Statistical Mechanics

4. Consider an ideal monatomic gas used as the working fluid in a thermodynamic cycle. The number of particles is \( n_0 \). It follows a cycle consisting of one adiabat, one isochore and one isotherm, as shown below.

(a) Calculate the pressure, temperature, and volume at each corner of the cycle, A, B, and C, expressing your answer in terms of \( P_0 \), \( V_0 \), \( n_0 \) and perhaps \( R \), the ideal gas constant. Note that point A the pressure is \( P_0 \) and the volume is \( V_0 \). (3pts)

(b) Calculate the work done on the system, the heat into the system and the change in the internal energy of the system for each process step. (4.5pts)

(c) What direction around the cycle must the system follow to be used as a functional heat engine? (1/2pt)

(d) What is the efficiency of the cycle, run as an engine? (1pt)

(e) What is the efficiency of an ideal Carnot engine run between reservoirs B and C? (1pt)
@ use ideal gas equation

2

\[
\begin{align*}
\frac{P_B}{P_c} V_B &= n_o K T_B \\
\frac{P_c V_c}{P_c} &= n_o K T_c \quad (V_B = V_c)
\end{align*}
\]

\[
\frac{P_B}{P_c} = \frac{T_B}{T_c} \quad \rightarrow \quad T_B = \frac{P_B}{P_c} T_c \quad \rightarrow \quad T_c = \frac{T_B P_c}{P_B}
\]

3

\[
\begin{align*}
\frac{P_A V_A}{P_B V_B} &= \frac{n_o K T_A}{n_o K T_B} \quad (T_A = T_B) \\
\frac{P_A}{P_B} &= \frac{V_A}{V_B} = 1
\end{align*}
\]

or

\[
\frac{P_A V_A}{P_B V_B} = \left( \frac{P_A}{P_B} \right)^\gamma = \frac{V_A}{V_B} \quad \Rightarrow \quad P_A = P_o \left( \frac{V_A}{V_B} \right) = P_o \left( \frac{1}{2} \right) = \frac{1}{2} P_o
\]

\[
\frac{P_c}{P_B} = 0.3 \quad \Rightarrow \quad P_c = 0.3 P_o
\]

\[
V_A = V_B \quad \Rightarrow \quad V_A = V_o
\]

\[
\frac{V_B}{V_o} = 2 \quad \Rightarrow \quad V_B = V_c = 2 V_o
\]

Reading off the graph:

\[
\frac{P_B}{P_o} = 0.5 \quad \Rightarrow \quad P_B = \frac{1}{2} P_o
\]

... can't solve this way

use:

\[
\begin{align*}
T_A &= \frac{P_A V_A}{n_o K} = \frac{P_o V_o}{n_o K} \\
T_B &= \frac{P_B V_B}{n_o K} = \frac{P_o 2 V_o}{3 n_o K} = \frac{T_A}{2} \\
T_c &= \frac{P_c V_c}{n_o K} = \frac{P_o V_o}{3 n_o K} = \frac{2}{3} T_A
\end{align*}
\]

as expected
Using $\Delta E = \Delta Q + \Delta W$ done on system

\[
\begin{align*}
\text{step} & & \Delta Q & & \Delta W & & \Delta E \\
\text{adiabat} & & & & W = \frac{1}{2} P_0 V_0 & & \frac{3}{2} N_0 k (T_A - T_e) \\
1 & & Q = 0 & & & & = \frac{3}{2} N_0 k (T_A - \frac{2}{3} T_B) \\
& & & & & & = \frac{3}{2} N_0 k \frac{P_0 V_0}{n_0 k} (1 - \frac{2}{3}) = \frac{1}{2} P_0 V_0 \\
\text{isochore} & & Q = \Delta E & & W = p dV = 0 & & \frac{3}{2} N_0 k (T_e - T_B) \\
2 & & & & & & = \frac{3}{2} N_0 k \left( \frac{2}{3} T_B - T_B \right) = \frac{3}{2} N_0 k \frac{T_B}{8} \\
& & & & & & = -\frac{1}{2} P_0 V_0 \\
\text{isotherm} & & Q = \Delta W & & W = \int_{V_i}^{V_f} p dV & & \Delta E = 0 \\
3 & & & & W = p V_A \ln \left( \frac{V_f}{V_i} \right) & & = -N_0 k T_A \ln \left( \frac{V_f}{V_i} \right) \\
& & & & & & = -V_0 P_o \ln \frac{V_f}{V_i} \\
& & & & & & \Delta E = 0 \\
& & & & & & \frac{P_0 V_0 \ln 2 - \frac{1}{2} P_0 V_0}{P_0 V_0 \ln 2} \geq 88\% \\
& & & & & & = 0.27 = 27\% \\
& & & & & & \text{clockwise (A \rightarrow B \rightarrow C \rightarrow A)} \\
& & & & & & \eta = \frac{|W_{out}|}{|Q_{in}|} = \frac{18.5 T_B L m_{2}}{P_0 V_0 L m_{2}} \\
& & & & & & \text{as expected } \eta_c > \eta \text{ because } \eta_c \text{ is the best possible efficiency between two given heat reservoirs.} \\
& & & & & & \left( T_e = \frac{1}{3} \right) \approx 33.3\% \\
\end{align*}
\]
Problem 4 (10 Points):

