accelerations.

As another example, let's consider a plane wave incident on an atom. Since the proton mass is approximately 2000 times more than the electron mass, and most atoms of interest contain a large number of protons and neutrons, we can assume that the electric field in the wave acts primarily on the electron. Therefore, we can assume that the oscillating electric field acts only on the electrons. The equation of motion for the electron due to the driving electric field is

$$m\frac{d^2 z}{dt^2} + m\omega_0^2 z = qE_z \cos\omega t \tag{8}$$

where $\omega_0$ is the natural frequency of the oscillator which can be taken as the energy associated with the bound state energy of the electron, and $\omega$ is the driving frequency. If $\omega \ll \omega_0$, then we can ignore the inertial term $ma$, and the motion is entirely given by the electric field in the wave

$$z = \frac{qE_z}{m\omega_0^2}\cos\omega t \tag{9}$$

In order to calculate the radiated power by the atom, we need to determine the acceleration

$$\frac{d^2 z}{dt^2} = \frac{eE_z\omega^2}{m\omega_0^2}\cos\omega t \tag{10}$$

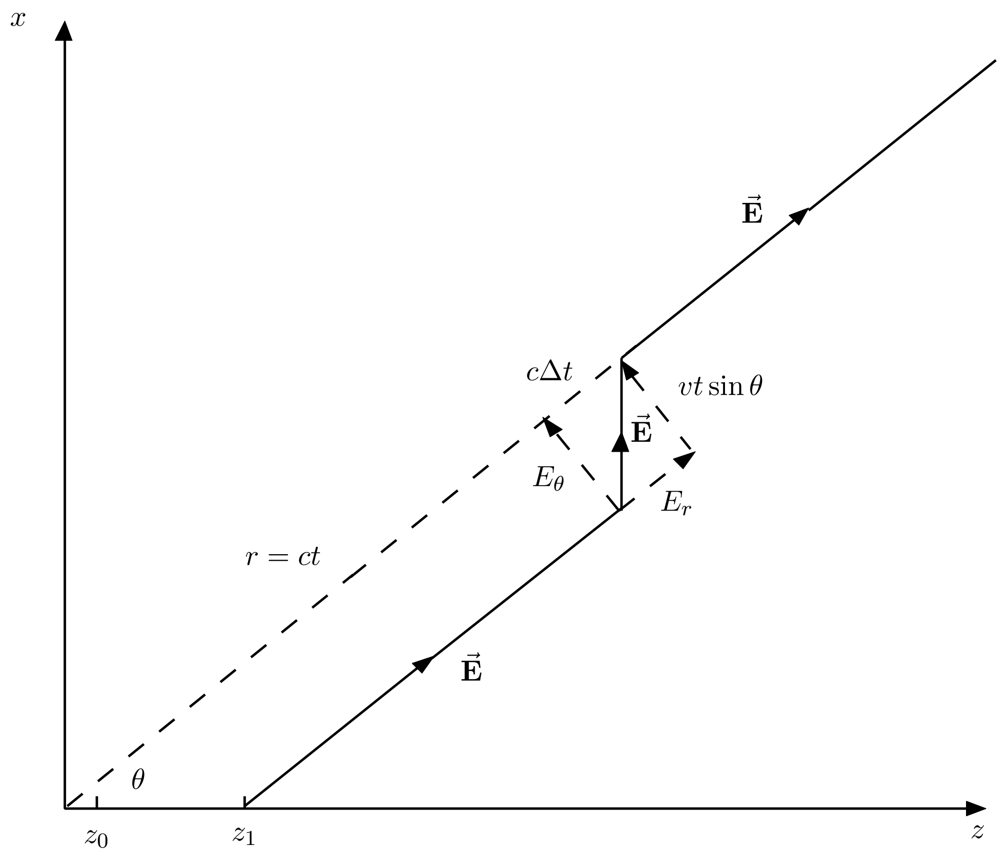


Figure 1: This figure depicts a single electric field line from a point source that is accelerating. The field line for distances less than $ct$ points back to the charge, but those that are farther than $c(t + \Delta t$ point back to the original position of the charge, with a transition period in between that corresponds to a wave front.

where we have made the substitution $q = -e$. Substituting the acceleration into the Larmor formula we get

$$P = \frac{dU}{dt} = \frac{1}{4\pi\epsilon_0}\frac{2e^4E^2}{3c^3m_e^2}\left(\frac{\omega}{\omega_0}\right)^4\cos^2\omega t \tag{11}$$

taking the time average

$$\langle P\rangle = \frac{1}{4\pi\epsilon_0}\frac{2e^4E^2}{3c^3m_e^2}\left(\frac{\omega}{\omega_0}\right)^4\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\cos^2\omega t\,dt = \frac{1}{4\pi\epsilon_0}\frac{e^4E^2}{3c^3m_e^2}\left(\frac{\omega}{\omega_0}\right)^4 \tag{12}$$

Therefore, the higher the frequency, the more power radiated by the atom. Taking light with a uniform frequency distribution incident on our atmosphere, the light along the direction of the source will appear red, while that away from the source is blue; note that $\omega_{\text{blue}} = 1.8\omega_{\text{red}}$. This is why the sky is blue.

### 1.2 Electrodynamics and Potentials

All electrodynamic phenomena can be derived either using the fields, as given by the Maxwell equation, or using potentials. When discussing radiation, it is usually easier to start with the potentials, and then derive the field from the potentials if necessary. Let's start by writing down the Maxwell equations

$$\vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{13}$$
$$\vec{\nabla} \cdot \vec{\mathbf{B}} = 0 \qquad \vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}} + \mu_0 \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

Since the divergence of a curl is zero, we can define the magnetic field in terms of a vector potential

$$\vec{\mathbf{B}} = \vec{\nabla} \times \vec{\mathbf{A}} \tag{14}$$

Now substitute this expression for the magnetic field into Faraday's law

$$\vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial}{\partial t} \vec{\nabla} \times \vec{\mathbf{A}} \quad \Rightarrow \quad \vec{\nabla} \times \left( \vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{A}}}{\partial t} \right) = 0 \tag{15}$$

Since the gradient of a curl is zero, we can write the electric field as the gradient of a scalar potential and the time derivative of the vector potential

$$-\vec{\nabla} V = \vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{A}}}{\partial t} \quad \Rightarrow \quad \vec{\mathbf{E}} = -\vec{\nabla} V - \frac{\partial \vec{\mathbf{A}}}{\partial t} \tag{16}$$

If $\vec{\mathbf{A}}$ is a constant in time, then we get back the electrostatic form of the field potential relation. Equations 14 and 16 satisfy the homogeneous Maxwell equations. The remaining two equations impose additional conditions on the potentials. Substituting Eq. 16 into Gauss's law yields

$$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{\mathbf{A}}) = -\frac{\rho}{\epsilon_0} \tag{17}$$

which reduces to Poisson's equation for a time independent potential $\vec{\mathbf{A}}$. Next, we substitute Eqs. 14 and 16 into the Ampere/Maxwell law

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{A}}) = \mu_0 \vec{\mathbf{J}} - \mu_0 \epsilon_0 \vec{\nabla} \left( \frac{\partial V}{\partial t} \right) - \mu_0 \epsilon_0 \frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} \tag{18}$$
$$\Rightarrow \quad \left( \nabla^2 \vec{\mathbf{A}} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} \right) - \vec{\nabla} \left( \vec{\nabla} \cdot \vec{\mathbf{A}} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} \right) = -\mu_0 \vec{\mathbf{J}}$$

where the vector relation

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{A}}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{\mathbf{A}}) - \nabla^2 \vec{\mathbf{A}} \tag{19}$$

was used to get the second relation from the first in Eq. 18.

### 1.3 Gauge Transformation

In the case of electrostatics, we noted that the potential is not uniquely define. A constant can be added to the potential without changing the electric field, and since it is the electric field that determines the forces acting on a charged particle, the potential is not unique. The same holds in the general case, except that a change to the scalar potential might require a simultaneous change to the vector potential.

Let's start by examining what happens if we change the vector potential by a constant $\vec{\mathbf{A}}' = \vec{\mathbf{A}} + \vec{\boldsymbol{\alpha}}$. The relation between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ doesn't change, since the derivative of a constant is zero. But we can give a more general form for $\vec{\boldsymbol{\alpha}}$, since we have to satisfy a curl equation

$$\vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times (\vec{\mathbf{A}} + \vec{\boldsymbol{\alpha}}) = \vec{\mathbf{B}} + \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\alpha}} \quad \Rightarrow \quad \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\alpha}} = 0 \quad \Rightarrow \quad \vec{\boldsymbol{\alpha}} = \vec{\boldsymbol{\nabla}}\lambda \tag{20}$$

But Eq. 16 must still be satisfied. We start be requiring that the scalar potential transform as $V' = V + \beta$, which must be a constant in terms of the spatial coordinates, in order to satisfy the conditions we found for electrostatics; that is is $\vec{\mathbf{A}}$ is time independent. Under the "gauge" transformation, Eq. 16 becomes

$$\vec{\mathbf{E}} = -\vec{\boldsymbol{\nabla}}(V + \beta) - \frac{\partial}{\partial t}(\vec{\mathbf{A}} + \vec{\boldsymbol{\nabla}}\lambda) \quad \Rightarrow \quad \vec{\mathbf{E}} = -\vec{\boldsymbol{\nabla}}V - \frac{\partial \vec{\mathbf{A}}}{\partial t} - \left(\vec{\boldsymbol{\nabla}}\beta + \frac{\partial}{\partial t}(\vec{\boldsymbol{\nabla}}\lambda)\right) \tag{21}$$

$$\Rightarrow \quad \beta = -\frac{\partial \lambda}{\partial t}$$

Therefore, the fields are invariant to the gauge transformations defined as

$$\vec{\mathbf{A}} \to \vec{\mathbf{A}} + \vec{\boldsymbol{\nabla}}\lambda \qquad\qquad V \to V - \frac{\partial \lambda}{\partial t} \tag{22}$$

For electrodynamics, the importance of gauge transformations is that they simplify Eqs. 17 and 18. But more generally, gauge transformations are the backbone of all modern physical theories when applied to quantum mechanics. If we require that the Hamiltonian be invariant to a gauge transformation, then the wave function must be invariant to a local phase transformation $e^{i\phi(\vec{\mathbf{r}})}$. If we extend this, we find that any local phase transformation requires that the free particle Hamiltonian satisfy a gauge transformation, which leads to an interaction term, that is a potential.

In solving problems, the two most commonly used gauges are the Coulomb and the Lorentz gauges. Notice that for the vector potential, we only specified its curl, which leaves the divergence arbitrary (a field is completely defined only when its curl and divergence are specified). Therefore, we can select the divergence in such a way as to simplify the problem we are working on. Let's consider the case when $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}} = 0$ (this is referred to as the Coulomb gauge), which is the same condition as in magnetostatics. Remember, if this is not satisfied, we can always make the transformation $\vec{\mathbf{A}} \to \vec{\mathbf{A}} + \vec{\boldsymbol{\nabla}}\lambda$ such that this holds. In this case, Eq. 17 reduces to the Poisson equation

$$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}) = -\frac{\rho}{\epsilon_0} \quad \Rightarrow \quad \nabla^2 V = -\frac{\rho}{\epsilon_0} \tag{23}$$

for which we know the solution

$$V(\vec{\mathbf{r}}, t) = \int_V \frac{\rho(\vec{\mathbf{r}}', t)}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}\, dV' \tag{24}$$

To calculate $\vec{\mathbf{E}}$, we use Eq. 16, which still require us to calculate $\vec{\mathbf{A}}$. Before doing so, notice that the potential calculated in Eq. 24, even though it depends on the time, changes instantaneously everywhere in space. This obviously doesn't make sense, except that what matters is the field and not the absolute value of the potential (the time dependence in the electric field comes from the time dependence on $\vec{\mathbf{A}}$). To derive the vector potential, we substitute $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}} = 0$ into Eq. 18

$$\begin{aligned} & \left(\nabla^2 \vec{\mathbf{A}} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2}\right) - \vec{\boldsymbol{\nabla}} \left(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \vec{\mathbf{J}} \\ \Rightarrow \quad & \left(\nabla^2 \vec{\mathbf{A}} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2}\right) = -\mu_0 \vec{\mathbf{J}} + \vec{\boldsymbol{\nabla}} \left(\mu_0 \epsilon_0 \frac{\partial V}{\partial t}\right) \end{aligned} \tag{25}$$

This equation is not particularly easy to solve.

The other commonly used guage is the Lorentz gauge. It imposes the requirement that the two differential equations (Eq. 17 and 18) decouple. Equation 18 tells us that this condition should be

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} = 0 \quad \Rightarrow \quad \left(\nabla^2 \vec{\mathbf{A}} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2}\right) = -\mu_0 \vec{\mathbf{J}} \tag{26}$$

If we impose this condition on Eq. 17, we arrive at

$$\nabla^2 V - \frac{\partial}{\partial t}(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}) = -\frac{\rho}{\epsilon_0} \quad \Rightarrow \quad \nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{27}$$

Therefore, we have two differential equations that represent waves with sources associated with them. If the potentials are in a form that does not satisfy the Lorentz condition, then we can carry out a gauge transformation

$$\begin{aligned} & \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} = 0 \\ \Rightarrow \quad & \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}' + \mu_0 \epsilon_0 \frac{\partial V'}{\partial t} + \nabla^2 \lambda - \frac{\partial^2 \lambda}{\partial t^2} = 0 \\ \Rightarrow \quad & \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}' + \mu_0 \epsilon_0 \frac{\partial V'}{\partial t} = -\left(\nabla^2 \lambda - \frac{\partial^2 \lambda}{\partial t^2}\right) \end{aligned} \tag{28}$$

where $\lambda$ satisfies the above inhomogeneous differential equation.

## 1.4 The Retarded Potentials

In the past we have made the comment that Coulomb's law and the Biot-Savart law are only valid for time independent sources (static charges and constant current respectively). But we have never actually stated the reason that this is so. In this section, we will generalize these laws to the case where the sources are not time independent, and in the process explain why the original forms are not valid for non-constant sources. We will start by making a physical argument to transform the potentials to the appropriate form, and follow this up by deriving the expression from the differential equations governing the potentials.

The potentials and fields are calculated at the field point, with the sources being some distance away. If the source changes, the field point will not know of the change for a time $\Delta t = R/c$, where

$R = |\vec{\mathbf{r}} - \vec{\mathbf{r}}'|$ is the distance from the source to the field point. Stated differently, it takes a finite amount of time for the signal to propagate between the two points. Therefore, we use the value of the source at the earlier time to calculate the field. But keep in mind, since each point in the source is a different distance from the field point, each source point has a different time associate with it. This time is referred to as the retarded time. Based on this simple argument, the potentials are

$$V(\vec{\mathbf{r}},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\mathbf{r}}',t_r)}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}\,d^3x \qquad \vec{\mathbf{A}}(\vec{\mathbf{r}},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{\mathbf{J}}(\vec{\mathbf{r}}',t_r)}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}\,d^3x \tag{29}$$

where $t_r$, the retarded time is

$$t_r \equiv t - \frac{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}{c} \tag{30}$$

the earlier time at the source, which accounts for the propagation speed of the signal. Note, that these arguments will not get you the correct fields, as can be verified by going back to the fields derived from using the Lorentz transformation.

Given the simplicity of the argument, one would like to verify that these in fact hold. One method of doing this, is to apply the differential equations associated with the potentials, and verify that these solutions hold. Alternatively, we can solve those differential equations, which we will do. We start with the equations using the Lorentz condition

$$\left.\begin{array}{l} \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0} \\ \nabla^2 \vec{\mathbf{A}} - \frac{1}{c^2}\frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = -\mu_0 \vec{\mathbf{J}} \end{array}\right\} \quad \Rightarrow \quad \nabla^2 f(\vec{\mathbf{r}},t) - \frac{1}{c^2}\frac{\partial^2 f(\vec{\mathbf{r}},t)}{\partial t^2} = -s(\vec{\mathbf{r}},t) \tag{31}$$

where all four equations have the same form, therefore we only need to solve one of them. Note also that the field and source point in these equations are the same. This equation can be simplified by expressing the field and source as an infinite series of harmonic terms (Fourier transform the time piece of the differential equation)

$$\nabla^2 F(\vec{\mathbf{r}},\omega) + k^2 F(\vec{\mathbf{r}},\omega) = -S(\vec{\mathbf{r}},\omega) \tag{32}$$

where $k = \omega/c$ is used and the time and frequency functions are

$$F(\vec{\mathbf{r}},\omega) = \int_{-\infty}^{+\infty} e^{i\omega t} f(\vec{\mathbf{r}},t)\,dt \quad \text{and} \quad f(\vec{\mathbf{r}},t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega t} F(\vec{\mathbf{r}},\omega)\,d\omega \tag{33}$$

with similar expressions for the source term. In arriving at the Eqs. 33, we used the following representation for the Dirac delta function

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega(t-t')}\,d\omega \tag{34}$$

The solution to Eq. 32 can be arrived at in the following manner: The source is a very large collection of point sources, determine the solution for a single point source then sum up the solutions for all the sources. A point source at $\vec{\mathbf{r}}'$ corresponds to a delta function

$$s(\vec{\mathbf{r}}') = -4\pi\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}') \tag{35}$$

where the $4\pi$ is a normalization factor that is necessary to satisfy Maxwell's equation when applied to Coulomb's law. Therefore, we solve the following differential equation

$$\nabla^2 G(\vec{\mathbf{r}}, \omega) + k^2 G(\vec{\mathbf{r}}, \omega) = -4\pi\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}') \tag{36}$$

which will only depend on the radial coordinate. The important component of the equation reduces to

$$\frac{\partial^2 G(\vec{\mathbf{r}}, \omega)}{\partial r^2} + \frac{2}{r}\frac{\partial G(\vec{\mathbf{r}}, \omega)}{\partial r} + k^2 G(\vec{\mathbf{r}}, \omega) = -4\pi\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}') \tag{37}$$

The solution to this equation requires a complex exponential to take care of the $k^2$ term, and a $1/r$ to handle the two derivatives. The solution is

$$G(\vec{\mathbf{r}}, \omega) = \frac{C_r e^{ikR} + C_a e^{-ikR}}{4\pi R} \tag{38}$$

where $R = |\vec{\mathbf{R}}| = |\vec{\mathbf{r}} - \vec{\mathbf{r}}'|$. This solution can be verified by substituting back into the differential equation (the differential equation is zero everywhere except $\vec{\mathbf{r}} = \vec{\mathbf{r}}'$). The potential at the point $\vec{\mathbf{r}}$ is then the sum of the field due to each point charge weighted by the source distribution

$$F(\vec{\mathbf{r}}, \omega) = \int \frac{C_r e^{ikR} + C_a e^{-ikR}}{4\pi R} S(\vec{\mathbf{r}}', \omega)\, d^3x' \tag{39}$$

where this is derived from

$$\begin{aligned}
&\int_V S(\vec{\mathbf{r}}', \omega)\left[\nabla^2 G(\vec{\mathbf{r}}, \omega) + k^2 G(\vec{\mathbf{r}}, \omega) = -4\pi\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}')\right]\, d^3x' \qquad (40)\\
&\nabla^2\left(\int_V S(\vec{\mathbf{r}}', \omega) G(\vec{\mathbf{r}}, \omega) d^3x'\right) + k^2\left(\int_V S(\vec{\mathbf{r}}', \omega) G(\vec{\mathbf{r}}, \omega)\, d^3x'\right) = -4\pi \int_V S(\vec{\mathbf{r}}', \omega)\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}')\, d^3x\\
&\nabla^2 F(\vec{\mathbf{r}}, \omega) + k^2 F(\vec{\mathbf{r}}, \omega) = -S(\vec{\mathbf{r}}, \omega)
\end{aligned}$$

noting that the derivative and integral are over different variables.