The coffee purchased at rest stops is often too hot to drink. One way to cool off your coffee is to add ice, but how much ice should you add? Take the initial conditions for the coffee to be $T_{0}^{\text{coffee}} = 80^\circ\text{C}$ and $V = 400\text{ ml}$. Take the initial conditions for the ice to be $T_{0}^{\text{ice}} = 0^\circ\text{C}$. The final temperature for the coffee and ice that you want to achieve is $T_f = 60^\circ\text{C}$. For the following questions assume that the coffee is pure water (a good assumption for most rest stop coffee) and the process is adiabatic with respect to the surroundings. Neglect volume changes of the coffee and ice and any temperature dependence of the heat capacity. The following thermodynamic properties of water may be useful:

- $M = 18.0\text{ g mole}^{-1}$, molar mass
- $\rho = 1.00\text{ g/cm}^3$, density
- $\Delta H_{fus} = 6.00\text{ kJ mole}^{-1}$, heat of fusion
- $C_p = 75.4\text{ J mole}^{-1}\text{ K}^{-1}$, heat capacity of liquid

For parts (a.)-(c.) your answers should be in terms of the variables described here.

a. Find a general (algebraic solution) expression for the mass of ice, $m$, that is needed to cool the coffee to $T_f$? (4 Points)

b. Calculate, numerically, how many grams of ice you should add to your coffee to lower the temperature to $T_f = 60^\circ\text{C}$. (1 Points)

c. What is the entropy change of the system (coffee + ice)? Find an algebraic solution. (3 Points)

d. What is the entropy change of the surroundings? (1 Points)

e. Is this a thermodynamically reversible process? Explain. (1 Points)
\( Q_T = 0 \quad \Delta T = T_f - T_i \)

Q\text{melting ice} + Q\text{cooling coffee} + Q\text{warming ice water} = 0

M_{\text{ice}} \cdot L_i + M_{\text{caf}} \cdot C_{\text{caf}} \cdot \Delta T_{\text{caf}} + M_{\text{ice}} \cdot C_{\text{water}} \cdot \Delta T_{\text{water}} = 0

M_i (L_i + C_p \Delta T_{\text{water}}) + M_{\text{caf}} \cdot C_p \Delta T_{\text{c}} = 0

\[ m = \frac{-m_{\text{caf}} \cdot C_p \cdot (T_f - T_{\text{cof}})}{\Delta H_{\text{fus}} + C_p (T_f - T_{\text{ice}})} \]

\[ m = -400 \text{ g} \times \frac{1}{18} \text{ mole} \times 75.4 \frac{J}{\text{mole} \cdot K} \times (-20 \text{ K}) \]

\[ = \frac{6 \times 10^3 \frac{J}{\text{mole}} + 75.4 \frac{J}{\text{mole} \cdot K} \times (60 \text{ K})}{2/\text{mole} \times 18 \frac{g}{\text{mole}}} \]

\[ = 3.184 \text{ mole} \times 18 \frac{g}{\text{mole}} = 57.3 \text{ g} \]

\[ dE = T dS - p dV + \mu dN \rightarrow dS = \frac{dE + p dV - \mu dN}{T} \]

page 30 Schroeders \( C_P - C_V = nR \) (\( n = \# \) of mols)

\( C_p = C_v \) for a liquid (incompressible)

\( C = \frac{Q}{\Delta T} = \frac{\Delta U - W}{\Delta T} \)

\[ \Delta S = \frac{Q}{T} \]