We next combine each of the harmonic components (take the inverse Fourier transform)

$$f(\vec{\mathbf{r}}, t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t} \int \frac{C_r e^{ikR} + C_a e^{-ikR}}{4\pi R}\, d^3x' \int_{-\infty}^{+\infty} e^{i\omega t'} s(\vec{\mathbf{r}}', t')\, dt' \tag{41}$$

We first calculate the frequency ($\omega$) integral. This requires changing $k$ to $\omega/c$ making the frequency integral

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\left[\frac{C_r e^{i\omega R/c} + C_a e^{-i\omega R/c}}{4\pi R}\right] e^{i\omega t'} s(\vec{\mathbf{r}}', t') = \\
&\qquad\qquad \left[\frac{C_r \delta\left(-t + \frac{R}{c} + t'\right) + C_a \delta\left(-t - \frac{R}{c} + t'\right)}{4\pi R}\right] s(\vec{\mathbf{r}}', t') \qquad (42)
\end{aligned}$$

We can now evaluate the integral over $t'$

$$f(\vec{\mathbf{r}}, t) = \int_V \frac{1}{4\pi R}\left[C_r s(\vec{\mathbf{r}}', t_r) + C_a s(\vec{\mathbf{r}}', t_a)\right] d^3x' \tag{43}$$

where $t_r = t - R/c$ is the retarded time, and $t_a = t + R/c$ is the advanced time. The advanced time makes no physical sense, since this corresponds to the source causing a change in the field at an earlier time. The potentials are therefore

$$V(\vec{\mathbf{r}},t) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{\mathbf{r}}',t_r)}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}\,d^3x' \qquad \vec{\mathbf{A}}(\vec{\mathbf{r}},t) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{\mathbf{J}}(\vec{\mathbf{r}}',t_r)}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}\,d^3x' \tag{44}$$

Having derived the potentials, we can now calculate the fields. We start with the electric field, which is found by calculating the following the following expression

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},t) = -\vec{\boldsymbol{\nabla}}V(\vec{\mathbf{r}},t) - \frac{\partial\vec{\mathbf{A}}(\vec{\mathbf{r}},t)}{\partial t} \tag{45}$$

We first calculate the gradient of the scalar potential, noting that $t_r$ is also a function of $\vec{\mathbf{r}}$. The gradient is

$$\vec{\boldsymbol{\nabla}}V(\vec{\mathbf{r}},t) = \frac{1}{4\pi\epsilon_0}\int_V \left[\frac{\vec{\boldsymbol{\nabla}}\rho(\vec{\mathbf{r}}',t_r)}{R} + \rho(\vec{\mathbf{r}}',t_r)\vec{\boldsymbol{\nabla}}\left(\frac{1}{R}\right)\right]d^3x' \tag{46}$$

The derivatives are

$$\vec{\boldsymbol{\nabla}}\left(\frac{1}{R}\right) = \frac{\hat{\mathbf{R}}}{R^2} \quad \text{and} \quad \vec{\boldsymbol{\nabla}}\rho(\vec{\mathbf{r}}',t_r) = \dot{\rho}(\vec{\mathbf{r}}',t_r)\vec{\boldsymbol{\nabla}}t_r = -\frac{\dot{\rho}(\vec{\mathbf{r}}',t_r)}{c}\vec{\boldsymbol{\nabla}}R \tag{47}$$

where the chain rule and the relation

$$\frac{\partial\rho}{\partial t} = \frac{\partial\rho}{\partial t_r} \tag{48}$$

since $t$ and $\vec{\mathbf{r}}$ are independent, are used to get the time derivative of the charge density. Next we calculate the time derivative of the vector potential

$$\frac{\partial}{\partial t}\vec{\mathbf{A}}(\vec{\mathbf{r}},t) = \frac{\mu_0}{4\pi}\int_V \frac{\dot{\vec{\mathbf{J}}}(\vec{\mathbf{r}}',t_r)}{R}\,d^3x' \tag{49}$$

Combining the two pieces, the electric field is

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},t) = \frac{1}{4\pi\epsilon_0}\int_V \left[\frac{\rho(\vec{\mathbf{r}}',t_r)}{R^2}\hat{\mathbf{R}} + \frac{\dot{\rho}(\vec{\mathbf{r}}',t_r)}{cR}\hat{\mathbf{R}} - \frac{\dot{\vec{\mathbf{J}}}(\vec{\mathbf{r}}',t_r)}{c^2R}\right]d^3x \tag{50}$$

Having calculated the electric field, we now proceed to calculate the magnetic field. The magnetic field is calculated by taking the curl of the vector potential

$$\vec{\mathbf{B}}(\vec{\mathbf{r}},t) = \vec{\boldsymbol{\nabla}}\times\vec{\mathbf{A}}(\vec{\mathbf{r}},t) = \frac{\mu_0}{4\pi}\int_V \left[\frac{1}{R}\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{J}}(\vec{\mathbf{r}}',t_r) - \vec{\mathbf{J}}(\vec{\mathbf{r}}',t_r)\times\vec{\boldsymbol{\nabla}}\left(\frac{1}{R}\right)\right]d^3x \tag{51}$$

To calculate the curl of the current density, we consider a single component and then generalize to all three components

$$\left(\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{J}}(\vec{\mathbf{r}}',t_r)\right)_x = \frac{\partial J_z}{\partial y} - \frac{\partial J_y}{\partial x} = -\frac{1}{c}\left(\dot{J}_z\frac{\partial R}{\partial y} - \dot{J}_y\frac{\partial R}{\partial z}\right) = \frac{1}{c}\left[\dot{\vec{\mathbf{J}}}(\vec{\mathbf{r}}',t_r)\times(\vec{\boldsymbol{\nabla}}R)\right]_x \tag{52}$$

where the chain rule is applied to get the time derivative of the current density

$$\frac{\partial J_z}{\partial y} = \frac{dJ_z}{dt_r}\frac{\partial t_r}{\partial y} = \frac{\partial J_z}{\partial t}\frac{\partial t_r}{\partial y} \tag{53}$$

In addition, the gradient of $R$ is $\hat{\mathbf{R}}$ ($\vec{\boldsymbol{\nabla}} R = \hat{\mathbf{R}}$), therefore

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{J}}(\vec{\mathbf{r}}', t_r) = \frac{1}{c}\dot{\vec{\mathbf{J}}}(\vec{\mathbf{r}}', t_r) \times \hat{\mathbf{R}} \quad \Rightarrow \quad \vec{\mathbf{B}}(\vec{\mathbf{r}}, t) = \frac{\mu_0}{4\pi}\int_V \left[\frac{\vec{\mathbf{J}}(\vec{\mathbf{r}}', t_r)}{R^2} + \frac{\dot{\vec{\mathbf{J}}}(\vec{\mathbf{r}}', t_r)}{cR}\right] \times \hat{\mathbf{R}}\, d^3x \tag{54}$$

# Physics 4183 Electricity and Magnetism II
# Radiation from a Point Charge

## 1 Introduction

In this lecture, we will discuss the fields generated by a point charge. We will start by calculating the potentials (Liénard-Wiechert Potentials), then give the fields associated with these potentials. From here we will derive expressions for the potentials and fields of a charge in uniform motion, and accelerated motion.

### 1.1 The Liénard-Wiechert Potentials

One of the important problems in electrodynamics is the calculation of the radiation field of a single particle. The has important applications in the design of particle accelerators, medical imaging and cancer radiation devices, astrophysics, all of which deal with synchrotron radiation[1].

We will start with the potential derived in the previous section. The charge and current densities for point sources are given by

$$\rho(\vec{\mathbf{r}}', t_r) = q\delta(\vec{\mathbf{r}}' - \vec{\mathbf{r}}_0(t_r)) \qquad \vec{\mathbf{J}}(\vec{\mathbf{r}}', t_r) = q\vec{\mathbf{v}}(t_r)\delta(\vec{\mathbf{r}}' - \vec{\mathbf{r}}_0(t_r)) \tag{1}$$

where $\vec{\mathbf{r}}_0(t_r)$ is the position of the charged particle at the retarded time $(t - |\vec{\mathbf{r}} - \vec{\mathbf{r}}'|/c)$, and

$$\vec{\mathbf{v}}(t_r) = \frac{d\vec{\mathbf{r}}_0(t_r)}{dt_r} \tag{2}$$

Using these sources, the potentials are

$$\begin{aligned} V(\vec{\mathbf{r}}, t) &= \frac{1}{4\pi\epsilon_0} \iiint_V \frac{q\delta(\vec{\mathbf{r}}' - \vec{\mathbf{r}}_0(t_r))}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}\, d^3x' \\ \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) &= \frac{\mu_0}{4\pi} \iiint_V \frac{q\vec{\mathbf{v}}(t_r)\delta(\vec{\mathbf{r}}' - \vec{\mathbf{r}}_0(t_r))}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}\, d^3x' \end{aligned} \tag{3}$$

Since the field time is fixed, integration over $\vec{\mathbf{r}}'$ means that $t_r$ also varies. Therefore both terms in the delta function depend on the integration variable and we cannot simply replace $\vec{\mathbf{r}}'$ with $\vec{\mathbf{r}}_0(t_r)$ after integration. Several methods exist to get around this problem. We will insert a second delta function that decouples $t_r$ from $\vec{\mathbf{r}}'$

$$\begin{aligned} V(\vec{\mathbf{r}}, t) &= \frac{1}{4\pi\epsilon_0} \iiint_V \frac{q\delta(\vec{\mathbf{r}}' - \vec{\mathbf{r}}_0(t_r))\delta(t_r - t + |\vec{\mathbf{r}} - \vec{\mathbf{r}}'|/c)}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}\, d^3x'\, dt_r \\ \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) &= \frac{\mu_0}{4\pi} \iiint_V \frac{q\vec{\mathbf{v}}(t_r)\delta(\vec{\mathbf{r}}' - \vec{\mathbf{r}}_0(t_r))\delta(t_r - t + |\vec{\mathbf{r}} - \vec{\mathbf{r}}'|/c)}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}\, d^3x'\, dt_r \end{aligned} \tag{4}$$

At this point we can carry out the volume integral, since $\vec{\mathbf{r}}'$ and $t_r$ are not formally related. This leads to

$$\begin{aligned} V(\vec{\mathbf{r}}, t) &= \frac{1}{4\pi\epsilon_0} \iiint_V \frac{q\delta(t_r - t + |\vec{\mathbf{r}} - \vec{\mathbf{r}}_0(t_r)|/c)}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}_0(t_r)|}\, dt_r \\ \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) &= \frac{\mu_0}{4\pi} \iiint_V \frac{q\vec{\mathbf{v}}(t_r)\delta(t_r - t + |\vec{\mathbf{r}} - \vec{\mathbf{r}}_0(t_r)|/c)}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}_0(t_r)|}\, dt_r \end{aligned} \tag{5}$$

[1] This is radiation from a point particle following an approximately circular orbit.

where

$$\vec{\mathbf{v}}(t_r) = \frac{d\vec{\mathbf{r}}_0(t_r)}{dt_r} \tag{6}$$

To finish the derivation of the potentials, we use the following delta function identity

$$\int_{-\infty}^{+\infty} \delta[h(x)] f(x)\, dx = \sum_i \frac{f(x_i)}{|dh/dx|_{x=x_i}} \tag{7}$$

where $x_i$ is the set of simple zeros of $h(x)$, where the identity can be readily established using

$$y = h(x) \qquad\qquad dy = \left|\frac{dh}{dx}\right| dx \tag{8}$$

where the absolute value insures that $dy > 0$ gives $dx > 0$. In our case

$$h(t_r) = t_r - t + R/c \quad \Rightarrow \quad \frac{dh}{dt_r} = 1 + \dot{R}/c \quad \text{where} \quad R = |\vec{\mathbf{r}} - \vec{\mathbf{r}}_0(t_r)| \tag{9}$$

The time derivative with respect to the retarded time of $R$ is given by

$$\begin{aligned} \frac{dR}{dt_r} &= \frac{d}{dt_r}\sqrt{r^2 + \vec{\mathbf{r}}_0(t_r)\cdot\vec{\mathbf{r}}_0(t_r) - 2\vec{\mathbf{r}}\cdot\vec{\mathbf{r}}_0(t_r)} \\ \dot{R} &= \frac{1}{2}\frac{2\vec{\mathbf{r}}_0(t_r)\cdot\dot{\vec{\mathbf{r}}}_0(t_r) - 2\vec{\mathbf{r}}\cdot\dot{\vec{\mathbf{r}}}_0(t_r)}{R} \\ \dot{R} &= -\hat{\mathbf{R}}\cdot\dot{\vec{\mathbf{r}}}_0(t_r) = -\hat{\mathbf{R}}\cdot\vec{\mathbf{v}} \end{aligned} \tag{10}$$

So finally, the potentials are given by

$$\begin{aligned} V(\vec{\mathbf{r}},t) &= \frac{1}{4\pi\epsilon_0}\frac{q}{[1-\hat{\mathbf{R}}(t_r)\cdot\vec{\boldsymbol{\beta}}(t_r)]|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0(t_r)|} \\ \vec{\mathbf{A}}(\vec{\mathbf{r}},t) &= \frac{\mu_0}{4\pi}\frac{q\vec{\mathbf{v}}(t_r)}{[1-\hat{\mathbf{R}}(t_r)\cdot\vec{\boldsymbol{\beta}}(t_r)]|\vec{\mathbf{r}}-\vec{\mathbf{r}}_0(t_r)|} \end{aligned} \tag{11}$$

With the potentials for a point particle having been calculated, we calculate the fields and investigate the properties of the radiated power. The fields are given by carrying out the derivatives

$$\vec{\mathbf{E}} = -\vec{\boldsymbol{\nabla}}V - \frac{\partial\vec{\mathbf{A}}}{\partial t} \qquad\qquad \vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}}\times\vec{\mathbf{A}} \tag{12}$$

and applying these to either Eq. 5 (easier method) or Eq. 11 (more difficult method). The fields are given by

$$\begin{aligned} \vec{\mathbf{E}} &= \frac{q}{4\pi\epsilon_0}\frac{1-\beta^2}{R^2[1-\hat{\mathbf{R}}\cdot\vec{\boldsymbol{\beta}}]^3}(\hat{\mathbf{R}}-\vec{\boldsymbol{\beta}}) + \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{R}}\times[(\hat{\mathbf{R}}-\vec{\boldsymbol{\beta}})\times\vec{\mathbf{a}}/c]}{Rc[1-\hat{\mathbf{R}}\cdot\vec{\boldsymbol{\beta}}]^3} \\ \vec{\mathbf{B}} &= \frac{1}{c}\hat{\mathbf{R}}\times\vec{\mathbf{E}} \end{aligned} \tag{13}$$

where the first term in the electric field corresponds to the static field, the field that has had time to catch up with the present position of the charge, and a term that depends on the acceleration and goes as $1/R$ the radiation field ($\vec{\mathbf{E}} = \vec{\mathbf{E}}_v + \vec{\mathbf{E}}_a$). These expressions are exact and valid for any relative velocity.

### 1.2 Uniform Motion

Let's examine the electric field for the case of uniform motion along the $z$ axis as defined in Fig. 1. This case corresponds to zero acceleration, therefore no radiation is expected.

We start with the electric field derived in Eq. 13 and set the acceleration equal to zero

$$\vec{\mathbf{E}} = \frac{q}{4\pi\epsilon_0}\frac{1-\beta^2}{R^2[1-\hat{\mathbf{R}}\cdot\vec{\boldsymbol{\beta}}]^3}(\hat{\mathbf{R}}-\vec{\boldsymbol{\beta}}). \tag{14}$$

Recall, the vector $\vec{\mathbf{R}}$ points from the retarded position to the field point. Because the acceleration is zero, the field is expected to point radially outward from the point charge. Therefore, it is best to calculate the field relative to the present position of the charge; the point from which the field originates. From Fig. 1, it is clear that the distance from the present position of the charge to the field point is $\vec{\mathbf{R}}_p = R(\hat{\mathbf{R}}-\vec{\boldsymbol{\beta}})$, which leads to the following form for the electric field

$$\vec{\mathbf{E}} = \frac{q}{4\pi\epsilon_0}\frac{1-\beta^2}{R^3[1-\hat{\mathbf{R}}\cdot\vec{\boldsymbol{\beta}}]^3}\vec{\mathbf{R}}_p. \tag{15}$$

The denominator can be manipulate as follows: Take the two terms and expand them out

$$R^2[1-\hat{\mathbf{R}}\cdot\vec{\boldsymbol{\beta}}]^2 = [R-\vec{\mathbf{R}}\cdot\vec{\boldsymbol{\beta}}]^2 = R^2 - 2R\vec{\mathbf{R}}\cdot\vec{\boldsymbol{\beta}} + (\vec{\mathbf{R}}\cdot\vec{\boldsymbol{\beta}})^2. \tag{16}$$

Next we evaluate the dot product using the expression for the present position and squaring it

$$R_p^2 = R^2 - 2R\vec{\mathbf{R}}\cdot\vec{\boldsymbol{\beta}} + R^2\beta^2 \quad\Rightarrow\quad -2R\vec{\mathbf{R}}\cdot\vec{\boldsymbol{\beta}} = R_p^2 - R^2 - R^2\beta^2, \tag{17}$$

which leads to the following expression for denominator

$$R^2 + (R_p^2 - R^2 - R^2\beta^2) + (\vec{\mathbf{R}}\cdot\vec{\boldsymbol{\beta}})^2 = R_p^2 - R^2\beta^2 + (\vec{\mathbf{R}}\cdot\vec{\boldsymbol{\beta}})^2. \tag{18}$$

This still needs further simplification. Next we notice that the two position vectors (retarded and present) form right triangles with the same height. Using the Pythagorean theorem we arrive at

$$R^2\beta^2 - (\vec{\mathbf{R}}\cdot\vec{\boldsymbol{\beta}})^2 = R_p^2\beta^2 - (\vec{\mathbf{R}}_p\cdot\vec{\boldsymbol{\beta}})^2 \quad\Rightarrow\quad (\vec{\mathbf{R}}\cdot\vec{\boldsymbol{\beta}})^2 = R^2\beta^2 - R_p^2\beta^2 + (\vec{\mathbf{R}}_p\cdot\vec{\boldsymbol{\beta}})^2 \tag{19}$$

Finally, we substitute Eq. 19 into Eq. 18

$$R_p^2 - R^2\beta^2 + (\vec{\mathbf{R}}\cdot\vec{\boldsymbol{\beta}})^2 = R_p^2 - R_p^2\beta^2 + R_p^2\beta\cos^2\theta_p = R_p^2(1-\beta^2\sin^2\theta_p) \tag{20}$$

where $\theta_p$ is the angle between $\vec{\mathbf{R}}_p$ and the $z$-axis.

Having calculated the denominator, the expression for electric field is

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},t) = \frac{q}{4\pi\epsilon_0}\frac{1-\beta^2}{R_p^2[1-\beta^2\sin^2\theta_p]^{3/2}}\hat{\mathbf{R}}_p \tag{21}$$

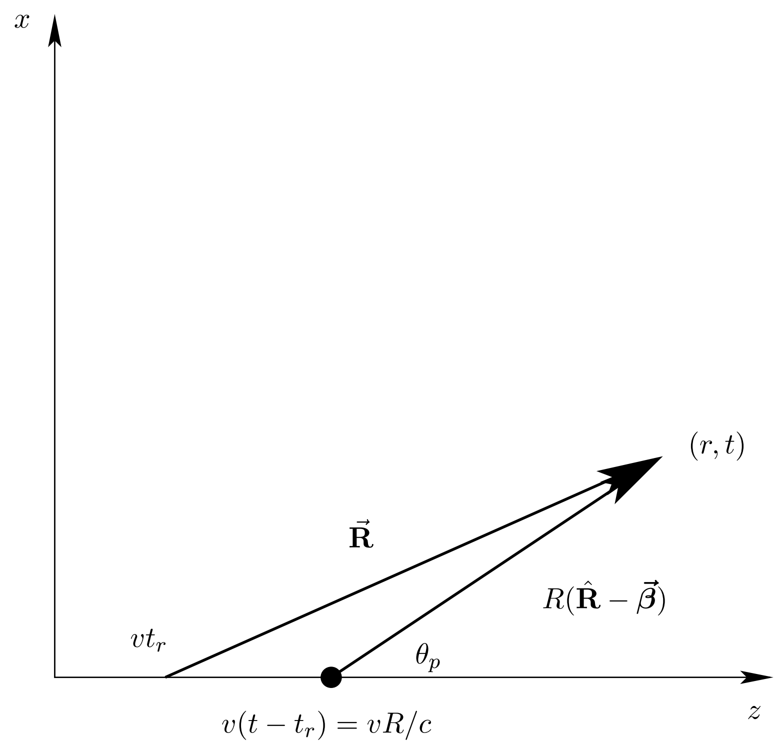


Figure 1: This figure shows the geometry of a charged particle moving with a uniform velocity. Point A corresponds to the retarded position, while point B is the present position.