\( Q_{\text{coffee}} = m c \Delta T = m c S_i \frac{dT}{dT} \)

\[ \Delta S = \frac{Q_{\text{ice}}}{T} + \frac{Q_{\text{water}}}{T} + \frac{Q_{\text{coffee}}}{T} \]
\[ \Delta S = \frac{M_{\text{ice}} \Delta H_{\text{fus}}}{T} + M_{\text{ice}} C_p \int_{T=0^\circ C}^{T=60^\circ C} \frac{1}{T} \, dT + M_c C_p \int_{T=80^\circ C}^{T=60^\circ C} \frac{1}{T} \, dT \]

\[ T = 0^\circ C = 273 \text{K} \]

(1) No change. No heat flow into/out of cup system from surroundings. (Adiabatic - no loss/gain of heat)

(2) \( \Delta S > 0 \) so not reversible, can't make ice again w/o work.
Problem 4 (10 Points):

Assume that air obeys the ideal gas equation. Take $M$ to be the molar mass, $P$ the pressure, $R$ the ideal gas constant, $T$ the temperature, $z$ the altitude, $\rho$ the density, and $g$ the acceleration due to gravity.

a. The density of our atmosphere decreases with increasing altitude. This is a consequence of hydrostatic equilibrium, where the pressure of the air at an altitude $z$, must balance the pressure from below and the weight of the column of air above. Given that air has a mass density $\rho = MP/RT$, find $dP/dz$. Assume that the atmosphere is isothermal. Neglect the curvature of the earth and the variation of $g$ with altitude. (4 Points)

b. Using the model in part (a.), consider a volume of air that is moved adiabatically within the atmosphere and able to do work on its surroundings; that is, expand and contract to maintain the same pressure as the surrounding air. If this section is moved upwards, it will cool as it is lifted, thus increasing in density compared to the surrounding air, and tend to sink back to its original altitude. Find $dT/dz$, the adiabatic lapse rate for the air. Assume the air is composed of diatomic molecules ($N_2$). (Hint: first find $dT/dP$). (4 Points)

The significance of the adiabatic lapse rate is that it determines the stability of the atmosphere to convection. The temperature in the lower part of the real atmosphere (troposphere) is not isothermal, but decreases with increasing altitude because it is heated by the ground. If the temperature gradient in the atmosphere is greater than the lapse rate, convection can occur.

c. If the section of air was wet so that condensation can occur, how does the lapse rate change? Explain your reasoning. (1 Points)

d. A helium balloon ascends in the atmosphere, expanding adiabatically just as the section of air in (b.). Will the lapse rate of helium be higher, the same, or lower than air? Explain. (1 Points)
Forces above and below a section of air must be equal; consider

Pressure from above = $P(z + dz)$

Area = $A$

$dz = \text{thickness}$

\[
\text{Weight} = mg = \rho V g = \rho A dz g = \frac{MP(z)}{RT} A g dz
\]

Pressure from below = $P(z)$

Force = Pressure x Area

Remember Taylor expansion \[ F(x + dx) = F(x) + dx \frac{dF}{dx} + \frac{dx^2}{2!} \frac{d^2F}{dx^2} + \ldots \]

\[ \sum \text{forces} = 0 = Ap(z) - Ap(z + dz) - \frac{MP(z)}{RT} A g dz \]

\[ 0 = Ap(z) - Ap(z) - dz \frac{d(p(z))}{dz} A - \frac{MP(z)}{RT} A g dz \]

\[ dz A \frac{dp}{dz} = - \frac{MP g A dz}{RT} \]

\[ \frac{dp}{dz} = - \frac{MP g}{RT} \]

Ideal gas law \[ PV = NkT \]

\[ \frac{d}{dz} \left( P(z) V(z) = NkT(z) \right) \]

\[ VdP + PdV = NkdT \]

Remember energy of diatomic gas \[ E_{N_2} = \frac{5}{2} NkT \]

\[ \frac{d}{dz} \left( E(z) = \frac{5}{2} NkT(z) \right) \]

\[ dE = \frac{5}{2} NkdT \quad \text{Remember} \quad dE = dW + d\Omega \quad \text{adiabatic} \]

so:

\[ dE = \frac{5}{2} NkdT = dW = -pdV \]

\[ VdP = -\frac{5}{2} NkdT = NkdT \]

plug in
\[ V_d P = N k d T + \frac{5}{2} N k d T \]
\[ V_d P = -\frac{7}{2} N k d T \]
\[ \frac{dT}{dP} = -\frac{2}{7} \frac{V}{N K} \]