### 1.3 Accelerated Point Charge

In this case our main interest is to examine the energy that is radiated, that is the energy that goes off to infinity. From our previous discussion, this is the term in the electric field that falls off as $1/R$, which is also the term that depends on the acceleration

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},t) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{R}}\times[(\hat{\mathbf{R}}-\vec{\boldsymbol{\beta}})\times\vec{\mathbf{a}}/c]}{Rc[1-\hat{\mathbf{R}}\cdot\vec{\boldsymbol{\beta}}]^3}, \tag{22}$$

where the field point is at $\vec{\mathbf{r}}$ and determined at time $t$ and $\vec{\mathbf{R}} = \vec{\mathbf{r}} - \vec{\mathbf{r}}'$ is the position vector from the retarded position. For simplicity and to get a feel for the physics, we will start with the case where $\beta \ll 1$. The field in this case is approximately given by

$$\vec{\mathbf{E}} \approx \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{R}}\times[\hat{\mathbf{R}}\times\vec{\mathbf{a}}/c]}{cR} = \frac{q}{4\pi\epsilon_0}\frac{(\hat{\mathbf{R}}\cdot\vec{\mathbf{a}})\hat{\mathbf{R}}-\vec{\mathbf{a}}}{c^2R} \tag{23}$$

If one takes the dot product of this expression with $\hat{\mathbf{R}}$, one find that it equals zero

$$\hat{\mathbf{R}}\cdot\vec{\mathbf{E}} = \frac{q}{4\pi\epsilon_0}\frac{(\hat{\mathbf{R}}\cdot\vec{\mathbf{a}})-(\hat{\mathbf{R}}\cdot\vec{\mathbf{a}})}{c^2R} = 0 \tag{24}$$

note $\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = 1$. Therefore, the field is normal to the radial direction from the retarded position. This is consistent with the simple picture that we started with, the radial component of the kink falls of as $1/R^2$ while the polar component, which depends on the acceleration, falls off as $1/R$. This later term is normal to $\hat{\mathbf{R}}$.

What we would like to calculate, this being the quantity of most interest for radiation, is the angular distribution of the power. For this we calculate the radial component of the Poynting vector, which is the only component that travels through any closed surface enclosing the charge. Using Eq. 13 we calculate the magnetic field, and therefore the Poynting vector

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}}\times\vec{\mathbf{B}} = \frac{1}{\mu_0 c}\vec{\mathbf{E}}\times(\hat{\mathbf{R}}\times\vec{\mathbf{E}}) = \frac{1}{\mu_0 c}\left[E^2\hat{\mathbf{R}} - (\vec{\mathbf{E}}\cdot\hat{\mathbf{R}})\vec{\mathbf{E}}\right] = \frac{1}{\mu_0 c}E^2\hat{\mathbf{R}} \tag{25}$$

Next we substitute the electric field into this expression

$$\vec{\mathbf{S}} = \frac{1}{\mu_0 c}\frac{q^2}{16\pi^2\epsilon_0^2}\left(\frac{(\hat{\mathbf{R}}\cdot\vec{\mathbf{a}})\hat{\mathbf{R}}-\vec{\mathbf{a}}}{c^2R}\right)\cdot\left(\frac{(\hat{\mathbf{R}}\cdot\vec{\mathbf{a}})\hat{\mathbf{R}}-\vec{\mathbf{a}}}{c^2R}\right)\hat{\mathbf{R}} = \frac{q^2}{16\pi^2c^3\epsilon_0}\frac{a^2-(\hat{\mathbf{R}}\cdot\vec{\mathbf{a}})^2}{R^2}\hat{\mathbf{R}}. \tag{26}$$

Next, we expand the dot product and calculate the dot product with $\hat{\mathbf{R}}$, arriving at the angular distribution of the radiated power

$$\vec{\mathbf{S}}\cdot\hat{\mathbf{R}} = \frac{dP}{R^2 d\Omega} = \frac{q^2}{16\pi^2c^3\epsilon_0}\frac{a^2\sin^2\theta}{R^2} \tag{27}$$

where $\theta$ is the angle between the direction of the acceleration and $\hat{\mathbf{R}}$, the direction from the retarded position to the field point. Notice that the radiation is emitted predominantly perpendicular to the direction of the acceleration and that it does not depend on the direction the particle travels in. To conclude this, we calculate the the total power radiated. This is given by integrating the expression for the differential power over the full solid angle

$$P = \frac{q^2}{16\pi^2 c^3 \epsilon_0} \int_0^{2\pi} d\phi \int_{-1}^{1} \frac{a^2 \sin^2\theta}{R^2} R^2 d\cos\theta = \frac{q^2 a^2}{6\pi c^3 \epsilon_0} \tag{28}$$

which is the Larmor formula that was derived earlier using the simple picture of radiation.

Next we calculate the exact formula for the angular distribution of power radiated from an accelerated point charge. The procedure is basically the same, except that now the fields depend on the velocity, both the direction and the magnitude (see Eq. 22). Therefore we will treat two separate cases: The first where the velocity and the acceleration are in the same direction, and the second where they are perpendicular to each other. In the case where $\vec{\boldsymbol{\beta}}$ and $\vec{\mathbf{a}}$ are parallel, the term $\vec{\boldsymbol{\beta}} \times \vec{\mathbf{a}} = 0$ in Eq. 22 meaning that the electric field is given by

$$\vec{\mathbf{E}} = \frac{q}{4\pi\epsilon_0} \frac{\hat{\mathbf{R}} \times [\hat{\mathbf{R}} \times \vec{\mathbf{a}}/c]}{Rc[1 - \hat{\mathbf{R}} \cdot \vec{\boldsymbol{\beta}}]^3} \tag{29}$$

The differential power, calculated as before, is given by

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2 \epsilon_0 c^3} \frac{a^2 \sin^2\theta}{[1 - \beta\cos\theta]^6} \tag{30}$$

where we have written the power at the field position at time $t$, which corresponds to the power radiated from the retarded position $\vec{\mathbf{R}}$ at time $t_r$. Now, the power radiated by the charge at time $t_r$ into the the solid angle $d\Omega$ is calculated through the following transformation

$$P = -\frac{dW}{dt} \rightarrow \frac{dW}{dt_r} = \frac{dW}{dt}\frac{dt}{dt_r} \quad \text{where} \quad \frac{dt}{dt_r} = 1 - \vec{\boldsymbol{\beta}} \cdot \hat{\mathbf{R}} = 1 - \beta\cos\theta. \tag{31}$$

where we used $t_r = t - R(t_r)/c$ and

$$\hat{\mathbf{R}} \cdot \hat{\mathbf{R}} = c^2(t - t_r)^2 \quad \Rightarrow \quad \left\{ \begin{array}{l} 2\hat{\mathbf{R}} \cdot \dfrac{d\vec{\mathbf{R}}}{dt} 2c^2(t - t_r)\left(1 - \dfrac{dt_r}{dt}\right) \\ \dfrac{d\vec{\mathbf{R}}}{dt} = -\dfrac{d\vec{\mathbf{r}}'(t_r)}{dt} = -\vec{\mathbf{v}}\dfrac{dt_r}{dt} \end{array} \right\} \quad \Rightarrow \quad \frac{dt_r}{dt} = \frac{R}{R - \vec{\mathbf{R}} \cdot \vec{\boldsymbol{\beta}}} \tag{32}$$

Therefore, the power radiated by the charge at time $t_r$ is

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2 \epsilon_0 c^3} \frac{a^2 \sin^2\theta}{[1 - \beta\cos\theta]^5} \tag{33}$$

where the angular distribution is modified relative to the case of small velocities by the angular term in the denominator, so that as $\beta \rightarrow 1$ the power is radiated in the forward direction. The total radiated power is calculated by integrating over the solid angle and found to be

$$P = \frac{q^2 a^2}{6\pi c^3 \epsilon_0} \gamma^6 \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \tag{34}$$

which is the same as the case for a slowly moving particle, but multiplied by $\gamma^6$.

The final example corresponds to the case where the velocity and acceleration are perpendicular to each other. The algebra is more involved, but straightforward. None-the-less I will simply state the answer. The angular distribution is given by

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0 c^3}\frac{(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\phi}{(1-\beta\cos\theta)^5} \tag{35}$$

Again as in the previous cases, the radiation is predominantly emitted perpendicular to the direction of the acceleration, and as the velocity increases the radiation is focused in the direction of motion.

# Physics 4183 Electricity and Magnetism II
# Radiation from Extended Sources

## 1 Introduction

We are now ready to start discussing radiation from a distributed source. We will do this for systems where the source has a harmonic time dependence and only consider the field at large distances from the source. To make the problem as general as possible, we will define the radiation integral, which contains all the information about the source. This integral can then be expanded in a multipole series and used to describe the source at different levels of approximation.

### 1.1 Radiation Integral

As stated in the introduction, we will assume that our sources have only a harmonic time dependence. Even though this may not seem general we can always add up different frequency components to build any other time dependence[1]. Given this, the charge[2] and current densities are described as follows

$$\rho(\vec{\mathbf{r}}', t_r) = \rho(\vec{\mathbf{r}}')e^{-i\omega t_r} = \rho(\vec{\mathbf{r}}')e^{ikR}e^{-i\omega t} \qquad \vec{\mathbf{J}}(\vec{\mathbf{r}}', t_r) = \vec{\mathbf{J}}(\vec{\mathbf{r}}')e^{-i\omega t_r} = \vec{\mathbf{J}}(\vec{\mathbf{r}}')e^{ikR}e^{-i\omega t} \tag{1}$$

where $t_r = t - R/c$ is the retarded time, and $R = |\vec{\mathbf{r}} - \vec{\mathbf{r}}'|$. In this case, both the scalar and vector potentials also have a harmonic time dependence, with the potentials being

$$\vec{\mathbf{A}}(\vec{\mathbf{r}}) = \frac{\mu_0}{4\pi}\iiint_V \frac{\vec{\mathbf{J}}(\vec{\mathbf{r}}')e^{ikR}}{R}\,dV' \qquad V(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho(\vec{\mathbf{r}}')e^{ikR}}{R}\,dV' \tag{2}$$

Remember that only the real part of these equations corresponds to the observable potentials.

To calculate the magnetic field, we take the curl of the vector potential

$$\vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}} \tag{3}$$

If there is no current density at the position of the field point, this is usually the case since we are interested in the fields at large distances from the source, then we can use the Ampere/Maxwell equation to calculate the electric field

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = \mu_0\vec{\mathbf{J}} + \mu_0\epsilon_0\frac{\partial\vec{\mathbf{E}}}{\partial t} \quad\Rightarrow\quad \frac{\partial}{\partial t}\vec{\mathbf{E}}e^{-i\omega t} = c^2\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}}e^{-i\omega t} \quad\Rightarrow\quad \vec{\mathbf{E}} = \frac{ic}{k}\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}}, \tag{4}$$

where we used $c = \omega/k$. Thus for sources that have a simple harmonic time dependence, both fields can be calculated from the vector potential in source free regions.

If the field is calculated for $kR \ll 1$, then we get back the approximate static solutions. This is referred to as the quasi-static limit. Note that this limit implies that the separation between source and field point is much less than a wavelength. If the field is calculated for $kR \gg 1$, then $R$ is large compared to a wavelength, and in this case

$$R = \sqrt{r^2 - 2\vec{\mathbf{r}}\cdot\vec{\mathbf{r}}' + r'^2} = r\sqrt{1 - 2\hat{\mathbf{r}}\cdot\frac{\vec{\mathbf{r}}'}{r} + \frac{r'^2}{r^2}} \approx r - \hat{\mathbf{r}}\cdot\vec{\mathbf{r}}' \tag{5}$$

[1] This is the basis of the Fourier series and transform.

[2] The form of the charge density assumes that it is stationary, but changes magnitude and sign over time.

where the last expression is a Taylor series expansion in the limit $r'/r \ll 1$. In this limit, the vector potential takes on the following form

$$\vec{\mathbf{A}}(\vec{\mathbf{r}}) \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\iiint_V \vec{\mathbf{J}}(\vec{\mathbf{r}}')e^{-ik\hat{\mathbf{r}}\cdot\vec{\mathbf{r}}'}\,dV' \tag{6}$$

This limit is referred to as the radiation zone approximation.

We can now calculate $\vec{\mathbf{B}}$, but first we will introduce some simplifying notation. The vector potential can be written as

$$\vec{\mathbf{A}}(\vec{\mathbf{r}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\vec{\mathbf{F}}(\hat{\mathbf{r}}) \quad \text{where} \quad \vec{\mathbf{F}}(\hat{\mathbf{r}}) = \iiint_V \vec{\mathbf{J}}(\vec{\mathbf{r}}')e^{-ik\hat{\mathbf{r}}\cdot\vec{\mathbf{r}}'}\,dV' \tag{7}$$

and is referred to as the radiation integral. Notice also that $\vec{\mathbf{F}}(\hat{\mathbf{r}})$ depends only on $\hat{\mathbf{r}}$, since $\vec{\mathbf{r}}'$ is integrated over, and in this form, the vector potential is the product of an outgoing spherical wave with a source term. To compute the magnetic field, we calculate the curl of the vector potential

$$\vec{\mathbf{B}} = \vec{\nabla}\times\vec{\mathbf{A}} = \frac{\mu_0}{4\pi}\vec{\nabla}\times\left[\frac{e^{ikr}}{r}\vec{\mathbf{F}}(\hat{\mathbf{r}})\right] = \frac{\mu_0}{4\pi}\left[\frac{e^{ikr}}{r}\vec{\nabla}\times\vec{\mathbf{F}}(\hat{\mathbf{r}}) - \vec{\mathbf{F}}(\hat{\mathbf{r}})\times\vec{\nabla}\left(\frac{e^{ikr}}{r}\right)\right] \tag{8}$$

Recall that this result is only valid in the limit $kR \gg 1$ or $r \gg r'$, therefore we are only concerned with the leading order $(1/r)$ terms above. The various derivatives are

$$\begin{aligned}\frac{e^{ikr}}{r}\vec{\nabla}\times\vec{\mathbf{F}}(\hat{\mathbf{r}}) &= \frac{e^{ikr}}{r}\vec{\nabla}\times\vec{\mathbf{F}}(\vec{\mathbf{r}}/r) \propto \frac{\hat{\mathbf{r}}}{r^2} \\ \vec{\nabla}\left(\frac{e^{ikr}}{r}\right) &= \frac{1}{r}\vec{\nabla}\left(e^{ikr}\right) + e^{ikr}\vec{\nabla}\left(\frac{1}{r}\right) = ik\frac{e^{ikr}}{r}\hat{\mathbf{r}} - e^{ikr}\frac{1}{r^2}\hat{\mathbf{r}} \approx ik\frac{e^{ikr}}{r}\hat{\mathbf{r}}\end{aligned} \tag{9}$$

therefore the magnetic field is approximately

$$\vec{\mathbf{B}} \approx \frac{\mu_0}{4\pi}ik\frac{e^{ikr}}{r}\hat{\mathbf{r}}\times\vec{\mathbf{F}}(\hat{\mathbf{r}}) \tag{10}$$

with the electric field in the same approximation

$$\vec{\mathbf{E}} = \frac{ic}{k}\vec{\nabla}\times\vec{\mathbf{B}} \approx -c\hat{\mathbf{r}}\times\vec{\mathbf{B}} \tag{11}$$

Notice that the electric and magnetic fields are perpendicular to each other, and perpendicular to the direction of propagation $\hat{\mathbf{r}}$. Furthermore, the field for large values of r in small regions of space looks like a plane wave.

Finally, we calculate the energy being radiated. We start by giving an expression for the Poynting vector

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}}\times\vec{\mathbf{B}} = -\frac{c}{\mu_0}\left(\hat{\mathbf{r}}\times\vec{\mathbf{B}}\right)\times\vec{\mathbf{B}} = \frac{c}{\mu_0}B^2\hat{\mathbf{r}} \tag{12}$$

with the power being propagated outward as expected. Since the important term in the magnetic field is the source $(\vec{\mathbf{F}}(\hat{\mathbf{r}}))$, we will express the Poynting vector in terms of this quantity. But first recall that only the real part of the magnetic field matters, therefore we have to put back the time dependence and extract the real part of the magnetic field and then carry out the multiplication

$$\text{Re}(\vec{\mathbf{B}}) = \frac{\mu_0 k}{4\pi r}\hat{\mathbf{r}} \times \text{Re}\left[ie^{i(kr-\omega t)}\vec{\mathbf{F}}(\hat{\mathbf{r}})\right] = \frac{\mu_0 k}{4\pi r}\hat{\mathbf{r}} \times \left[-\text{Re}(\vec{\mathbf{F}}(\hat{\mathbf{r}}))\sin(kr-\omega t) - \text{Im}(\vec{\mathbf{F}}(\hat{\mathbf{r}}))\cos(kr-\omega t)\right] \quad (13)$$

Squaring the magnetic field

$$B^2 = \vec{\mathbf{B}} \cdot \vec{\mathbf{B}} = \left(\frac{\mu_0 k}{4\pi r}\right)^2 \left|\hat{\mathbf{r}} \times \left[\text{Re}(\vec{\mathbf{F}}(\hat{\mathbf{r}}))\sin(kr-\omega t) + \text{Im}(\vec{\mathbf{F}}(\hat{\mathbf{r}}))\cos(kr-\omega t)\right]\right|^2 \quad (14)$$

Since the time dependent Poynting vector is not of practical use, we calculate the time average value, which is of practical use. The important quantities are the trig functions, since the contain the time. These are given by

$$\frac{1}{\tau}\int_0^\tau \sin^2(kr-\omega t)\,dt = \frac{1}{\tau}\int_0^\tau \cos^2(kr-\omega t)\,dt = \frac{1}{2} \quad (15)$$
$$\frac{1}{\tau}\int_0^\tau \cos(kr-\omega t)\sin(kr-\omega t)\,dt = 0$$

Therefore the time averaged Poynting vector is

$$\left\langle\vec{\mathbf{S}}\right\rangle = \frac{\mu_0}{16\pi^2}\frac{k^2 c}{2r^2}\left[(\hat{\mathbf{r}} \times \text{Re}\vec{\mathbf{F}}(\hat{\mathbf{r}}))^2 + (\hat{\mathbf{r}} \times \text{Im}\vec{\mathbf{F}}(\hat{\mathbf{r}}))^2\right]\hat{\mathbf{r}} \quad (16)$$

with the radiated power into a differential solid angle $d\Omega$

$$\langle dP\rangle = \left\langle\vec{\mathbf{S}}\right\rangle \cdot \hat{\mathbf{r}}\, r^2 d\Omega = \frac{\mu_0}{16\pi^2}\frac{k^2 c}{2}\left[(\hat{\mathbf{r}} \times \text{Re}\vec{\mathbf{F}}(\hat{\mathbf{r}}))^2 + (\hat{\mathbf{r}} \times \text{Im}\vec{\mathbf{F}}(\hat{\mathbf{r}}))^2\right] d\Omega \quad (17)$$

which is independent of the distance of the field point from the source as expected.

### 1.2 Electric Dipole Radiation

If the dimensions of the source are small compared to the wavelength of the emitted radiation $|\vec{\mathbf{r}}'| \ll \lambda$, where $\lambda = \nu/c$ is the wavelength and $\nu = \omega/2\pi$ is the frequency, the exponential term in Eq. 7, can be expanded in a Taylor series and depending on the system, only a few terms will contribute to the radiation field

$$e^{-ik\hat{\mathbf{r}}\cdot\vec{\mathbf{r}}'} = \sum_{n=0}^{\infty}\frac{1}{n!}(-ik\vec{\mathbf{r}}\cdot\vec{\mathbf{r}}')^n \quad (18)$$

This is the multipole expansion and in this case the radiation integral takes on the following form

$$\vec{\mathbf{F}}(\hat{\mathbf{r}}) = \sum_{n=0}^{\infty}\frac{1}{n!}\iiint_V \vec{\mathbf{J}}(\vec{\mathbf{r}}')(-ik\vec{\mathbf{r}}\cdot\vec{\mathbf{r}}')^n dV' \quad (19)$$

with the most important cases being the $n = 0$ (electric dipole), and $n = 1$ (magnetic dipole plus electric quadrupole) terms.

For $n = 0$, the radiation integral is

$$\vec{\mathbf{F}}(\hat{\mathbf{r}}) = \iiint_V \vec{\mathbf{J}}(\vec{\mathbf{r}}')\, dV' \tag{20}$$

To show that it corresponds to electric dipole radiation, we use the following relation

$$\iiint_V \vec{\boldsymbol{\nabla}}' \cdot (f(\vec{\mathbf{r}}')\vec{\mathbf{J}}(\vec{\mathbf{r}}'))\, dV' = \oiint_S f(\vec{\mathbf{r}}')\vec{\mathbf{J}}(\vec{\mathbf{r}}') \cdot d\vec{\mathbf{a}}' = 0 \tag{21}$$

where the derivatives act at the source points, and the surface is selected such that it encloses the current distribution meaning that the flux through the bounding surface is zero. Expanding the divergence

$$\vec{\boldsymbol{\nabla}}' \cdot (f(\vec{\mathbf{r}}')\vec{\mathbf{J}}(\vec{\mathbf{r}}')) = (\vec{\boldsymbol{\nabla}}' f(\vec{\mathbf{r}}')) \cdot \vec{\mathbf{J}}(\vec{\mathbf{r}}') + f(\vec{\mathbf{r}}')\vec{\boldsymbol{\nabla}}' \cdot \vec{\mathbf{J}}(\vec{\mathbf{r}}') \tag{22}$$

and using the continuity equation on a source with a harmonic time dependence

$$e^{-i\omega t}\vec{\boldsymbol{\nabla}}' \cdot \vec{\mathbf{J}}(\vec{\mathbf{r}}') = -\frac{\partial}{\partial t}\left[\rho(\vec{\mathbf{r}}')e^{-i\omega t}\right] = i\omega\rho(\vec{\mathbf{r}}')e^{-i\omega t} \tag{23}$$

leads to the identity

$$\iiint_V (\vec{\boldsymbol{\nabla}}' f(\vec{\mathbf{r}}')) \cdot \vec{\mathbf{J}}(\vec{\mathbf{r}}')\, dV' = \iiint_V f(\vec{\mathbf{r}}')\frac{\partial\rho(\vec{\mathbf{r}}')}{\partial t}\, dV' = -i\omega \iiint_V f(\vec{\mathbf{r}}')\rho(\vec{\mathbf{r}}')\, dV' \tag{24}$$

Since this equation is valid for any $f(\vec{\mathbf{r}}')$, we select $x_i'$

$$\iiint_V J_i(\vec{\mathbf{r}}')\, dV' = -i\omega \iiint_V x_i'\rho(\vec{\mathbf{r}}')\, dV' \quad \Rightarrow \quad \vec{\mathbf{F}}(\hat{\mathbf{r}}) = \iiint_V \vec{\mathbf{J}}(\vec{\mathbf{r}}')\, dV' = -i\omega\vec{\mathbf{p}} \tag{25}$$

where $\vec{\mathbf{p}}$ is the electric dipole moment.

Having found the value of the radiation integral, the vector potential is immediately given by

$$\vec{\mathbf{A}}(\vec{\mathbf{r}}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\vec{\mathbf{F}}(\hat{\mathbf{r}}) = -\frac{i\mu_0\omega}{4\pi}\frac{e^{ikr}}{r}\vec{\mathbf{p}} \tag{26}$$

In addition, the magnetic field, which is given by Eq. 10, is

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}ik\frac{e^{ikr}}{r}\hat{\mathbf{r}} \times \vec{\mathbf{F}}(\hat{\mathbf{r}}) = k\omega\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\hat{\mathbf{r}} \times \vec{\mathbf{p}} \tag{27}$$

and the electric field is

$$\vec{\mathbf{E}} = -c\hat{\mathbf{r}} \times \vec{\mathbf{B}} = -\omega^2\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \vec{\mathbf{p}}) \tag{28}$$

Since an electric dipole has a single defined direction, we can choose this to be along the $z$-axis. Then the cross products reduce to

$$\hat{\mathbf{r}} \times \vec{\mathbf{p}} = -p\sin\theta\,\hat{\boldsymbol{\phi}} \quad \text{and} \quad \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \vec{\mathbf{p}}) = p\sin\theta\,\hat{\boldsymbol{\theta}} \tag{29}$$

leading to the following expressions for the fields

$$\vec{\mathbf{B}} = -\frac{\mu_0}{4\pi}\frac{\omega^2 p}{c}\frac{e^{ikr}}{r}\sin\theta\,\hat{\boldsymbol{\phi}} \qquad\qquad \vec{\mathbf{E}} = -\frac{\mu_0}{4\pi}\omega^2 p\frac{e^{ikr}}{r}\sin\theta\,\hat{\boldsymbol{\theta}} \tag{30}$$

To calculate the radiated power per unit of solid angle, we use Eq. 17 with $\vec{\mathbf{F}}(\hat{\mathbf{r}}) = -i\omega p\hat{\mathbf{z}}$

$$\langle dP \rangle = \frac{1}{4\pi\epsilon_0}\left(\frac{\omega^4 p^2}{8\pi c^3}\right)\sin^2\theta\, d\Omega \quad \Rightarrow \quad \langle P \rangle = \frac{1}{4\pi\epsilon_0}\left(\frac{\omega^4 p^2}{8\pi c^3}\right) \tag{31}$$

Notice that the direction of the radiated power is a maximum in the direction perpendicular to the dipole (this is the direction the charges are accelerated). The result is similar to that derived from our simple model of radiation.

### 1.3 Magnetic Dipole Radiation

To calculate magnetic dipole radiation, we repeat the process that we just carried out for electric dipole radiation, except that we use the $n = 1$ term. The radiation integral in this case is

$$\vec{\mathbf{F}}(\hat{\mathbf{r}}) = -ik\iiint_V (\hat{\mathbf{r}}\cdot\vec{\mathbf{r}}')\vec{\mathbf{J}}(\vec{\mathbf{r}}')\, dV' \quad \Rightarrow \quad F_i = -\frac{ik}{r}\sum_j \iiint_V x_j x'_j J_i\, dV' \tag{32}$$

where the second expression is in Cartesian coordinates to make the identification of the multipole terms easier. The integral can be expressed as a sum and a difference

$$\sum_j x_j \int x'_j J_i\, dV' = \sum_j x_j \int \frac{1}{2}(x'_j J_i - x'_i J_j)\, dv' + \sum_j x_j \int \frac{1}{2}(x'_j J_i + x'_i J_j)\, dv' \tag{33}$$

The first term on the right hand side can be expressed as a cross product

$$\vec{\mathbf{m}} = \frac{1}{2}\iiint_V \vec{\mathbf{r}}' \times \vec{\mathbf{J}}(\vec{\mathbf{r}}')\, dV' \tag{34}$$

this is the magnetic dipole moment. *The other term is the electric quadrupole moment.*

The radiation integral for the magnetic dipole moment is

$$\vec{\mathbf{F}}(\hat{\mathbf{r}}) = -ik\vec{\mathbf{m}} \times \hat{\mathbf{r}} \tag{35}$$

Therefore the magnetic and electric fields associated with magnetic dipole radiation are

$$\left.\begin{aligned} \vec{\mathbf{B}} &= \frac{\mu_0}{4\pi}k^2\frac{e^{ikr}}{r}\hat{\mathbf{r}}\times(\vec{\mathbf{m}}\times\hat{\mathbf{r}}) \\ \vec{\mathbf{E}} &= \frac{1}{4\pi\epsilon_0}\omega k\frac{e^{ikr}}{r}\vec{\mathbf{m}}\times\hat{\mathbf{r}} \end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned} \vec{\mathbf{B}} &= -\frac{\mu_0 k^2 m}{4\pi}\frac{e^{ikr}}{r}\sin\theta\,\hat{\boldsymbol{\theta}} \\ \vec{\mathbf{E}} &= \frac{\mu_0 c k^2 m}{4\pi}\frac{e^{ikr}}{r}\sin\theta\,\hat{\boldsymbol{\phi}} \end{aligned}\right. \tag{36}$$

We next calculate the time averaged Poynting vector, which we find to be

$$\left\langle \vec{\mathbf{S}} \right\rangle = \frac{dP}{r^2 d\Omega} = \frac{\mu_0 m^2\omega^4}{32\pi^2 c^3}\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}} \tag{37}$$

and the total radiated power is

$$\langle P \rangle = \frac{\mu_0 m^2 \omega^4}{12\pi^2 c^3}. \tag{38}$$

It is interesting to compare the radiated power for the electric and magnetic dipole. The ratio of magnetic to electric dipole radiation is

$$\frac{\langle P_{mag}\rangle}{\langle P_{elect}\rangle} = \left(\frac{m}{pc}\right)^2. \tag{39}$$

If the two system are comparable in size, then it is clear that magnetic dipole radiation is much less than electric dipole radiation

$$\left.\begin{array}{ll} m = \pi b^2 I & I = q\omega \\ p = qd = q\pi b & \end{array}\right\} \Rightarrow \frac{\langle P_{mag}\rangle}{\langle P_{elect}\rangle} = \left(\frac{m}{pc}\right)^2 = \left(\frac{\omega b}{c}\right)^2 \tag{40}$$

# Physics 4183 Electricity and Magnetism II
# Special Theory of Relativity—The Basics

## 1 Introduction

Based on the idea that electromagnetic waves need a media for them to propagate there energy through like sound waves, and the realization that the components of the stress tensor are non-zero, Faraday among others proposed that an all encompassing ether permeates all space. Given this picture, the speed of light varies from reference frame to reference frame depending on the relative speed of the observer and the ether. In In 1904, Albert Einstein proposed

### 1.1 Michelson-Morely Experiment

To determine the velocity of the earth relative to the ether, Michelson and Morely devised an experiment to show that the velocity of light was different along the direction of the earth's motion and perpendicular to it. The setup an interferometer into which light entered and was split along to paths. If the two paths are of equal length, then the recombined light will have no phase difference, and an intensity maximum will be observed.

Assume that the interferometer is setup such that one path is along the direction of motion of the earth through the ether and the other is perpendicular to it. Further, assume that the velocity of light relative to the ether is fixed, then the travel time through the path perpendicular the motion is

$$(ct_1)^2 = (vt_1)^2 + \ell^2 \quad \Rightarrow \quad t_1 = 2\frac{\ell}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \approx \frac{2\ell}{c}\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) \tag{1}$$

where $v$ is the velocity of the earth through the ether, and the extra factor of two accounts for the round trip time. Along the path parallel to the earth's motion, the velocity of light in one direction is increase by $v$, while in the other it is decreased by $v$, therefore the total travel time is

$$t_2 = \frac{\ell}{c-v} + \frac{\ell}{c+v} = \frac{2\ell}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1} \approx \frac{2\ell}{c}\left(1 + \frac{v^2}{c^2}\right) \tag{2}$$

The phase difference is given by

$$\frac{\delta t}{\tau} = c\frac{t_2 - t_1}{\lambda} = \frac{\ell}{\lambda}\frac{v^2}{c^2} \tag{3}$$

where $\tau$ is the period of oscillation of the light. Assuming that the ether is fixed to the center of mass of the solar system, the $v$ is the velocity of the earth orbiting the earth ($v = 3 \times 10^6$ m/s), and using a light source (sodium) of wavelength $\lambda = 6 \times 10^{-5}$ cm as Michelson and Morely did, and a path length of $10^3$ cm, the phase difference is is approximately 0.17, they found a null result. The accuracy of there measurement was 0.3% of the expected value.

Clearly the Michelson and Morely experiment shows that the speed of light is independent of direction, either along the direction of motion of the earth or perpendicular to it. The experiment was carried out at different times of the year to insure that no accidental cancellation of velocities occurred; earth, solar system, and galaxy happen to just be moving in directions that cancel out their velocities relative to the ether. Again the results were null, the velocity of light is the same in all direction. A number of models were proposed to explain the observation without eliminating the ether. In 1905, Albert Einstein eliminated the ether by proposing that the speed of light is a constant independent of inertial frame of reference.

## 1.2 Galilean Relativity and Electrodynamics

Before proceeding to discuss Einstein's proposal, let's discuss where physics stood before 1905. The laws of physics were assumed to be invariant (covariant) with respect to a Galilean transformation. Recall that under a Galilean transformation the coordinates transform like

$$x' = x - v_0 t \quad \text{or in three dimensions} \quad \vec{\mathbf{r}}' = \vec{\mathbf{r}} - \vec{\mathbf{v}}_0 t \tag{4}$$

and $t = t'$, where the primed frame is moving to the right with a velocity $v_0$ relative to the unprimed frame. Under this transformation, we can see that the Newton's second law is invariant to this transformation

$$\begin{aligned} \vec{\mathbf{F}}(\vec{\mathbf{r}}') &= m\vec{\mathbf{a}}' & (5)\\ &= m\frac{d^2\vec{\mathbf{r}}'}{dt^2} & (6)\\ &= m\vec{\mathbf{a}} - m\frac{d\vec{\mathbf{v}}_0}{dt} \\ &= m\vec{\mathbf{a}} = \vec{\mathbf{F}}(\vec{\mathbf{r}}) \end{aligned}$$

The force acting on the object is the same in both frames, as long as they are moving at a constant velocity relative to each other.

Let's now ask the same question of the Maxwell equations. Consider Faraday's law in integral form

$$\oint \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{\ell}} = \iint_S \frac{d\vec{\mathbf{B}}}{dt} \cdot d\vec{\mathbf{a}} \tag{7}$$

The total time derivative is given by

$$df(\vec{\mathbf{r}}, t) = \frac{\partial f(\vec{\mathbf{r}}, t)}{\partial t}dt + \frac{\partial f(\vec{\mathbf{r}}, t)}{\partial x}dx + \frac{\partial f(\vec{\mathbf{r}}, t)}{\partial y}dy + \frac{\partial f(\vec{\mathbf{r}}, t)}{\partial z}dz \quad \Rightarrow \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \vec{\mathbf{v}} \cdot \vec{\boldsymbol{\nabla}} \tag{8}$$

Therefore, Faraday's law becomes

$$\oint \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{\ell}} = \oint_S \left[\frac{\partial \vec{\mathbf{B}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{B}}\right] \cdot d\vec{\mathbf{a}} \tag{9}$$

Using the vector relation

$$\vec{\boldsymbol{\nabla}} \times (\vec{\mathbf{B}} \times \vec{\mathbf{v}}_0) = (\vec{\mathbf{v}}_0 \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{B}} + \vec{\mathbf{B}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{v}}_0) - \vec{\mathbf{v}}_0(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}}) - (\vec{\mathbf{B}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{v}}_0 = (\vec{\mathbf{v}}_0 \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{B}} \tag{10}$$

where the last thee terms in the middle expression are zero since the velocity is independent of the coordinates, and the divergence of $\vec{\mathbf{B}}$ is zero. Substituting the curl expression into Eq. 9, and converting the surface integral of the curl to a line integral, we get

$$\oint \left(\vec{\mathbf{E}} - \vec{\mathbf{v}}_0 \times \vec{\mathbf{B}}\right) \cdot d\vec{\boldsymbol{\ell}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{11}$$

The form of the Faraday's law has changed with a velocity dependent term being added; similar terms are added to the other Maxwell equations. Therefore, Faraday's law is not invariant to a Galilean transformation, so either classical mechanics has to change or electrodynamics must change.

### 1.3 Frame Transformations

Given that we wish to satisfy the results of the Michelson and Morely experiment, and that we want the Maxwell equations to remain invariant, what are the conditions that we need to impose on our description of physical phenomena. Einstein proposed the following two conditions:

1. The laws of physics apply in all inertial frames of reference—*this is due to Galileo's observations about mechanics*;
2. The speed of light in vacuum is the same for all inertial observers, regardless of the relative motion of the source.

The first law states that there is no absolute rest (preferred) frame, while the second is a statement of the results of the Michelson and Morely experiment.

In addition to Einstein's two postulates, we need to impose the following two conditions:

1. Space is isotropic;
2. Space is homogeneous.

We also need to define an inertial frame of reference. This is rather complicated, and there are many ways of doing this. One way of defining an inertial frame is to say that it is a frame that moves at a constant velocity relative to the most distance stars that can be observed. A second and more rigors method of defining an inertial frame, is a frame where Newton's first law applies. In a non-inertial frame, there are fictitious forces that cause objects to not travel in straight lines even though not acted on by forces. Finally, we can give a very general definition of an inertial frame:

1. The distance between points $P_1$ and $P_2$ is independent of time;
2. The clocks that sit at every point ticking of the time coordinate $t$ are synchronized and all run at the same rate;
3. The geometry of space at any constant time $t$ is Euclidean.

What conditions do the postulates impose on coordinate transformations? Assume that we are at rest relative to a source of light, and we define our coordinates such we and the source are at the origin. Then the following condition must be satisfied

$$(ct)^2 - x^2 - y^2 - z^2 = 0 \tag{12}$$

An observer in a second frame with relative $v_0$ must satisfy the same condition, but with coordinates relative to his frame (assume that the light is emitted when the two observers are at the same position, and the second observer is standing at the origin of his coordinate system). This observer must satisfy the condition

$$(ct')^2 - x'^2 - y'^2 - z'^2 = 0 \quad \Rightarrow \quad (ct')^2 - x'^2 - y'^2 - z'^2 = (ct)^2 - x^2 - y^2 - z^2 \qquad (13)$$

the last condition corresponds to the position and time the same object is hit in both frames, in terms of the coordinates of each frame. Even though this condition holds, it doesn't tell us how to transform between frames.

Let's go through a series of thought experiments to determine how the coordinates transform. The first experiment involves the emission of light and its detection as viewed from different inertial frames. Assume that we have a railroad car with an isotropic light source in the middle. After the light source flashes, the light will be detected at both ends of the car simultaneously as view by an observer on the car (we still need to define an observer). An observer on the ground will not see the same thing. We will assume that the car is traveling to the right with a relative velocity $v$. The ground observer will see the light strike the rear of the car first, then the front of the car. The reason is that the car during the transit time of the light has moved forward by a distance $vt$. The light has a longer path to travel in the forward direction and shorter in the backward direction

$$\begin{aligned} ct = \ell - vt_b \quad &\Rightarrow \quad t_b = \frac{\ell}{c+v} \\ ct = \ell + vt_f \quad &\Rightarrow \quad t_f = \frac{\ell}{c+v} \end{aligned} \qquad (14)$$

where $t_b$ corresponds to the transit time to the back of the car and $t_f$ toward the front of the car. Events are only simultaneous in a given frame, not in all frames.

Before we go on, we need to define an observer. Our definition of an observer will not be what one sees with ones eyes, but will correspond to an inertial frame where all the clocks in the frame have been synchronized. Notice that the clocks will only be synchronized when examined in this one frame. Also notice that this will be different that what a camera sees. In addition, we will at this point define a frame as primed if it travels to the right relative to another frame (the unprimed frame).