**Chain rule:**
\[ \frac{dT}{d\bar{z}} = \frac{dP}{d\bar{z}} \frac{d\bar{z}}{dP} \]
\[ = (-\frac{MP}{RT g})(-\frac{2}{7} \frac{V}{N K}) = \frac{MPa 2V}{RT 2 K} \]
\[ PV = N K T \]

\[ \frac{dT}{d\bar{z}} = \frac{2Mg N K T}{7 R N K T} = \frac{2 Mg}{7 R} \]

- **Molar mass**
  - \( N_2 = \frac{14}{2} \times 1 + 2 = 28 \)
  - \( H_2O = \frac{8}{2} \times 2 + 16 + 2 \times 1 = 18 \)
  - \( He = 4 \)

Molar mass of wet air is higher than dry air \( (N_2) \) so \( \frac{dT}{d\bar{z}} \) is greater for wet air.

- Molar mass of He is lower than \( N_2 \) so \( \frac{dT}{d\bar{z}} \) is lower for He.
Statistical Mechanics

4. It can be shown that the Helmholtz free energy for a photon gas is given by:

\[ F(T, V, N) = -\frac{1}{3} \sigma VT^4 \]

where \( \sigma \) is the Stefan-Boltzmann constant. Using this relation, answer the following:

(a) What are the equations of state (that is, \( P, S \), and \( \mu \) as functions of \( T, V \) and \( N \))? (3pts.)

(b) Consider a Carnot cycle using a photon gas as its working fluid. The cycle is driven by one hot and one cold temperature reservoir, with temperatures \( T_h \) and \( T_c \) respectively. Draw the cycle in the \( P-V \) plane. **Caution:** This is not an ideal gas! Think carefully about the steps in a Carnot cycle and use your results from above to determine what the cycle will look like. (2pts.)

(c) Solve for the heat exchanged in each leg of your Carnot cycle. Your answer may depend upon \( T_h, T_c, \) and any other variables you might choose in defining your cycle. (2pts.)

(d) Using these values for the heat exchanged, calculate the efficiency of a Carnot cycle that uses a photon gas as its working fluid. If you cannot calculate it, devise a careful argument for its value. (3pts.)
\[ S = - \frac{\partial F}{\partial T(n,V)} \quad P = - \frac{\partial F}{\partial V(n,T)} \quad \mu = \frac{\partial F}{\partial N(v,T)} \]

\[ P = - \frac{\partial}{\partial V} \left( \frac{1}{3} \sigma V T^4 \right) = \frac{1}{3} \sigma V^3 T^4 \]

\[ S = - \frac{\partial}{\partial T} \left( -\frac{1}{3} \sigma V T^4 \right) = \frac{1}{3} \sigma V^4 T^3 - \frac{4}{3} \sigma V T^3 \]

\[ \mu = \frac{\partial}{\partial N} \left( -\frac{1}{3} \sigma V T^4 \right) = 0 \]

\[ F = E - TS \rightarrow dF = dE - TdS - SdT \]

\[ dF = TdS - pdV + \mu dN - TdS - SdT \]

\[ dF = -pdV + \mu dN - SdT \]

\[ P \propto T^4 \text{ in the } Y \text{ gas} \]

\[ \mu \propto V T^3 \]

\[ dS = \int \frac{dQ}{T} \]

\[ dS = 0 \]

\[ dP = \frac{1}{3} \sigma \left( \frac{2}{4} \frac{1}{\rho dV} \right) V^5 \]

\[ dP = \left( \frac{1}{\rho \frac{dV}{dV}} \right) V^5 \]
\[ ds = 0 = d \left( \frac{4}{3} \sigma VT^3 \right) \]
\[ = \frac{4}{8} \sigma (dVT^3 + V^3 \cdot T^2 dT) \]

so:
\[ T^2 \, dV = -3VT^2 \, dT \]
\[ \frac{dV}{\sigma T^4} = \frac{3V}{T^4} dT \]
\[ \frac{dP}{\frac{1}{3} \sigma T^3} = dT \]
\[ dV = -\frac{9}{4} \frac{dP}{4 \sigma T^4} \]

\[ dV = -\frac{9}{4} \frac{dP}{4 \sigma T^4} \]

\[ \int dV = \int \left[ \frac{3V}{4} \left. \ln \frac{p^2}{p_1^2} \right] \right] \]

\[ \int V\,dV = \int-\frac{3}{4} p^{-1} \, dP \]

\[ m \left. V \right|_{v_1}^{v_2} = -\frac{3}{4} \ln \frac{p_2}{p_1} \]

\[ -\frac{4}{3} \ln V = \ln P \rightarrow \ln \left( V^{\frac{4}{3}} \right) = \ln (P) \]