Let's consider a second thought experiment. Assume that we are standing in the unprimed frame, and that the primed frame is moving relative to the unprimed frame with relative velocity $\beta = v/c$ along the $x$-axis. The $x$-axis lie on top of each other. Consider an observer standing at the origin of each frame, with both observers being the same height. If the observer in the unprimed frame sees the observer in the primed frame to have shrunk, then he can hold out a stick just above the head of the primed observer. The first postulate of relativity claims the the primed observer must be able to do the same, but this forms a contradiction, since the primed observer must be able to go under the stick according to the unprimed observer, but not according to the primed observer. Therefore, coordinates transverse to any motion must not change—if I measure something at $y = 1$ m in the unprimed frame, I measure it at $y' = 1$ m in the primed frame.

What about the time coordinate? Since the speed of light is the same in all frames, we can use it to setup a clock. We will define a unit of time as the time interval that it takes light to travel between a source and a mirror a distance $\ell$ above the source (perpendicular to the direction of relative motion between frames) and back to the source. In the rest frame of the observer, one unit of time is $\Delta t' = 2\ell/c$. But if the unprimed observer looks in the primed frame, he has to account for the transit time for the light and the fact that the train has travel some distance during this time. Therefore he sees one unit of time as

$$(c\Delta t)^2 = (v\Delta t)^2 + \ell^2 \quad \Rightarrow \quad \Delta t = \frac{2\ell}{c\sqrt{1-\beta^2}} \quad \Rightarrow \quad \Delta t = \frac{\Delta t'}{\sqrt{1-\beta^2}} \tag{15}$$

where the factor of 2 in the second equation accounts for the fact that the first equation corresponds only to the time to travel from the source to the mirror. Notice that for small velocities relative to $c$, a unit of time in both frames is the the same, but as we approach the speed of light, time appears to slow down in the other frame, note that primed observer sees the same thing in the opposite frame.

How do coordinates (lengths) transform along the direction of relative motion? To do this, rotate the clock in the unprimed frame by 90° to lie along the direction of motion. An observer standing on the train sees the light pulse complete a round trip in time $\Delta t' = 2\ell'/c$. Let's ask the same of an observer on the ground. Assume that the two origins coincide when the pulse of light is emitted. As observed in the unprimed frame, the mirror that is used to build the clock is moving away from the pulse at velocity $v$. The time to reach the mirror is

$$ct_r = \ell + vt_r \quad \Rightarrow \quad t_r = \frac{\ell}{c(1-\beta)} \tag{16}$$

where $\ell$ is the length of the clock as observed by the unprimed observer. The time for the light to return must take into account that the point of the source is moving toward the light pulse

$$ct_l = \ell - vt_l \quad \Rightarrow \quad t_l = \frac{\ell}{c(1+\beta)} \tag{17}$$

We can now equate the time interval calculated ($t_r + t_\ell$) to the transformation between time intervals given by Eq. 15

$$\Delta t = t_r + t_l = \frac{2\ell}{c}\frac{1}{1-\beta^2} = \Delta t' \frac{1}{\sqrt{1-\beta^2}} \quad \Rightarrow \quad \ell = \ell'\sqrt{1-\beta^2} \tag{18}$$

where $\Delta t' = 2\ell'/c$. The unprimed observer measures a shorter length in the primed frame, and visa-versa.

## 1.4 The Lorentz Transformations

Having determined how intervals transform, we are now ready to calculate the coordinate transformations. Let's assume that an event occurs at time $t$ and position $x$ in the unprimed frame. The primed frame is traveling with a relative velocity $v$ along the positive $x$ axis. Both observers are standing at the origin of their coordinate systems. The observer in the primed frame sees the following:

1. When both clocks display zero time (this corresponds to the time when both origins overlap), the point $x$ will appear to be at
$$x' = x\sqrt{1-\beta^2} \tag{19}$$
2. During the time interval $t'$ (the interval before the event occurs as seen by the primed observer) the unprimed system moves by $vt'$ to the left

3. At time $t'$ when the event occurs, the position of the event is

$$x' = x\sqrt{1-\beta^2} - vt' \quad \Rightarrow \quad x = \frac{x' + vt'}{\sqrt{1-\beta^2}} \tag{20}$$

By symmetry, we arrive at the inverse transformation

$$x' = \frac{x - vt}{\sqrt{1-\beta^2}} \tag{21}$$

Combining Eq. 20 and 21, then solving Eq. 21 for $x$ and equating the two we get

$$\left.\begin{array}{l} x = \frac{x'+vt'}{\sqrt{1-\beta^2}} \\ x = x'\sqrt{1-\beta^2} + vt \end{array}\right\} \quad \Rightarrow \quad ct = \frac{ct' + x'\beta}{\sqrt{1-\beta^2}} \tag{22}$$

and by symmetry, we get the inverse transformation

$$ct = \frac{ct' - x'\beta}{\sqrt{1-\beta^2}} \tag{23}$$

The Lorentz transformations can be put into a simple form using matrix multiplication as follows

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{24}$$

where $\gamma \equiv 1/\sqrt{1-\beta^2}$

### 1.5 Simultaneity—An Example

From the previous section we found that time in a moving frame is dilated by

$$\Delta t = \Delta t' \gamma \tag{25}$$

while lengths are contracted in the moving frame

$$\Delta x = \frac{\Delta x'}{\gamma} \tag{26}$$

where in both cases the frame is moving with a velocity $\beta$ relative to the observer. Now consider an observer in the moving frame making a measurement of the speed of light by emitting a light pulse and measuring the time interval $\Delta t$ for it to cover a distance $\Delta x$. Therefore, we get

$$c = \frac{\Delta x'}{\Delta t'} \tag{27}$$

If we blindly apply the time dilation and length contraction formulas, we get

$$c = \frac{\Delta x}{\Delta t} = \gamma^{-2} \frac{\Delta x'}{\Delta t'} \tag{28}$$

the speed of light is not the same in both frames, yet we required it be the same in the derivation of the two formulas. So what is wrong?

In order to determine what went wrong, we need to determine the requirements to make a measurement. In the rest frame, the pulse will be emitted from $x' = 0$ at $t' = 0$ and received at $x' = \ell'$ at $t' = t'_f$. This leads to $c = \ell'/t'_f$. Now let's ask what happens in the unprimed frame. Assume for simplicity that the light pulse is emitted when the origins of the two frames coincide. The Lorentz transformations tell us that the position $x' = \ell$ corresponds to

$$\ell = (\ell' + c\beta t'_f)\gamma \tag{29}$$

while the time transformation is

$$ct_f = (ct'_f + \ell'\beta)\gamma \tag{30}$$

Now use the relation $t'_f = \ell'/c$. The transformations then become

$$\left.\begin{array}{l}\ell = (\ell' + \beta\ell')\gamma \\ ct_f = (\ell' + \ell'\beta)\gamma\end{array}\right\} \quad \Rightarrow \quad \frac{\ell}{t_f} = c \tag{31}$$

Therefore, the speed of light is the same in both frames.

The question that we must now answer is what went wrong with the first approach? In our original derivation of time dilation and length contraction, we assumed that the time interval was measured at the same position in the primed frame. Using the Lorentz transformations, the time dilation formula is

$$\Delta t = \gamma\left[(t'_f + c\beta x') - (t'_i + c\beta x')\right] = \gamma\left[t'_f - t'_i\right] = \gamma\Delta t' \tag{32}$$

For the case of length contraction, we again measured the transit time of the light back to its starting point in the primed frame. During its transit in the unprimed frame, it has traveled a distance that is longer than the separation of observer and mirror; $\ell + vt_f$ where $\ell$ is the measured separation in the unprimed frame and $vt_f$ is the distance the frame travels to the right. This doesn't give us the contraction directly, we must still include the time dilation between primed and unprimed frames. This still doesn't answer the question. What we need to determine is what we mean by a length measurement. A length measurement requires measuring the end points simultaneously ($\Delta t = 0$). In this case we get

$$\Delta x' = \gamma\left[(x_f - c\beta t) - (x_i - c\beta t)\right] = \gamma(x'_f - x'_i) = \gamma\ell \quad \Rightarrow \quad \ell = \frac{\ell'}{\gamma} \tag{33}$$

### 1.6 Example—*Transforming a Fast Moving Cube*

Let's consider a cube that is traveling close to the speed of light at a distance that is large compared to its dimensions, and ask the question what does a person (camera) see when the edge parallel to the direction of motion is perpendicular to the persons eye. Figure 1 shows the variables needed to define this problem.

We first note that the length $\overline{AB}$, which corresponds to the rest frame length, is Lorentz contracted to length $\overline{AB'}$

$$\overline{AB'} = \ell/\gamma = \ell\sqrt{1-\beta^2} \tag{34}$$

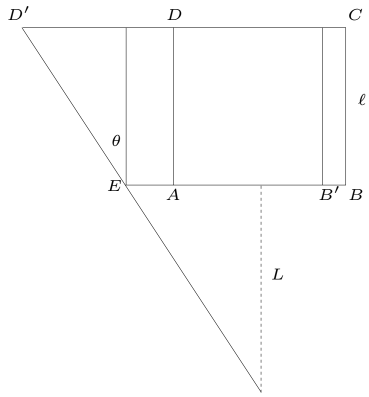


Figure 1: Geometry of fast moving cube. The variables are defined in the text.

Next we note that the light is emitted from the upper left hand corner earlier than from the lower left hand corner. This is due to the finite propagation time of the light and that the eye combines all the light at one instant of time. The light is emitted from the position $D'$ and follows the trajectory $\overline{D'E}$ to the eye. The distance $\overline{D'D}$ corresponds to the distance the cube has traveled in the time it takes the light to reach point $E$

$$\left.\begin{array}{l} \overline{D'D} = vt \\ \overline{D'E} = ct = \dfrac{\ell}{\cos\theta} \end{array}\right\} \quad \Rightarrow \quad \frac{\overline{D'D}}{v} = \frac{\overline{EA}}{c} = \frac{\ell}{c\cos\theta} \tag{35}$$

Finally, the distance $\overline{EA}$, the separation between where the upper left corner appears to be and the lower left corner appears, is given by

$$\overline{EA} = \overline{D'D} - \ell\tan\theta = \left(\frac{\ell\beta}{\cos\theta} - \frac{\ell\sin\theta}{\cos\theta}\right) \approx \ell\beta \tag{36}$$

where we use the assumption that $L \gg \ell$ in the last step. Therefore, the two lower corners will be separated by a distance

$$\overline{AB'} = \ell/\gamma = \ell\sqrt{1-\beta^2} \tag{37}$$

and the upper left hand corner will appear at a distance

$$\overline{EA} = \ell\beta \tag{38}$$

therefore the back end of the cube is visible. One can consider the cube to be rotated by an angle $\alpha$ where $\beta = \sin\alpha$ which gives

$$\overline{EA} = \ell\sin\alpha = \ell\beta \quad \text{and} \quad \overline{AB'} = \ell\cos\alpha = \ell\sqrt{1-\beta^2} \tag{39}$$

where the relation $\cos^2\alpha = \sqrt{1-\sin^2\alpha}$ is used in the last step of the second expression.

### 1.7 Invariants

We have previously shown that the interval

$$\Delta s^2 = -(ct)^2 - x^2 - y^2 - z^2 = -(ct')^2 - x'^2 - y'^2 - z'^2 = 0 \tag{40}$$

for a pulse of light, since the $c$ has the same value in all frames of reference. Since we used this information to arrive at the Lorentz transformations, we might ask if this holds for all intervals between events. Let's assume that in the unprimed frame an event occurs at $(x, t)$, therefore the interval between the event and the origin is

$$\Delta s^2 = -(ct)^2 - x^2 \tag{41}$$

where we assume that the event occurs at $z = y = 0$ for simplicity. In the primed frame, which is traveling to the right at a velocity $\beta = v/c$, this interval corresponds to

$$\begin{aligned} -(ct')^2 - x'^2 &= -(ct - \beta x)^2 \gamma^2 + (x - c\beta t)^2 \gamma^2 \\ &= \frac{-(ct)^2 (1 - \beta^2) + x^2 (1 - \beta^2)}{1 - \beta^2} = -(ct)^2 + x^2 \end{aligned} \tag{42}$$

as we would expect, this interval forms an invariant[1].

A more convenient way of writing the interval is $\Delta s^2 = (\Delta x^\mu)(\Delta x_\mu)$ which defines the dot product in a 4-dimensional space-time. Physically the invariant is the proper time (time in the objects rest frame). The quantity $x^\mu = (ct, x, y, z)$ is referred to as a contravariant vector, while the quantity $x_\mu = (-ct, x, y, z)$ is a covariant vector. The invariant interval is therefore

$$\Delta s^2 = -(c\Delta t)^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \tag{43}$$

Notice that the invariant interval can be positive, zero, or negative. The following terminology

1. $\Delta s^2 < 0$ the interval is timelike, a frame can be found where events occur at the same location;
2. $\Delta s^2 = 0$ the interval is lightlike, events are connected by a signal traveling at the speed of light;
3. $\Delta s^2 > 0$ the interval is spacelike, a frame can be found where events occur at the same time but separated spatially.

Timelike intervals are connected by frames that travel at relative velocities less than the speed of light, while spacelike intervals are connected by frames with relative velocities greater than the speed of light (*this last statement should be obvious since* $\Delta x > c\Delta t \Rightarrow \Delta x/\Delta t > c$).

If we are in the rest frame of an event, the invariant interval is

$$\Delta s^2 = -(c\Delta t)^2 \tag{44}$$

which is referred to as the proper time $c\tau^2 = -\Delta s^2$. Therefore, the invariant interval always gives the proper time (time in the events rest frame) since it is an invariant. Notice also that the proper time of an object is shorter if it is moving relative to another observer (see Fig. 3). In this case the proper time is given by

$$cd\tau = \sqrt{(cdt)^2 - (dx)^2} = \sqrt{1 - \beta^2}\, dt \tag{45}$$

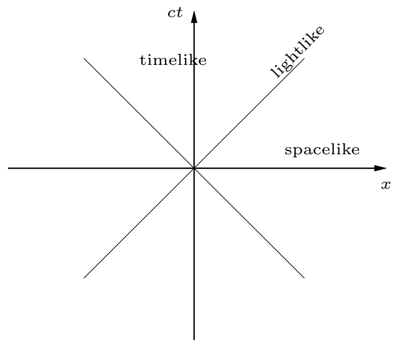


Figure 2: This figures shows the different regions of space-time.

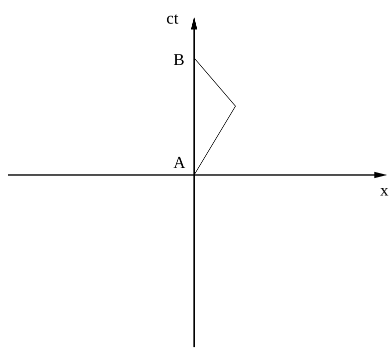


Figure 3: This figure show the space-time path of two travelers: One that stays at $x = 0$ and travels from $ct = A$ to $ct = B$, and a second that traveler that travels both spatially and in time. Note that the second traveler will record a shorter proper time.

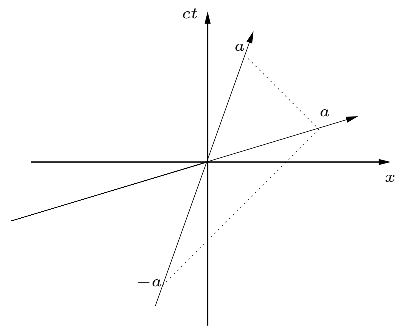


Figure 4: The figure shows the emission of light from the $x' = 0$ at $ct' = -a$, reflected from $x' = a$ at $ct' = 0$ and received back at $x' = 0$ at $ct' = a$. Since $a$ is arbitrary, this procedure can be used to define the $x'$ axis. Notice that light path can be extended to the $x$ and $ct$ axis. In this case it is easy to deduce that the angle that the $t'$ axis makes with the $t$ axis is the same as the $x'$ axis makes with the $x$ axis.

The last issue that we wish to explore before moving on, is the issue of how the axis transform between frames. Consider the axis for the unprimed frame given in Fig 4, which are orthogonal. Now ask what they look like for an observer moving to the right with a velocity $v$ (primed frame). The time axis corresponds to the locus of points that remain at the origin of the moving frame, therefore it will be moving at a velocity $v$ and will have a slope $1/v$ relative to the unprimed frame. The $x'$ axis is found by starting in the primed rest frame and determining the locus of points that correspond to zero time. This is done by emitting a pulse at time $ct' = -a$ $x' = 0$ reflecting it at time $ct' = 0$ (this corresponds to $x' = a$) and recieving it back at $x' = 0$ at $ct' = a$. When viewed from the unprimed frame, the light signal still has to hit the same points, but the path of the light has a slope of 45°. The light paths form two isosceles triangles, therefore the angle that the $x'$ axis makes relative to the $x$ axis is the same as the angle that the $ct'$ axis makes to the $ct$ axis. Notice that we can calibrate the primed axis by using the invariant interval. Start on the $ct$ axis with the invariant interval beng $s^2 = -(ct_0)^2$ any other point that satisfies $-(ct_0)^2 = -(ct)^2 + x^2$ has the same invariant interval and can be reached by a Lorentz transformation. If we transform to the primed frame, and ask what point this corresponds to on the $ct'$ axis, we find that the distance from the origin is longer than it is on the $ct$ axis. Therefore one unit of time in the unprimed frame is shorter than one unit in the primed frame. The same arguements can be used for the $x$ and $x'$ axis with the same conclusion.

Another set of four-vectors that lead to an invariant, is the four-velocity. The 4-velocity is defined by taking the time derivative of the coordinates of an object relative to a specific reference frame. The derivative is taken relative to the proper time, the time in the rest frame of the object

$$u^\mu = \frac{dx^\mu}{d\tau} \tag{46}$$

In the rest frame of the object, the 4-velocity is

$$u^\mu = (c, 0, 0, 0) \quad \Rightarrow \quad u^\mu u_\mu = -c^2 \tag{47}$$

[1] This quantity can be generalized so that it holds in non-inertial frames also.

The 4-momentum is defined from the 4-velocity by multiplying by the rest mass of the object

$$p^\mu = m_0 c u^\mu \tag{48}$$

In the objects rest frame, the 4-momentum is

$$p^\mu = (m_0 c^2, 0, 0, 0) \quad \Rightarrow \quad p^\mu p_\mu = -m_0^2 c^4 \tag{49}$$

The 4-momentum when the object is moving relative to a second frame is

$$p^\mu = m_0 c \left( c \frac{dt'}{d\tau} \right) \left( \frac{1}{c} \frac{dx^\mu}{dt'} \right) \quad \Rightarrow \quad p^\mu = (m_0 c^2 \gamma, m_0 c \beta_x \gamma, m_0 c \beta_y \gamma, m_0 c \beta_z \gamma) \tag{50}$$

where the time dilation formula is used to transform between frames

$$dt' = \gamma d\tau \tag{51}$$

In addition to those invariants defined above, any quantity that is a pure number must remain invariant. For instance, the number of particles contained in a box must be the same in both frames. The density may change, but the number remains the same.

# Physics 4183 Electricity and Magnetism II
# Vectors, Forms and Tensors

## 1 Introduction

Before we start our discussion of special relativity as it applies to electromagnetism, we introduce the concept of 4-vectors and tensors. We will base our discussion on the only 4-vector that we have defined so far, the coordinates. We will also need the Lorentz transformations, in order to understand how various 4-vectors transform. We will start with a the definition of a linear vector spaces, the move on to define and discuss tensors.