\[ V^{\frac{4}{3}} = P \quad \text{the adiabatic lines are (-) exponential slopes} \]

\[ \frac{|Q_H|}{T_H} = \frac{|Q_L|}{T_L} \quad (21-8) \quad \text{or} \quad Q_1 = Q_3 = 0 \quad \text{for adiabats} \]

\[ Q_2 = W_2 = P_2 (V_3 - V_1) \]

\[ U = \frac{1}{2} N k \Delta T = 0 \quad \text{for isotherms} \]

\[ U_a = Q_4 - W_4 \]

\[ Q_4 = -P_1 \left( V_4 - V_2 \right) \]
\[ \eta_c = 1 - \frac{T_c}{T_H} \quad (21.11) \]

or \[ \eta_c = \frac{1}{\text{Qin}} \left( \frac{1}{\text{Qout}} - 1 \right) = \frac{\frac{P_4}{P_3} (V_4 - V_2)}{P_2 (V_3 - V_1)} = \frac{Q_H - Q_c}{Q_H} \]

\[ = 1 - \frac{|Q_4|}{|Q_2|} \]
4. The gas turbine (jet engine) can be modeled as a Brayton cycle. Below is the P-V diagram for this process.

Assume that the working fluid is an ideal monatomic gas.

(a) Calculate the work done by the gas on each step in the cycle. (3 pts.)

(b) Find the heat for each step in the cycle. (3 pts.)

(c) Find the efficiency of this engine. Your answer should be in terms of the pressures ($P_1$ and $P_2$) and the volumes ($V_1$, $V_2$, $V_3$, and $V_4$). (3 pts.)

(d) To produce work, which way does the cycle operate? Clockwise or counter clockwise? (1 pt.)
Work is done on/by gas in all four steps.

\[ W = \int_{V_i}^{V_f} p \, dV \]

\[ W_b = p_2 \int_{V_2}^{V_4} dV = p_2 (V_4 - V_2) \]

\[ W_d = p_1 \int_{V_3}^{V_1} dV = p_1 (V_1 - V_3) \]

\[ \Delta S = \int_{i}^{f} \frac{dQ}{T} \]

\[ dE = d\Omega + dW \]

so:

\[ dE = dW \]

\[ dE = \frac{3}{2} Nk \, dT \]

\[ a, c: \Delta S = 0 \implies Q = 0 \]

Isothermal (i.e. adiabatic)

Energy of monatomic ideal gas

\[ E = \frac{3}{2} NkT \]

\[ W_c = \frac{3}{2} Nk T_c = \frac{3}{2} Nk \left( \frac{P_1 V_3 - P_2 V_4}{Nk} \right) \]

\[ + W \text{ is work done by gas } W_b, W_c \]

\[ - W \text{ is work on gas } W_d, W_a \]

\[ Q_c = Q_a = 0 \]

\[ Q_b = C_p \Delta T_b = C_p \left( \frac{P_2 V_4 - P_2 V_2}{Nk} \right) = \frac{C_p P_2}{Nk} (V_4 - V_2) \quad (+) \]

\[ Q_d = C_p \Delta T_d = C_p \left( \frac{P_1 V_1 - P_1 V_3}{Nk} \right) = \frac{C_p P_1}{Nk} (V_1 - V_3) \quad (-) \]

\[ \eta = \frac{|W_{at}|}{|Q_{in}|} = \frac{|W_b| + |W_c|}{|Q_b|} \]

\[ \text{clockwise} \]
Statistical Mechanics

4. **Helmholtz Free Energy:** The Helmholtz free energy of an ideal monoatomic gas can be written as

\[ F(T, V, N) = NkT \left( A - \log \left( \frac{T^{3/2}V}{N} \right) \right) \]

where \( N \) is the total number of gas atoms, \( V \) is the volume, \( T \) is temperature, \( k \) is Boltzmann’s constant and \( A \) is a dimensionless constant.

Consider a piston separating a system into two parts, with equal numbers of particles on the left and the right hand side. The whole system is in good thermal contact with a reservoir at constant temperature \( T \). Initially, \( V_1 = 2V_2 \). The total volume, \( V_{\text{tot}} = V_1 + V_2 \), is fixed for this whole problem.