### 1.1 Linear Vector Spaces

Before we start our discussion, we will define a linear vector space. First of all, a vector space is

1. If $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are elements of the space, then $\vec{\mathbf{A}} + \vec{\mathbf{B}}$ is also an element;
2. Addition is associative $(\vec{\mathbf{A}} + \vec{\mathbf{B}}) + \vec{\mathbf{C}} = \vec{\mathbf{A}} + (\vec{\mathbf{B}} + \vec{\mathbf{C}})$;
3. The space contains an identity $\vec{\mathbf{A}} + \vec{\mathbf{0}} = \vec{\mathbf{A}}$;
4. The space contains an inverse $\vec{\mathbf{A}} + \vec{\mathbf{B}} = \vec{\mathbf{0}}$;
5. Addition is commutative $\vec{\mathbf{A}} + \vec{\mathbf{B}} = \vec{\mathbf{B}} + \vec{\mathbf{A}}$;
6. For a real scalar $a$ $a\vec{\mathbf{A}}$ is in the vector space;
7. Scalar multiplication is associative $(ab)\vec{\mathbf{A}} = a(b\vec{\mathbf{A}})$;
8. Scalar multiplication is distributive $(a+b)\vec{\mathbf{A}} = a\vec{\mathbf{A}} + b\vec{\mathbf{A}}$ and $a(\vec{\mathbf{A}} + \vec{\mathbf{B}}) = a\vec{\mathbf{A}} + a\vec{\mathbf{B}}$;
9. There is an identity under scalar multiplication $1\vec{\mathbf{A}} = \vec{\mathbf{A}}$.

### 1.2 Coordinate 4-Vector

The coordinate 4-vector is defined as follows

$$x^\mu = (ct,\ x,\ y,\ z) \tag{1}$$

and transform under the Lorentz transformation as

$$x^{\nu'} = \Lambda^{\nu'}{}_{\mu} x^\mu \quad \Rightarrow \quad \Lambda^{\nu'}{}_{\mu} = \frac{\partial x^{\nu'}}{\partial x^\mu} \quad \Rightarrow \quad x^{\nu'} = \frac{\partial x^{\nu'}}{\partial x^\mu} x^\mu \tag{2}$$

where the second equation comes from differentiating the first equation and noting that $\Lambda^{\nu'}{}_{\mu}$ is independent of the coordinates (constant). The primed frame is traveling with a velocity $\beta$ to the right relative to the unprimed frame. The Lorentz transformation between the two frames is given by

$$\left[\Lambda^{\nu'}{}_{\mu}\right] = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

We can transform in the opposite direction by using the following relations

$$\Lambda^{\mu}{}_{\nu'} x^{\nu'} = \Lambda^{\mu}{}_{\nu'} \Lambda^{\nu'}{}_{\alpha} x^{\alpha} = x^{\mu} \tag{4}$$

where we deduce that $\Lambda^{\mu}{}_{\beta'}\Lambda^{\beta'}{}_{\alpha} = \delta^{\mu}{}_{\alpha}$; this is the 4-dimensional version of the Kronecker delta function, which equal 1 if the indicies are the same and zero otherwise. That the product of the transform and inverse transform is the unit matrix should not come as a surprise, since the inverse should simply undo what the transform does. The inverse transformation in matrix notation is

$$\left[\Lambda^{\mu}{}_{\nu'}\right] = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

In a Euclidean vector space, vectors are written as the product of a unit vector (basis vector) and the magnitude of the component. We can do the same thing in 4-dimensions

$$\vec{\mathbf{x}} = x^{\mu}\vec{\mathbf{e}}_{\mu} = x^{\mu'}\vec{\mathbf{e}}_{\mu'} \tag{6}$$

where the second equality state that the vector is independent of what coordinates are selected; the value of the components, and the unit vectors are not but the vector corresponds to the same physical object. Given that vectors are independent of coordinates, the transformation of the unit vectors are the inverse of the transformation of the components

$$\vec{\mathbf{e}}_{\nu'} = \Lambda^{\mu}{}_{\nu'}\vec{\mathbf{e}}_{\mu} \quad \text{while} \quad x^{\nu'} = \Lambda^{\nu'}{}_{\mu} x^{\mu} \tag{7}$$

therefore vectors are call contravariant vectors; the components transform opposite to the unit vectors[1]

### 1.3 The Metric Tensor

We have now defined a vector along with its transformation properties. Next we would like to define the inner (dot) product in 4-dimensions. To do so, we introduce the metric tensor. A tensor of type $\binom{0}{N}$ is a function that transforms $N$ vectors into a real number. In addition, the tensor is linear in its $N$ arguments. Clearly the dot product can be defined in terms of a tensor of type $\binom{0}{2}$, since it takes two vectors and transforms them into a real number. Therefore, we define the metric

[1] In this set of notes I will refer to objects with upper indicies as vectors, which is a more modern notation. Just keep in mind that the text, as do many other books, refer to them as contravariant vectors.

tensor as taking two vectors (this will be generalized later) that produce the inner product of the two

$$\mathbf{g}(\vec{\mathbf{x}},\vec{\mathbf{x}}) = x^\mu x^\nu \mathbf{g}(\vec{\mathbf{e}}_\mu,\vec{\mathbf{e}}_\nu) = x^\mu x^\nu g_{\mu\nu} = -(ct)^2 + x^2 + y^2 + z^2 \quad \Rightarrow \quad [g_{\mu\nu}] = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

The metric tensor defines how lengths are measured in a specific space. In this case the form is simple, but for curvilinear coordinates or for an non-flat space-time as in those found in general relativity, the metric is much more complicated depending on the coordinates and having off-diagonal elements. In a Euclidean space using Cartesian coordinates, the metric is the unit matrix.

Let's now answer the question of how a tensor transforms. We start with the metric tensor applying it to the basis vectors

$$\mathbf{g}(\vec{\mathbf{e}}_\mu,\vec{\mathbf{e}}_\nu) = g_{\mu\nu} \tag{9}$$

Now transform each of the basis vectors

$$\mathbf{g}(\Lambda^\mu{}_{\alpha'}\vec{\mathbf{e}}_\mu,\Lambda^\nu{}_{\beta'}\vec{\mathbf{e}}_\nu) = \Lambda^\mu{}_{\alpha'}\Lambda^\nu{}_{\beta'}\mathbf{g}(\vec{\mathbf{e}}_\mu,\vec{\mathbf{e}}_\nu) = \Lambda^\mu{}_{\alpha'}\Lambda^\nu{}_{\beta'}g_{\mu\nu} = g_{\alpha'\beta'} \tag{10}$$

therefore each index must be transformed, since each index corresponds to a basis vector.

### 1.4 Tensors—A 1-Form

Let's define a tensor with a single slot (1-form), that produces a real number

$$\tilde{\mathbf{p}}(\vec{\mathbf{x}}) = x^\mu \tilde{\mathbf{p}}(\vec{\mathbf{e}}_\mu) = x^\mu p_\mu \quad \text{where} \quad p_\mu \equiv \tilde{\mathbf{p}}(\vec{\mathbf{e}}_\mu) \tag{11}$$

The Lorentz transformation properties of this object can be derived as follows

$$\tilde{\mathbf{p}}(\vec{\mathbf{e}}_{\nu'}) = \tilde{\mathbf{p}}(\Lambda^\mu{}_{\nu'}\vec{\mathbf{e}}_\mu) = \Lambda^\mu{}_{\nu'}\,\tilde{\mathbf{p}}(\vec{\mathbf{e}}_\mu) = \Lambda^\mu{}_{\nu'}\,p_\mu \tag{12}$$

This transforms in the same manner as the unit vectors, which is opposite to the components. This being the reason that these objects are often called covariant vectors, that is they transform like the unit vectors. In the modern language of differential geometry they are called forms, in this particular case it is a 1-form.

As in the case of vectors, we can define unit 1-forms that can be used to expand 1-forms in terms of their components

$$\tilde{\mathbf{p}} = p_\mu \tilde{\boldsymbol{\omega}}^\mu = p_\mu\,\Lambda^\mu{}_{\nu'}\Lambda^{\nu'}{}_\mu\,\tilde{\boldsymbol{\omega}}^\mu = p_{\nu'}\tilde{\boldsymbol{\omega}}^{\nu'} \tag{13}$$

Therefore, the unit 1-form transforms like the components of a vector.

To complete the discussion of 1-forms, let's determine how a unit 1-form acts on a unit vector. We know that a 1-form acting on a vector gives the sum of the product of the components

$$\tilde{\mathbf{p}}(\vec{\mathbf{x}}) = x^\mu\,p_\mu \tag{14}$$

If we write the 1-form and vector in components, we get

$$\tilde{\mathbf{p}}(\vec{\mathbf{x}}) = p_\mu \tilde{\boldsymbol{\omega}}^\mu(x^\nu \vec{\mathbf{e}}_\nu) = p_\mu x^\nu \tilde{\boldsymbol{\omega}}^\mu(\vec{\mathbf{e}}_\nu) \quad \Rightarrow \quad \tilde{\boldsymbol{\omega}}^\mu(\vec{\mathbf{e}}_\nu) = \delta^\mu{}_\nu \tag{15}$$

### 1.5 Vector to 1-Form Mapping

We have shown that the metric tensor defines the inner product between two vectors. In addition, we have shown that a 1-form acting on a vector gives something that looks like an inner product. Is there a connection between vectors and 1-forms, and is a 1-form acting on a vector an inner product?

Let's start with the metric tensor acting on a single vector. Since it acts on a single vector, it still has an slot open, which when filled gives a real number. Therefore, the metric tensor acting on a single vector is a 1-form

$$\mathbf{g}(\vec{\mathbf{x}},\ ) \equiv \tilde{\mathbf{x}}(\ ) \tag{16}$$

Now we have the 1-form act on a unit vector

$$\tilde{\mathbf{x}}(\vec{\mathbf{e}}_\mu) = \mathbf{g}(\vec{\mathbf{x}}, \vec{\mathbf{e}}_\mu) = \mathbf{g}(x^\nu \vec{\mathbf{e}}_\nu, \vec{\mathbf{e}}_\mu) = x^\nu \mathbf{g}(\vec{\mathbf{e}}_\nu, \vec{\mathbf{e}}_\mu) = x^\nu\, g_{\nu\mu} = x_\mu \tag{17}$$

which says that the metric tensor provides a mapping between vectors and 1-forms. Therefore, the inner product of two vectors is written as the product of the components of the vector and the 1-form obtained from the other vector

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A^\mu B^\nu g_{\nu\mu} = A^\mu B_\mu \tag{18}$$

where the components of the 1-form are given by

$$B^\nu g_{\nu\mu} = B_\mu = (-B^0, B^1, B^2, B^3) \tag{19}$$

### 1.6 Index Gymnastics

We can continue to increase the number of indicies on our tensor in a formal manner, but since our goal is to discuss the application of special relativity to electrodynamics, we will take the rules deduced so far and generalize them. To convert a vector into a 1-form or a 1-form into a vector, we use the metric tensor

$$A^\mu = g^{\mu\nu}\, A_\nu \quad \text{or} \quad A_\mu = g_{\mu\nu}\, A^\nu \tag{20}$$

keeping in mind that repeated up-down indicies are summed over. The components of vectors and forms transform opposite to each other

$$x^{\nu'} = \Lambda^{\nu'}{}_{\mu}\, x^\mu \quad \text{and} \quad x_{\nu'} = \Lambda^{\mu}{}_{\nu'}\, x_\mu \tag{21}$$

The product of two tensors of different rank, gives a third tensor of a rank one smaller than the larger of the two (the rank of a tensor is the total number of indicies)

$$A^\mu_\nu B^{\alpha\nu\beta}_{\mu\gamma\delta\eta} = C^{\alpha\beta}_{\gamma\delta\eta} \tag{22}$$

### 1.7 Derivatives

Let's consider the world line (path through space-time) of a particle, which we denote by $\Phi(x^\mu)$. Since $x^\mu$ can be written as a function of the proper time $\tau$, we can also write the functional

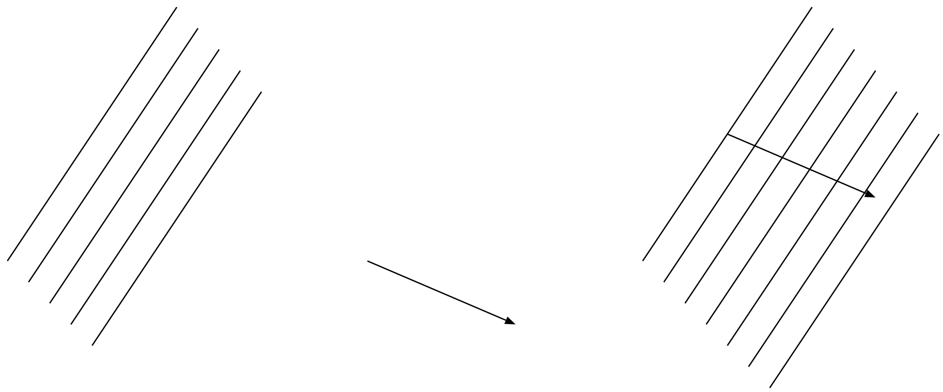


Figure 1: Graphical representation of a 1-form, a vector, and the inner (dot) product.

dependence of the world line in terms of the proper time $\Phi(\tau)$. The rate of change of the world line lies given by taking the total derivative with respect to the proper time

$$\frac{d\Phi(\tau)}{d\tau} = \frac{\partial\Phi(\tau)}{\partial x^0}\frac{dx^0}{d\tau} + \frac{\partial\Phi(\tau)}{\partial x^1}\frac{dx^1}{d\tau} + \frac{\partial\Phi(\tau)}{\partial x^2}\frac{dx^2}{d\tau} + \frac{\partial\Phi(\tau)}{\partial x^3}\frac{dx^3}{d\tau} \tag{23}$$

Notice that the left hand side is a scalar function, and that the gradient converts the 4-velocity into a scalar, which is the definition of a 1-form. Based on this, we rewrite the total derivative in a more suggestive manner

$$\frac{d\Phi(\tau)}{d\tau} = [\partial_\mu \Phi(\tau)]\, U^\mu \quad \text{where} \quad \partial_\mu = \left\{ \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right\} \tag{24}$$

### 1.8 Final Comments on Vectors and 1-forms

Based on the fact that the gradient behaves like a 1-form, 1-forms are graphically depicted as a series of surfaces, where the separation is smallest for 1-forms having large components and largest for 1-forms having small components, similar to how a gradient behaves. The inner product is then depicted as the number of surface pierced by the vector (see Fig. 1).

Instead of defining the inner product of a 1-form as a tensor that transforms a vector into a real number, we could have defined the inner product the other way around. A vector is a tensor that converts a 1-form into a real number. The fact that the two are equivalent shows a duality between vectors and 1-forms. In fact, the space of 1-forms is usually referred to as a dual vector space.

# Physics 4183 Electricity and Magnetism II
# Some Thought Experiments on Field Transformations

## 1 Introduction

In the last section, we developed the tools that will be used to study how the electric and magnetic fields transform between different inertial frames. We will start with a couple of thought experiments in this sections, then move on to a more formal development of the transformations.

### 1.1 Origins of the Magnetic Field

To show that the origins of the magnetic field are due to relativity, we will assume that the force a charged particle sees in its own rest frame is given by the electric field that it sees. We will also assume that the electric field is calculated in the same manner in all frames.

In a frame where a charge $q$ is traveling to the right with a velocity $u$ parallel to a wire, the wire carries a current composed of negative charges with linear charge density $-\lambda$ traveling to the left with a velocity $-v$ and positive charges with linear charge density $\lambda$ traveling to the right with velocity $v$. We also assume that $u < v$, so that in both frames we will consider, there are charges flowing in both directions. The current as seen in this frame is $2\lambda v$, and the net charge is zero. Therefore, in this reference frame the electric field is zero[1].

Let's now ask what the an observer traveling with the charge $q$ sees. The first question we ask is what happens to the charge density. In the frame $S$, the magnitude of the charge densities are same. In the frame $S'$, the charge densities will not be the same since the separation between charges (amount of Lorentz contraction) depends on the relative velocity of the charges with respect to the observer. The charge density, in any frame, is

$$\lambda = \frac{Q}{\ell} \tag{1}$$

assuming a uniform charge distribution where $\ell$ is the uniform separation between charges. In the frame $S'$, the separation is Lorentz contracted $\ell' = \gamma^{-1}\ell$ where we take $\ell$ to be the separation in the $S$ frame and $\lambda$ the linear charge density in this frame as well. This leads to the transformation for the charge density to be

$$\lambda' = \gamma\lambda \tag{2}$$

To determine the linear charge density in our example, we need to calculate the velocities of the charges in the $S'$ frame. The velocities of each group of charges in this frame are given by the velocity addition law since the charges have velocities relative to the $S$ frame

$$v_{\pm} = \frac{v \mp u}{1 \mp uv/c^2} \tag{3}$$

This allows us to calculate the charge density in the $S'$ relative to the charge density in the rest frame of the charge densities

$$\lambda'_{\pm} = \pm\gamma_{\pm}\lambda_0 \tag{4}$$

[1] In addition to no net charge (zeroth order moment), we assume that the are no higher order moments in the charge distribution

where $\lambda_0$ is the charge density in the rest frame of the charge distribution, which is given in the $S$ frame by

$$\lambda = \gamma\lambda_0 \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{5}$$

and

$$\gamma_\pm = \frac{1}{\sqrt{1 - v_\pm^2/c2}} \tag{6}$$

The term $\gamma_\pm$ can be simplified as follows

$$\begin{aligned}
\gamma_\pm = \frac{1}{\sqrt{1 - v_\pm^2}} &= \frac{1}{\sqrt{1 - (v \mp u)^2(1 \mp uv/c^2)^{-2}}} = \frac{(c^2 \mp uv)}{\sqrt{(c^2 \mp uv)^2 - c^2(v \mp u)^2}} \\
&= \frac{(c^2 \mp uv)}{\sqrt{c^4 + u^2v^2 - v^2c^2 - u^2c^2}} = \frac{(c^2 \mp uv)}{\sqrt{(c^2 - u^2)(c^2 - v^2)}} \\
&= \frac{1}{\sqrt{1 - v^2/c^2}} \frac{(1 \mp uv/c^2)}{\sqrt{1 - u^2/c^2}} = \gamma \frac{(1 \mp uv/c^2)}{\sqrt{1 - u^2/c^2}}
\end{aligned} \tag{7}$$

Therefore, the charge density in the $S'$ frame is give by

$$\lambda'_\pm = \gamma \frac{(1 \mp uv/c^2)}{\sqrt{1 - u^2/c^2}} \lambda_0 = \frac{(1 \mp uv/c^2)}{\sqrt{1 - u^2/c^2}} \lambda \tag{8}$$

where the charge density for the positive and negative charges is seen to be different. The net charge density is given by

$$\lambda_{\text{net}} = \lambda_+ - \lambda_- = -\frac{2\lambda uv}{c^2\sqrt{1 - u^2/c^2}} \tag{9}$$

The electric field from the charge density is

$$E_r = \frac{\lambda_{\text{net}}}{2\pi\epsilon_0 r} \tag{10}$$

and the force acting on the charge $q$ is

$$F' = qE = -\frac{\lambda v}{\pi\epsilon_0 c^2 r} \frac{qu}{\sqrt{1 - u^2/c^2}} \tag{11}$$

This force causes the charge to move toward the wire. Both frames must observe this.