![Thermally conducting piston](image)

(a) Calculate the equilibrium position of the piston, once it is released. You must prove your answer, and not simply assert it. (3 points)

(b) Calculate the maximum available work the system can perform as it changes from the initial condition to the equilibrium position. (3 points)

(c) Calculate the change in the internal energy, \( U \) of gas 1 and gas 2 in the process. (2 points)

(d) Given your answers above, explain the source of energy for the work done during the expansion. (2 points)
Assuming $T_1^f \neq T_2^f \neq T_R$

Known

- $V_1 = 2V_2^f$
- $P_2^f = P_1^f$
- $N_1 = N_2$
- $T_1^f = T_2^f = T_R$
- $V_{tot} = V_1 + V_2$

**Side 1**

$$P_1^i V_1 = N_1 k T_1$$
$$\frac{P_1^i V_1^i}{T_1^i} = N_1 k = \frac{P_1^f V_1^f}{T_1^f}$$

**Side 2**

$$\frac{P_2^i V_2^i}{T_2^i} = N_2 k = \frac{P_2^f V_2^f}{T_2^f}$$

At Eqn. 4:

$$P_i = P_2^f$$

Using $p = -\frac{\partial F}{\partial V_{(N,T)}}$

$$\frac{\partial}{\partial V_1} N_1 k T_1 \left[ A - \ln \left( \frac{T_1^{3/2} V_1}{N_1} \right) \right] = \frac{\partial}{\partial V_2} N_2 k T_2 \left[ A - \ln \left( \frac{T_2^{3/2} V_2}{N_2} \right) \right]$$

$$- \frac{N_1 k T_1}{V_1} = - \frac{N_2 k T_2}{V_2}$$

$$\frac{T_1^f}{V_1^f} = \frac{T_2^f}{V_2^f}$$

Since $T_1^f = T_2^f \Rightarrow V_1^f = V_2^f$

and piston is in the middle of the system.

Work done by gas $= -pSV$

Consider side 2 as doing all the work b/c piston moves from right to left. But $p$ is not constant during process.

$$W = -\int P(V) dV = -\int_{V_i}^{V_f} \frac{N k T}{V} dV$$
Assuming $T'_1 = T'_2 = T_R = T$

Known: $V'_1 = 2V'_2$

$p^f_1 = p^f_2$

$T^f_1 = T^f_2 = T_R = T$

$V_{tot} = V_1 + V_2$

$N'_1 = N'_2$

$p^i_1 V'_1 = NKT = p^f_1 V'_1$

$p^i_2 V'_2 = p^f_2 V'_2$

Remember: $p = -\frac{\partial F}{\partial V(n,T)}$

at eqn: $p^f_1 = p^f_2 \rightarrow \frac{\partial F_1}{\partial V'_1} = \frac{\partial F_2}{\partial V'_2}$

\[
\frac{\partial}{\partial V'_1} \left( NKT \left( A - \ln \left( \frac{T^{3/2} V'_1}{N} \right) \right) \right) = \frac{\partial}{\partial V'_2} \left( NKT \left( A - \ln \left( \frac{T^{3/2} V'_2}{N} \right) \right) \right)
\]

\[-NKT \left( -\frac{1}{V'_1} \right) = -NKT \left( -\frac{1}{V'_2} \right)\]

$V'_2 = V'_1$

Since $T$ is const, and side 2 does the work:

$W = \int_{V'_1}^{V'_2} P(v) dv = \int_{V'_1}^{V'_2} \frac{NKT}{V} dv = NKT \ln \left( \frac{V'_2}{V'_1} \right) = NKT \ln \left( \frac{3}{2} \right)$

$F = E - TS$

$E = F + TS$ \quad \rightarrow \quad dE = dF + SdT + TdS$

$\text{using: } F = NkT \left( \frac{1}{V} \right) - NKT \ln \left( \frac{3}{2} \right)$

$\frac{dF}{dV} = -NkT \frac{1}{V} \quad \rightarrow \quad dF = -NkT \frac{1}{V} dv$

$\text{using: } S = -\frac{dF}{dT} = - \left( Nk A - \left( Nk \frac{3}{2} \ln T + Nk \frac{3}{2} \frac{T}{T} \right) - NKT mV + NKT mN \right)$

$\frac{dS}{dV} = -\left( -Nk \frac{1}{V} \right) \quad \rightarrow \quad dS = Nk \frac{1}{V} dv$
\[ dE = - \frac{NkT}{V} dV + \frac{TNk}{V} dV = 0 \]

since \( T, N \) are constants, then side 2 must start out at a higher pressure than side 1.
Statistical Mechanics

4. Heat Engines: A pulse jet operates under a Lenoir cycle. This consists of an adiabat, an isobar, and an isochore, as shown.