To see whether this occurs in the $S$ frame, we transform the force from the $S'$ frame. The transformation law for the force can be calculated in the following manner: The force in the rest frame of an observer can be written in terms of the rate of change in the 4-momentum

$$F^\mu = \frac{dp^\mu}{d\tau} \tag{12}$$

Since this quantity is a 4-vector (this being the case since the 4-momentum is a 4-vector and $\tau$ is an invariant), its components transform as follows

$$F^{\mu'} = \Lambda^{\mu'}{}_\nu F^\nu \tag{13}$$

The force is applied in a direction that is perpendicular to the force, therefore the component of the 4-force has the same value in both frames. We take the direction of the force as being along the $y$ axis. The $y$ component of the 3-force is then given by

$$F^2 = \frac{dp^2}{d\tau} \quad \Rightarrow \quad F_y = F'_y \gamma^{-1} \tag{14}$$

where we use $d\tau = \gamma^{-1}\, dt$. Having found the elements needed to transform the force from the $S'$ to the $S$ frame, we fine the force acting on the charged particle in the $S$ frame to be

$$F = -\frac{\lambda v}{\pi \epsilon_0 c^2} \frac{qu}{r} = -qu \frac{\mu_0 I}{2\pi r} \tag{15}$$

where the current is $I = 2\lambda v$, and the relation $c^2 = 1/\epsilon_0 \mu_0$ was used. The expression we arrive at is the Lorentz force on acting on a moving charge in a magnetic field

$$B = \frac{\mu_0 I}{2\pi r} \tag{16}$$

which is exactly expression one would derive for an infinitely long wire. Therefore, we started by examining the force in a frame where only the electric field acts, we transform to a second frame where only the magnetic field acts.

### 1.2 Field Transformations—*Some Thought Experiments*

Let's consider the following thought experiment: Take a parallel plate capacitor with plate separation small compared to the size of the plates so that we can ignore the fringe fields at the edges of the capacitor. In the rest frame of the capacitor ($S_0$), the field is given by

$$\vec{\mathbf{E}}_0 = \frac{\sigma_0}{\epsilon_0} \hat{\mathbf{y}} \tag{17}$$

The field as viewed from the $S$ frame, which is traveling to the right along the $x$-axis with velocity $v_0$, will have the same form (*give discussion*)

$$\vec{\mathbf{E}} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{y}} \tag{18}$$

except that the value of the charge density changes because of Lorentz contraction. Since only the $x$ dimension of the plate is contracted, only one factor of $\gamma$ is introduced. The charge density as view by an observer in $S$ is

$$\sigma = \gamma_0 \sigma_0 \quad \Rightarrow \quad \vec{\mathbf{E}}^{\perp} = \gamma_0 \vec{\mathbf{E}}_0^{\perp} \quad \text{where} \quad \gamma_0 = \frac{1}{\sqrt{1 - v_0^2/c^2}} \tag{19}$$

To determine how the component of the field parallel to the motion transforms, we rotate the capacitor by 90°. In this case the charge density remains the same, therefore the field is the same for both observers

$$E^{\parallel} = E_0^{\parallel} \tag{20}$$

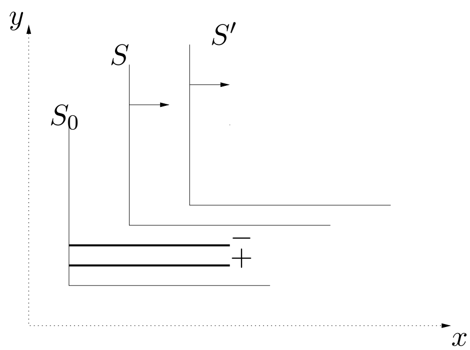


Figure 1: Reference frames used to calculate the fields in seen by different observerss ($x$-$y$ plane).

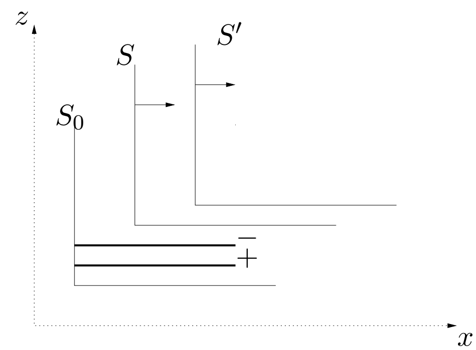


Figure 2: Reference frames used to calculate the fields in seen by different observers ($x$-$z$ plane).

These of course are not the most general transformations, since we have not included the magnetic field. To include magnetic fields, we calculate the transformations between a frame where the capacitor is not at rest ($S$), and a second frame ($S'$) that is moving relative to the frame $S$.

To generalize the transformations, we need to have both electric and magnetic fields in a single frame and then transform to a second frame. We can do this as shown in Fig 1, take a capacitor with surface charge density $\sigma_0$ in its rest frame $S_0$. Place an observer in frame $S$ that moves to the right with velocity $\beta_0$. The observer in this frame sees a surface charge density

$$\sigma = \gamma_0\sigma_0 \quad \text{where} \quad \gamma_0 = \frac{1}{\sqrt{1-\beta_0^2}} \tag{21}$$

The observer also sees a surface current density given by

$$\vec{\mathbf{K}}_{\pm} = \mp\sigma v_0\hat{\mathbf{x}} = \mp\gamma_0\sigma_0 v_0\hat{\mathbf{x}} \tag{22}$$

where the minus sign is due to the frame $S_0$ traveling to the left relative to frame $S$. The electric and magntic fields in frame $S$ are given by

$$\vec{\mathbf{E}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{y}} = \gamma_0\frac{\sigma_0}{\epsilon_0}\hat{\mathbf{y}} \qquad\qquad \vec{\mathbf{B}} = -\mu_0\sigma v_0\hat{\mathbf{z}} = -\gamma_0\mu_0\sigma_0 v_0\hat{\mathbf{z}} \tag{23}$$

where Gauss's and Ampere's laws are used to calculate the fields.

Now we ask what does an observer in frame $S'$ see, which is traveling at a velocity $v$ to the right relative to $S$. We wish to write our answer in terms of the fields seen in frame $S$, which means we write the fields in terms of $\sigma$. So far we only know how to transform the charge density from its rest frame. Therefore, we calculate the transformation of the charge density from its rest frame $S_0$ to the $S'$ frame. The velocity that an observer in the $S'$ frame measures for the capacitor is given by

$$\beta' = \frac{\beta+\beta_0}{1+\beta_0\beta} \tag{24}$$

Therefore transformation of the charge density from $S_0$ to $S'$ is given by

$$\sigma' = \gamma'\sigma_0 = \frac{\gamma'}{\gamma_0}\sigma \quad \text{where} \quad \gamma' = \frac{1}{\sqrt{1+\beta'^2}} \tag{25}$$

The ratio of the $\gamma$'s can be simplified as follows

$$\frac{\gamma'}{\gamma_0} = \frac{\sqrt{1-\beta_0^2}}{\sqrt{1-\beta'^2}} = \frac{\sqrt{1-\beta_0^2}}{\sqrt{1-\left(\frac{\beta+\beta_0}{1+\beta\beta_0}\right)^2}} = \frac{\sqrt{1-\beta_0^2}(1+\beta\beta_0)}{\sqrt{(1+\beta\beta_0)^2-(\beta+\beta_0)^2}}$$
$$= \frac{\sqrt{1-\beta_0^2}(1+\beta\beta_0)}{\sqrt{1+2\beta\beta_0+\beta^2\beta_0^2-(\beta^2+\beta_0^2-2\beta\beta_0)}} = \frac{\sqrt{1-\beta_0^2}(1+\beta\beta_0)}{\sqrt{(1-\beta_0^2)(1-\beta^2)}}$$
$$= \frac{(1+\beta\beta_0)}{\sqrt{1-\beta^2}} \quad (26)$$

Based on this transformation, the electric field in $S'$ in terms of the fields in the $S$ frame are given by

$$E'_y = \frac{(1+\beta\beta_0)}{\sqrt{1-\beta^2}}\frac{\sigma}{\epsilon_0} = \gamma(E_y - \beta c B_z) \quad (27)$$

where Eq. 23 was used to convert from charge densities to fields. The magnetic field is given by

$$B'_z = -\mu_0\sigma' v' = -\frac{(1+\beta\beta_0)}{\sqrt{1-\beta^2}}\mu_0\sigma c\frac{\beta+\beta_0}{1+\beta\beta_0} = \gamma(B_z - \beta/cE_y) \quad (28)$$

Notice that these are not simple transformations like those for 4-vectors.

To calculate the transformation of the fields $E_z$ and $B_y$, we rotate the capacitor by 90° (see Fig. 2). The transformation are arrived at following the same procedure as before replacing $E_y$ by $E_z$ in Eq. 27 and $-B_z$ by $B_y$ in Eq. 28. The field transformation are therefore

$$E'_z = \gamma(E_z + \beta c B_y) \qquad B'_y = \gamma(B_y + \beta/cE_z) \quad (29)$$

We have already shown that the $x$ component of the electric field stays the same ($E'_x = E_x$). The only thing that remains is to calculate the $x$ component of the magnetic field. To carry this out, we will use a solenoid with its axis along the $x$ direction. The solenoid's rest frame is the $S$ frame, and its field is chosen to be in the positive $x$ direction. The field of the solenoid is given by

$$B_x = \mu_0 n I \quad (30)$$

in the $S$ frame. In the $S'$ frame, the field is

$$B'_x = \mu_0 n' I' \quad (31)$$

Since $n$ is the number of current loops per unit length, we expect in the $S'$ frame that $n'$ will be larger due to length contraction. The transformation is $n' = \gamma n$. The current is the amount of charge crossing a point per unit time. Since the clock in the rest frame of the solenoid runs slower, the current will be lower in the $S'$ frame. The transformation is $I' = I/\gamma$. The two factors of $\gamma$ cancel, therefore the $x$ component of the magnetic field is the same in both frames $B'_x = B_x$.

The transformation equations are given by

$$E'_x = E_x \quad E'_y = \gamma(E_y - \beta c B_z) \quad E'_z = \gamma(E_z + \beta c B_y) \quad (32)$$
$$B'_x = B_x \quad B'_y = \gamma(B_y + \beta/cE_z) \quad B'_z = \gamma(B_z - \beta/cE_y) \quad (33)$$

# A Addition of Velocities

The relativistic addition of velocities can be easily performed using the 4-velocity

$$\eta^\mu = \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx^\mu}{dt} = \gamma\frac{dx^\mu}{dt} = \gamma v^\mu \tag{34}$$

Assume two observers, one in the frame $S$, which we take as the stationary frame, and the second in the frame $S'$, which is moving with a relative velocity $u$ with respect to $S$ with the motion parallel to their respective $x$-axis. The 4-velocity $\eta^\mu$ is measured with respect to $S$. The problem will be to determine the velocity as seen by the observer in $S'$. We will only concern ourselves with calculating the $x$ component of the velocity, therefore, the only transformation that we concern ourselves with is

$$\eta^{1'} = \gamma(\eta^1 - (u/c)\eta^0) = \frac{\eta^1 - (u/c)\eta^0}{\sqrt{1-(u/c)^2}}, \tag{35}$$

which is the standard transformation for any 4-vector. The next step is two write $\eta^\mu$ in terms of the normal velocity $v$

$$\frac{v'_x}{\sqrt{1-(v'_x/c)^2}} = \frac{\frac{v_x}{\sqrt{1-(v_x/c)^2}} - (u/c)\frac{c}{\sqrt{1-(v_x/c)^2}}}{\sqrt{1-(u/c)^2}}. \tag{36}$$

The final step is to solve for $u'_x$

$$v'_x = \frac{v_x - u}{1 - v_x u/c^2} \tag{37}$$

# Physics 4183 Electricity and Magnetism II
# Covariant Formulation of Electrodynamics

## 1 Introduction

Having briefly discussed the origins of relativity, the Lorentz transformations, 4-vectors and tensors, and invariants, we are now prepared to discuss the relativity as it applies to electromagnetism. We will start with the formal aspects, that is we will discuss the covariant forms of the continuity equation, and the Maxwell equations. From this point we will move on to discussing how the fields transform.

### 1.1 The Continuity Equation

We will use the continuity equation (charge conservation) as the starting point in our development of a covariant form of electrodynamics. The continuity equation has the simple form of time derivative of a scalar function plus the divergence of a vector function equal to a constant

$$\frac{\partial \rho}{\partial t} + \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{J}} = 0 \tag{1}$$

The derivatives can be written in the compact form $\partial_\mu$

$$\partial_\mu = \left( \frac{\partial}{\partial ct},\ \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z} \right) \tag{2}$$

If we assume that a current density 4-vector can be define as $J^\mu = (c\rho, \vec{\mathbf{J}})$, then the continuity equation takes on the very compact form

$$\partial_\mu J^\mu = 0 \tag{3}$$

which is an invariant. This states that charge is conserved locally in all Lorentz frames; note the derivatives and the components of the $J^\mu$ are not the same in all frames, but the form of the equation remains the same, the continuity equation is covariant. Based on the invariance of the continuity equation, the 4-vector $J^\mu$ transforms like $J^\nu = \Lambda^{\nu'}{}_{\mu} J^\mu$

$$\partial_\mu J^\mu = \partial_\mu \Lambda^{\mu}{}_{\nu'} \Lambda_{\alpha}{}^{\nu'} J^\alpha = \partial_{\nu'} J^{\nu'} \tag{4}$$

An alternate way of show that the charge and current densities form a 4-vector, is to recall that in a given frame where the charge density has a velocity relative to the observer the current density is given by

$$\vec{\mathbf{J}} = \rho \vec{\mathbf{v}} \tag{5}$$

where $\rho$ corresponds to the charge density as observed in the observers frame. The charge and current densities written in terms of the charge density in its rest frame is

$$\rho = \gamma \rho_0 \qquad\qquad J_x = \gamma v_x \rho_0 \tag{6}$$

where for simplicity we have oriented our axis parallel to the direction of motion. Again, recall that the boost factor can be written in terms of derivatives

$$\gamma = \frac{dt}{d\tau} \quad \Rightarrow \quad \gamma v_x = \frac{dt}{d\tau}\frac{dx}{dt} = \frac{dx}{d\tau} \tag{7}$$

but this implies that $\gamma$ is the zeroth component of the 4-velocity and $v_x\gamma$ is the first component of the 4-velocity. Therefore, the charge and current densities can be written as

$$J^\mu = \rho_0 u^\mu \tag{8}$$

The first question that we must ask ourselves, does this make sense? We start by looking at the charge density. Transforming from its rest frame, the charge density is (note in the rest frame of the charge density the current density is zero)

$$\rho = \gamma\rho' \quad \text{while the current density is} \quad J^1 = c\beta_x\gamma\rho' \tag{9}$$

To see whether this makes sense or not, let's consider a charge distribution place in a cube of length $\ell'$ in its rest frame. The cube is assumed to be traveling along the $x$-axis at a velocity $\beta_x$, with one of the sides parallel to the $x$-axis (see Fig. 1). The length of the cube is therefore contracted along the $x$-axis

$$\ell = \frac{\ell'}{\gamma} \tag{10}$$

Next we assume that the container has impenetrable walls, therefore the total charge remains within the cube in both frame. The charge density as observed relative to the unprimed coordinates is

$$\rho = \frac{Q}{\ell'^3/\gamma} = \gamma\rho' \tag{11}$$

as we had already deduced from the invariance of the continuity equation. The current density is the charge density multiplied by the velocity $J_x = c\beta_x\gamma\rho'$.

The continuity equation implies charge conservation. The total charge is the volume integral over the charge density. In the rest frame, the total charge is

$$Q' = \iiint_V \rho' dx'\, dy'\, dz' \tag{12}$$

In the unprimed frame, the total charge is

$$Q = \iiint_V \rho dx\, dy\, dz = \iiint_V \gamma\rho' \frac{dx'}{\gamma}\, dy\, dz = Q' \tag{13}$$

therefore both observers see the same total charge, and our original assumption of treating the charge and current densities as components of a 4-vector is correct.

## 1.2 The Maxwell Equations

As was previously shown, the Maxwell equations are not covariant. The example we used was Faraday's law. In the rest frame of a loop, the induced EMF is equal to minus the rate of change

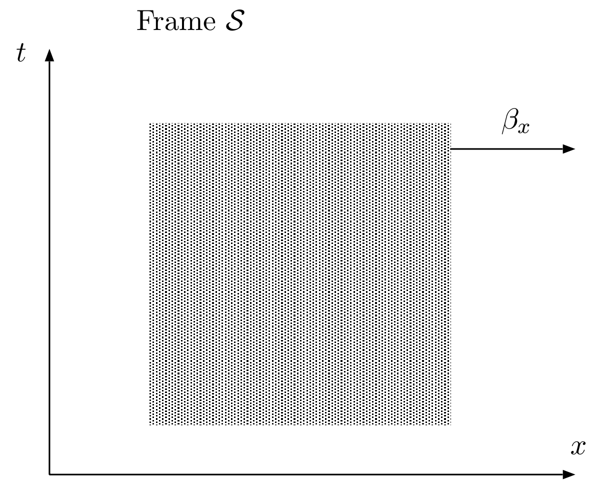


Figure 1: Box in uniform motion relative to the frame $\mathcal{S}$.

of the magnetic flux, while in a moving frame, the induce EMF has an added term that depends on the relative velocity of the two frames

$$\oint \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{\ell}} = -\frac{\partial \Phi_B}{\partial t} \quad \Rightarrow \quad \oint (\vec{\mathbf{E}} - \vec{\mathbf{v}} \times \vec{\mathbf{B}}) \cdot d\vec{\boldsymbol{\ell}} = -\frac{\partial \Phi_B}{\partial t} \tag{14}$$

Even though this is correct and consistent with what relativity tells us, it is not in the spirit of Einstein's first postulate, which states the the laws of physics are independent of the inertial frame they are viewed from. In this instance, we expect that Maxwell's equations retain the same form in all frames; the equations must be covariant (form invariant), the values of the fields can change, but the form of the equation cannot. So how do we show the equations to be covariant?

To begin with, let's write the Maxwell equations in free space

$$\begin{aligned} \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} &= \frac{\rho}{\epsilon_0} & \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} &= -\frac{\partial \vec{\mathbf{B}}}{\partial t} \\ \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}} &= 0 & \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} &= \mu_0 \vec{\mathbf{J}} + \mu_0 \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t} \end{aligned} \tag{15}$$

These equation can be expressed in terms of potential

$$\begin{aligned} \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}} = 0 \quad &\Rightarrow \quad \vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}} \\ \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \quad &\Rightarrow \quad \vec{\mathbf{E}} = -\left( \vec{\boldsymbol{\nabla}} \Phi + \frac{\partial \vec{\mathbf{A}}}{\partial t} \right) \end{aligned} \tag{16}$$

Let's assume that these equations can be rewritten in a covariant form by defining the 4-potential

$$A^\mu = \left( \Phi/c, \vec{\mathbf{A}} \right) \tag{17}$$

where we justify this by noting that using the 4-current density leads back to the equations for the potentials in terms of there sources[1]

$$A^\mu = \frac{\mu_0}{4\pi}\int \frac{J^\mu}{|\vec{\mathbf{R}}|}\, d^3\vec{\mathbf{r}} \tag{18}$$

It is clear that the current density is a 4-vector, based on our previous arguments, but for the 4-potential to be a 4-vector the ratio $d^3r/|\vec{\mathbf{R}}|$ must be an invariant. We start with an observer in the frame $S$, which is stationary with respect to the field point. The field point is located at $(0,0,0,0)$. The source point, in this frame, is located at $(ct,x,y,z)$ where $ct = \sqrt{x^2+y^2+z^2}$ corresponds to the propagation time between the two points, in other words, we have to take into account that any change that occurs at the source takes some amount of time to be felt elsewhere. The component of $\vec{\mathbf{R}}$ along its direction is $R = -ct$. Let's now ask what an observer sees in the $S'$ frame that is traveling at a velocity $\beta$ to the right relative to $S$. We will also assume that the time origins coincide $t = t' = 0$. The value of the component of $\vec{\mathbf{R}}$ along its direction is given by

$$R' = -ct' = -\gamma(ct - \beta x) = R\gamma\left(1 + \frac{x}{R}\beta\right) = R\gamma(1+\beta\cos\theta) \tag{19}$$

(see Fig. 2)

The quantity $d^3r'$ must take into account that changes that occur in the back end of the volume element takes a time $cdt$ longer to propagate to the field point. The differential element is $d^3r' = dx'dy'dz'$, where $dy'$ and $dz'$ are invariant. The other element is given by

$$dx' = \gamma(dx - \beta c\, dt) \tag{20}$$

but as we stated, the time difference between front and back has to be account for

$$dR = -c\, dt \quad \Rightarrow \quad dx' = \gamma(dx + \beta dR) = \gamma dx\left(1 + \beta\frac{dR}{dx}\right) = \gamma dx(1+\beta\cos\theta) \tag{21}$$

Therefore

$$\frac{d^3r'}{R'} = \frac{d^3r}{R} \tag{22}$$

The 4-potential as defined above it a 4-vector, since it is composed of a 4-vector $J^\mu$ and an invariant $d^3r/R$. This 4-vector is given by

$$A^\mu = (\phi/c, A_x, A_y, A_z) \tag{23}$$

where the factor of $1/c$ comes from the definition of the zeroth component of the 4-current $(c\rho)$ and the units used in Eq. 18.