Assuming that the working fluid is an ideal 3D monoatomic gas of $N$ particles:

(a) Find the work done in one complete cycle. (3 points)
(b) Find the heat exchanged in each step in the cycle. (3 points)
(c) Find the efficiency of the engine. Express your answer in terms of pressures and volumes. (3 points)
(d) To produce work, should the engine cycle operate clockwise ($A \rightarrow B \rightarrow C \rightarrow A$) or counterclockwise ($A \rightarrow C \rightarrow B \rightarrow A$)? (1 point)
Work = |Q_{in}^{(iii)}| - |Q_{out}^{(ii)}|
\quad = \left| C_V \Delta T_{(iii)}^{(iii)} \right| - \left| C_p \Delta T_{(ii)}^{(ii)} \right|
\quad = C_V \left| T_A - T_c \right| - C_p \left| T_c - T_b \right|
\quad \geq \left[ T_A - T_c \right] \left[ \Delta T_c \Delta\Delta T_b \right]

\textit{For an ideal gas, } C_V \approx C_p = C

b)
(i) none, it is an adiabatic
(ii) as above, \( Q_{out} = C_p \Delta T_{(ii)} = C_p \left| T_c - T_b \right| \)
(iii) \( Q_{in} = C_V \Delta T_{(iii)} = C_V \left| T_A - T_c \right| \)

Express \( a \) and \( b \) in terms of \( P, V \) using:
\( p_1 V_2 = nRT_b \quad (iii) \quad p_1 V_1 = nRT_c \)
\( \frac{V_2}{V_1} = \frac{T_b}{T_c} \quad \frac{P_1}{P_2} = \frac{T_c}{T_a} \)

\textit{then:}\n\( T_A - T_c = \frac{P_2}{P_1} T_c - T_c = T_c \left( \frac{P_2}{P_1} - 1 \right) = \frac{p_2 V_1}{Nk} \left( \frac{P_2}{P_1} - 1 \right) \)
\( T_c - T_b = \frac{p_2 V_1}{P_2} \Delta T_b = T_b \frac{V_2}{V_1} - T_b = T_b \left( \frac{V_2}{V_1} - 1 \right) \)

\( \eta = \frac{Q_{out}}{Q_{in}} = \frac{Q_{id} - Q_{out}}{Q_{in}} = 1 - \frac{C_p \left| T_c - T_b \right|}{C_V \left| T_A - T_c \right|} \)

\( n = \frac{Q_{out}}{Q_{in}} = \frac{|Q_{id} - Q_{out}|}{|Q_{in}|} = 1 - \frac{C_p \left| T_c - T_b \right|}{C_V \left| T_A - T_c \right|} \)

\( \text{clockwise} \)
Problem 4 (10 Points):

This problem considers a photon gas. A blackbody cavity can be considered to contain a gas that obeys the equations of state:

\[ U = b \cdot V \cdot T^4 \]

\[ P \cdot V = \frac{1}{3} \cdot U \]

where \( U \) is the internal energy, \( V \) is the volume, \( P \) is the pressure, \( T \) is the temperature and \( b = 7.56 \times 10^{-18} \text{ J/(m}^3\text{K}^4) \). Note that there is no dependence on \( N \), the number of particles.

a. Show that the fundamental equation for the entropy, \( S(U,V) \) is:

\[ S(U,V) = \frac{4}{3} \cdot b^{1/4} \cdot U^{3/4} \cdot V^{1/4} \]

(3 Points)

b. The universe can be treated as an expanding electromagnetic cavity at a temperature of \( T = 2.7 \text{ K} \). Assume the expansion of the universe is isentropic. What will the temperature of the universe be when it is twice its current size? (2 Points)

c. What is the pressure associated with the electromagnetic radiation? (1 Points)

d. What is the Helmholtz potential for this system as a function of \( U \) and \( P \)? (3 Points)

e. Why is there no dependence upon \( N \) in the fundamental equation for the photon gas? (1 Points)
Basic thermo is all MCE

\[ \frac{\partial s}{\partial E(v, u)} = \frac{1}{T} \quad (E = U) \]

\[ \int ds = \int \frac{1}{T} dE \quad \Rightarrow \quad T = \left( \frac{U}{bV} \right)^{1/4} \]

\[ S = \int \left( \frac{U}{bV} \right)^{1/4} dU = b^{1/4} V^{1/4} \left( \frac{U^{3/4}}{3^{3/4}} + \alpha \right) \]

\[ S = \frac{4}{3} b^{1/4} V^{1/4} U^{3/4} \]

\[ dU = d\alpha + dV \]