Using the 4-potential, the relation between the electric field and the potential is written as follows

$$\frac{E^j}{c} = -\left(\partial_j A^0 + \partial_0 A^j\right) = -\left(\partial^j A^0 - \partial^0 A^j\right) = \left(\partial^0 A^j - \partial^j A^0\right) = F^{0j} \tag{24}$$

where we use

$$\partial^\mu = g^{\mu\nu}\partial_\nu \quad \Rightarrow \quad \partial^\mu = (-\partial_0, \partial_1, \partial_2, \partial_3) \tag{25}$$

[1] In order to prove that the 4-potential as defined has the correct transformation properties, requires the introduction of the retarded potential. That is the propagation speed of the field lines has to be taken into account. We will return to this subject later.

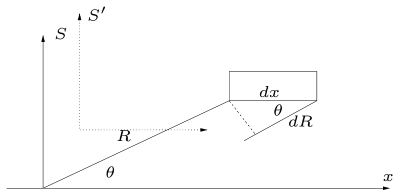


Figure 2: This shows the geometry used to show the invariance of $d^3r/R$.

The magnetic field becomes

$$B^k = \left(\partial_i A^j - \partial_j A^i\right) = \left(\partial^i A^j - \partial^j A^i\right) = F^{ij} \tag{26}$$

Combining the two sets, leads to

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \tag{27}$$

where $\mu$ corresponds to the row and $\nu$ to the column, and this tensor is called the Maxwell field strength tensor. Note that we started out with electric and magnetic fields, and ended up with a tensor that combines the two fields together.

To get back to the Maxwell equations, we need to find mathematical operations that give back the curl and divergence of the fields. Notice that if we apply $\partial_\mu$ on $F^{0\mu}$ we get the divergence of the electric field

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon_0} \quad \Rightarrow \quad c\,\partial_\mu F^{0\mu} = \partial_x E_x + \partial_y E_y + \partial_z E_z = \mu_0 J^0 = \mu_0 c \epsilon_0 c^2 \frac{\rho}{\epsilon_0} = \frac{\rho}{\epsilon_0} \tag{28}$$

while $\partial_j$ acting on $F^{ij}$ we get the curl of the magnetic field, as an example take $i = 1$

$$\begin{aligned} \left[\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}}\right]_x &= \mu_0 J_x + \epsilon_0 \mu_0 \frac{\partial E_x}{\partial t} \\ \Rightarrow \quad \partial_\nu F^{1\nu} &= \partial_0 F^{10} + \partial_1 F^{11} + \partial_2 F^{12} + \partial_3 F^{13} = \partial_y B_z - \partial_z B_y - \frac{1}{c^2}\partial_t E_x = \mu_0 J_x \end{aligned} \tag{29}$$

Therefore, taking the derivative of the field strength tensor gives

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu \tag{30}$$

To arrive at the homogeneous set of the Maxwell equations, expand them using the field strength tensor

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}} = 0 \quad \Rightarrow \quad \partial^1 F^{32} + \partial^2 F^{13} + \partial^3 F^{21} = 0 \tag{31}$$

and

$$\left[\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{B}}}{\partial t}\right]_x = 0 \quad \Rightarrow \quad \partial^2 F^{30} + \partial^3 F^{02} + \partial^0 F^{23} = 0 \tag{32}$$

The two equations can be combined to give

$$\partial^\alpha F^{\beta\gamma} + \partial^\gamma F^{\alpha\beta} + \partial^\beta F^{\gamma\alpha} = 0 \quad \Rightarrow \quad \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = 0 \tag{33}$$

where the totally antisymmetric tensor $\epsilon^{\alpha\beta\gamma\delta}$ is defined as

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{if } \alpha=0,\ \beta=1,\ \gamma=2,\ \delta=3 \text{ and even permutations of the indices} \\ -1 & \text{for odd permutations of the indices} \\ 0 & \text{if any indices are the same} \end{cases} \tag{34}$$

This equation can be put into a simpler form by using a duality transformation as discussed earlier in the semester. We make the replacements $\vec{\mathbf{E}} \to \vec{\mathbf{B}}$ and $\vec{\mathbf{B}} \to -\vec{\mathbf{E}}$; note that since there is no magnetic charge, it transform into no magentic charge after the transformation. The new tensor, $G^{\mu\nu}$, leads to the following form for the homogeneous Maxwell equation

$$\partial_\nu G^{\mu\nu} = 0 \quad \Rightarrow \quad G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_y/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix} \tag{35}$$

Note that the equation for $G^{\mu\nu}$ is the same equation as Eq. 33 for $F^{\mu\nu}$ using $G^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$.

### 1.3 The Lorentz Force

To complete the discussion of the equations of electrodynamics, we examine the Lorentz force, and put it into a covariant form. The standard form of the Lorentz force is

$$\frac{d\vec{\mathbf{p}}}{dt} = q\left(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) \tag{36}$$

Clearly this is not in a form that allows for simple transformations. Therefore, we rewrite the Lorentz force in replacing the time derivatives with the proper time derivatives, and add in the proper time derivative of $p^0$

$$\frac{d\vec{\mathbf{p}}}{d\tau} = q\left(u^0\vec{\mathbf{E}}/c + \vec{\mathbf{u}} \times \vec{\mathbf{B}}\right) \quad \text{and} \quad \frac{dp^0}{d\tau} = q\vec{\mathbf{u}} \cdot \vec{\mathbf{E}} \tag{37}$$

where the term $dp^0/d\tau$ is the rate of change of energy of the particle

$$W = \vec{\mathbf{F}} \cdot \ell \quad \Rightarrow \quad \frac{dW}{dt} = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}} \tag{38}$$

note there is no magnetic field term since it doesn't change the energy of a electrically charged particle. In the rest frame of the observer, this reduces to the standard form of the Lorentz force. Combining the Lorentz force with the components of the field strength tensor, we get

$$\frac{dp^\alpha}{d\tau} = qu_\beta F^{\alpha\beta} \tag{39}$$

which is in a covariant form (the product of tensor and 4-vector).

### 1.4 Field Transformations

The final item that needs to calculated is the transformation equations of the electric and magnetic fields. We will determine these by examining how the Maxwell field strength tensor transforms. If we blindly apply the rules that we have discussed, we find that the field strength tensor transforms like

$$F^{\alpha'\beta'} = \Lambda^{\alpha'}_{\ \mu}\Lambda^{\beta'}_{\ \nu}F^{\mu\nu} \tag{40}$$

one transformation tensor per index. To verify that this is correct, we write the field strength tensor in terms of the potential

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{41}$$

and transform each of the 4-vectors

$$F^{\alpha'\beta'} = \partial^{\alpha'}A^{\beta'} - \partial^{\beta'}A^{\alpha'} = \Lambda^{\alpha'}_{\ \mu}\partial^\mu\Lambda^{\beta'}_{\ \nu}A^\nu - \Lambda^{\beta'}_{\ \nu}\partial^\nu\Lambda^{\alpha'}_{\ \mu}A^\mu = \Lambda^{\beta'}_{\ \mu}\Lambda^{\alpha'}_{\ \nu}\left[\partial^\mu A^\nu - \partial^\nu A^\mu\right] = \Lambda^{\beta'}_{\ \mu}\Lambda^{\alpha'}_{\ \nu}F^{\mu\nu} \tag{42}$$

where we have used the known 4-vector transformations.

Based on the transformation of the field strength tensor, we can now derive the transformations for the electric and magnetic fields. As usual, we will assume that the primed frame travels with velocity $\beta$ along the $x$-axis of the unprimed frame. The $y$ and $y'$ axis are assumed to be parallel as are the $z$ and $z'$ axis. In matrix notation, the Lorentz transformation is

$$[\Lambda^{\mu\nu}] = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{43}$$

We can start by calculating $E'_x$ (where we only write down the terms that are non-zero)

$$E'_x = F^{0'1'} = \Lambda^{0'}_{\ \mu}\Lambda^{1'}_{\ \nu}F^{\mu\nu} = \Lambda^{0'}_{\ 1}\Lambda^{1'}_{\ 0}F^{10} + \Lambda^{0'}_{\ 0}\Lambda^{1'}_{\ 1}F^{01} = (1-\beta^2)\gamma^2 E_x = E_x \tag{44}$$

therefore the component of the electric field along direction of relative motion stays the same. Next we $E'_y$ (the transformation for $E'_z$ will have a similar form)

$$E'_y = F^{0'2'} = \Lambda^{0'}_{\ \mu}\Lambda^{2'}_{\ \nu}F^{\mu\nu} = \Lambda^{0'}_{\ 0}\Lambda^{2'}_{\ 2}F^{02} + \Lambda^{0'}_{\ 1}\Lambda^{2'}_{\ 2}F^{12} = \gamma E_y - \beta\gamma B_z = \gamma\left(E_y - c\beta B_z\right) \tag{45}$$

while the $E_z$ transforms like

$$E'_z = F^{0'3'} = \Lambda^{0'}_{\ \mu}\Lambda^{3'}_{\ \nu}F^{\mu\nu} = \Lambda^{0'}_{\ 0}\Lambda^{3'}_{\ 3}F^{03} + \Lambda^{0'}_{\ 1}\Lambda^{3'}_{\ 3}F^{13} = \gamma\left(E_z + c\beta B_y\right) \tag{46}$$

To calculate the transformations for the magnetic field, we proceed in the same manner as for calculating the electric fields. Start with $B_x$

$$B'_x = F^{2'3'} = \Lambda^{2'}_{\ \mu}\Lambda^{3'}_{\ \nu}F^{\mu\nu} = F^{23} = B_x \tag{47}$$

as was the case for $E_x$. The $B_y$ component transforms as

$$B'_y = F^{31} = \Lambda^{3'}_{\ \mu}\Lambda^{1'}_{\ \nu}F^{\mu\nu} = \Lambda^{3'}_{\ 3}\left(\Lambda^{1'}_{\ 0}F^{30} + \Lambda^{1'}_{\ 1}F^{31}\right) = \gamma\left(B_y + \beta E_z/c\right) \tag{48}$$

and finally the transformation for $B_z$

$$B'_z = F^{1'2'} = \Lambda^{1'}{}_{\mu}\Lambda^{2'}{}_{\nu}F^{\mu\nu} = \Lambda^{2'}{}_{2}\left(\Lambda^{1'}{}_{0}F^{02} + \Lambda^{1'}{}_{1}F^{12}\right) = \gamma\left(B_z - \beta E_y/c\right) \tag{49}$$

Combining all the transformation equations together we have

$$\begin{aligned} E'_x &= E_x \quad E'_y = \gamma(E_y - \beta c B_z) \quad\ \ E'_z = \gamma(E_z + \beta c B_y) \\ B'_x &= B_x \quad B'_y = \gamma(B_y + \beta/c E_z) \quad B'_z = \gamma(B_z - \beta/c E_y) \end{aligned} \tag{50}$$

# Physics 4183 Electricity and Magnetism II
# Field of a Relativistic Point Charge

## 1 Introduction

As an example of the field transformations, we will examine the field of a relativistic point charge.

### 1.1 Field of a Relativistic Point Charge

In this section, we will calculate the field of a point charge with a relative velocity $\beta$ parallel to the $x$-axis. We will assume that the charge is at rest in the primed frame, and that this frame travels to the right relative to the unprimed frame. We will calculate the field in the primed frame first, then transform it to the unprimed frame.

In the rest frame of the charge, there is no magnetic field, and the electric field is

$$\vec{\mathbf{E}}' = \frac{q}{4\pi\epsilon_0 r'^2}\,\hat{\mathbf{r}}' \tag{1}$$

The transformation of the electric field to the unprimed frame, which travels to the left, is

$$E_x = E'_x = \frac{qx'}{4\pi\epsilon_0 r'^3} \qquad \text{and} \qquad \vec{\mathbf{E}}_\perp = \gamma\vec{\mathbf{E}}'_\perp = \frac{q\gamma}{4\pi\epsilon_0 r'^3}(y'\hat{\mathbf{y}} + z'\hat{\mathbf{z}}) \tag{2}$$

To arrive at a useful result, we need to transform the coordinates to the unprimed frame also

$$x' = x\gamma - c\beta\gamma t \qquad y' = y \qquad z' = z \tag{3}$$

The components of the electric field become

$$E_x = \frac{q(x\gamma - c\beta\gamma t)}{4\pi r'^3} \quad \text{and} \quad \vec{\mathbf{E}}_\perp = \frac{q\gamma}{4\pi r'^3}(y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \quad \Rightarrow \quad \vec{\mathbf{E}} = \frac{q\gamma}{4\pi\epsilon_0 r'^3}\vec{\mathbf{r}}_p \tag{4}$$

where $\vec{\mathbf{r}}_p = (x - c\beta t)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ is a vector that points from the instantaneous position of the charge to the field point in the unprimed frame. Since a factor of $\gamma$ is introduced into each field component, in the $x$ direction it is due to the coordinate transformation, while for the transverse components it is due to the field transformations, the field remains radial in the unprimed frame. To transform the denominator, we insert the coordinate transformations, then convert to polar coordinates

$$r' = \sqrt{x'^2 + y'^2 + z'^2} \quad \Rightarrow \quad r = \sqrt{(\gamma x - c\beta\gamma t)^2 + y^2 + z^2} = |\vec{\mathbf{r}}_p|\sqrt{\gamma^2\cos^2\theta + \sin^2\theta}$$
$$= |\vec{\mathbf{r}}_p|\sqrt{(\gamma^2 - 1)\cos^2\theta + 1} = |\vec{\mathbf{r}}_p|\sqrt{\frac{1 - \beta^2\sin^2\theta}{1 - \beta^2}} \tag{5}$$

so finally the electric field in the unprimed frame is given by

$$\vec{\mathbf{E}} = \frac{q(1 - \beta^2)}{4\pi\epsilon_0(1 - \beta^2\sin^2\theta)^{3/2}}\frac{\vec{\mathbf{r}}_p}{|\vec{\mathbf{r}}_p|^3} \tag{6}$$

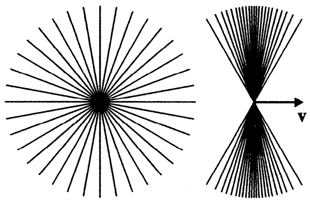


Figure 1: The electric field of a point charge as viewed from its rest frame and from a frame that travels to the right with velocity $\beta$ relative to the observer.

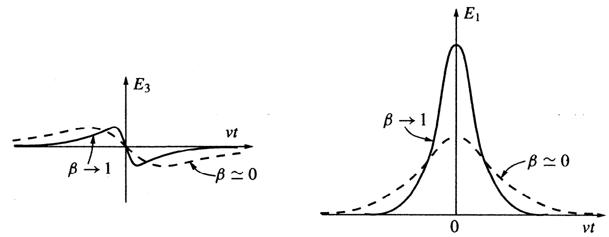


Figure 2: The magnitude of the electric fields as a function of position $vt$ of the point charge. The figure on the left is the component of the field along the direction of motion. The figure on the right is the component of the field perpendicular the the direction of motion. The fields are given for small velocities (dashed line) and for speeds close to the the speed of light.

This equation indicates that the field is largest perpendicular to the direction of relative motion, and weakest in the direction of motion.

Before proceeding to calculate the magnetic field generated by the point charge in the unprimed frame, we verify that Gauss's law is still satisfied. In the unprimed frame, we can draw a Gaussian surface about the instantaneous position of the point charge. As usual we take a sphere, since the field lines are radial, and, therefore are perpendicular to this surface

$$\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \oint_S \frac{q(1-\beta^2)}{4\pi\epsilon_0 |\vec{\mathbf{r}}_p|^2 (1-\beta^2\sin^2\theta)^{3/2}} |\vec{\mathbf{r}}_p|^2 \sin\theta \, d\theta \, d\phi = \int_0^\pi \frac{q(1-\beta^2)}{2\epsilon_0 (1-\beta^2\sin^2\theta)^{3/2}} \sin\theta \, d\theta \tag{7}$$

To simplify the integral, make a change of variables

$$u = \cos\theta \qquad\qquad du = -\sin\theta \, d\theta \tag{8}$$

which leads to

$$\int_{-1}^{1} \frac{q(1-\beta^2)}{2\epsilon_0 \left[1-\beta^2+\beta^2 u^2\right]^{3/2}} \, du = \frac{q(1-\beta^2)}{2\epsilon_0 \beta^3} \int_{-1}^{1} \frac{du}{(\beta^{-1}-1+u^2)^{3/2}} =$$

$$\frac{q(1-\beta^2)}{2\epsilon_0\beta^3} \frac{u}{(\beta^{-1}-1)\sqrt{\beta^{-2}-1+u^2}} \Bigg|_{-1}^{1} = \frac{q(1-\beta^2)}{2\epsilon_0\beta^3} \left[\beta^3 \frac{2}{1-\beta^2}\right] = \frac{q}{\epsilon_0} \tag{9}$$

Next we calculate the magnetic field due to the moving charge. In the primed frame we have no magnetic field since the charge is at rest. Using the transformation equations, we find that in the unprimed frame the magnetic field is given by

$$B_y = -\gamma\beta E_z/c \qquad\qquad B_z = \gamma\beta E_y/c \tag{10}$$

The magnetic field is azimuthally symmetric about the point charge. This is similar to what one expects from a wire carrying a current.

Given that the electric field lines are radial and the magnetic field lines are azimuthal, we expect the Poynting vector to be nonzero. This can also be seen by noting that as the charge approaches

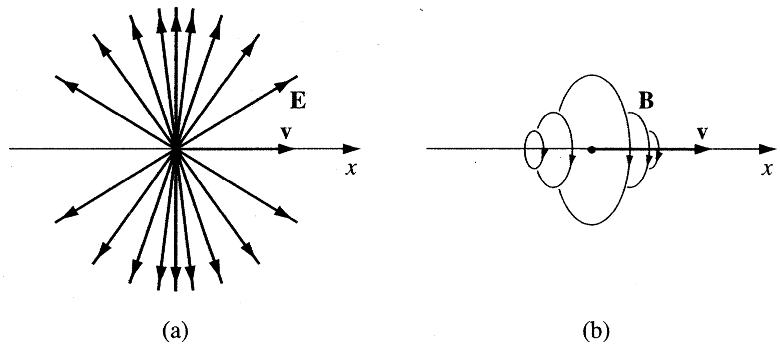


Figure 3: The electric and magnetic field lines as viewed by an observer watching a charge traveling to the right with velocity $\beta$.

a surface, there field strength increase, this corresponds to an energy flux. The Poynting vector is given by taking the cross product of the electric and magnetic fields given above. This leads to

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}} = -\frac{q^2}{16\pi^2\epsilon_0}\frac{(1-\beta^2)^2 c\beta\sin\theta}{|\vec{\mathbf{r}}_p|^4(1-\beta^2\sin^2\theta)^3}\,\vec{\boldsymbol{\theta}} \tag{11}$$

If we calculate the flux through some surface, the flux is proportional to $1/r^2$

$$\mathcal{F} = \oint_A \vec{\mathbf{S}}\cdot d\vec{\mathbf{a}} = \oint_A \vec{\mathbf{S}}\cdot\hat{\mathbf{n}}r^2\, d\cos\theta\, d\phi \tag{12}$$

Therefore the energy flux generated by the field does not propogate forever, that is a particle with uniform motion does not radiated.