Isoentropic, work done by universe

\[ \Delta S = 0, \quad Q = 0 \]

Isoentropic \( \equiv \Delta S = 0 \)

\[ S(U, V) = \frac{4}{3} b^{1/4} U^{3/4} V^{1/4} = \text{const.} = C \]

\[ C = \frac{4}{3} b^{1/4} (bVT^4)^{3/4} V^{1/4} = \frac{4}{3} bVT^3 \]

Find:

\[ \frac{4}{3} bV(T_1)^3 = \frac{4}{3} bV_2(T_2)^3 \quad \text{where} \quad V_2 = 2V_1 \]

\[ \frac{T_1(T_1)^3}{2V_1} = 2V_2(T_2)^3 \]

\[ \frac{T_1}{2V_1} = T_2^3 \quad \Rightarrow \quad T_2 = \sqrt[3]{\frac{T_1^3}{2}} \]

\[ du = Tds - pdV + \sum_i \mu_i dN_i \]

Holding said \( N \) const.

\[ du = -pdV \quad \Rightarrow \quad p = -\frac{du}{dV} \]

for 3 dimensions:

\[ p = \frac{U}{3V} \]

Pressure of thermal radiation

\[ \frac{\alpha}{\frac{1}{2}} \times \text{energy density} \]
(d) Helmholtz potential is Helmholtz free energy

\[ F = E - TS = U - TS = bVT^4 - T^4 B \frac{1}{4} n^{3/4} V^{1/4} \]

\[ = bVT^4 - \frac{T^4}{3} B^{1/4} (bVT^4)^{3/4} V^{1/4} \]

\[ = bVT^4 - \frac{T^4}{3} B^{1/4} B^{3/4} V^{3/4} T^{-3} V^{1/4} \]

\[ = bVT^4 - \frac{T^4}{3} B^{1} V = -\frac{1}{3} bVT^4 = -\frac{1}{3} U = -PV \]

(e) \( N \) is not defined for photons. \( N_y \) is determined from minimizing the Helmholtz free energy.

\[ dU = Tds - pdV + \mu_i dN_i \]
Problem 1: (10 Points)

We can use the temperature rise that results from the adiabatic compression of an ideal monoatomic gas to measure the velocity of a bullet. Suppose a piston of mass $M$ can move in a uniform frictionless tube of cross-sectional area $A$. The piston can only move in the direction of compression. The tube is closed at one end, and the piston is sealed so that no gas can escape. The cylinder is filled with He gas at temperature $T_0$ and pressure $P_0$, such that the initial position of the piston is $L_0$ from the closed end. A bullet of mass $m$ is fired from a gun and strikes the center of the piston. The bullet embeds itself in the piston, causing the piston to move and compress the gas in the tube. The maximum temperature of the gas in the cylinder is $T_f$. Assume that the piston compresses the gas adiabatically.

a. Find the initial velocity of the bullet, $v_0$, in terms of the given parameters. (2 Points)

b. What is the maximum displacement of the piston, $\Delta L$, in terms of the given parameters? (2 Points)

c. What is the maximum final pressure inside the cylinder, $P_f$, in terms of the given parameters? (2 Points)

d. Sketch the acceleration of the piston versus $\Delta L$ beginning at the moment the bullet hits the piston. Make sure that the sketch is qualitatively accurate. (2 Points)

e. Neglecting the exact time that the bullet impacts the piston, at what value of $\Delta L$ is the piston at when the magnitude of its acceleration is greatest? (1 Point)

f. We assumed that the gas was compressed adiabatically. If heat was lost to the walls of the cylinder, would the resulting value of $v_0$ be: (1.) too high, (2.) too low, or (3.) unchanged. To receive credit you must explain your answer. (1 Points)
\[ \frac{m_0 V_0}{V_o} = nRT_0 \quad \text{and} \quad \frac{p_f V_f}{V_o} = nRT_f \]

Conservation of \( P \) during collision:
\[ \frac{m_0 V_0}{(m+m)V_o} = V_o \]

Conservation of \( E \) during compression:
\[ KE_o + \frac{1}{2} m V_o^2 = \frac{1}{2} M V_f^2 + \frac{3}{2} nRT_f \]

\[ \frac{1}{2} \frac{m}{M} V_o^2 + \frac{3}{2} nRT_o = \frac{3}{2} \frac{m}{M} \frac{p_o V_o}{T_o} T_f \]

\[ \frac{1}{2} \frac{m^2}{M^2} V_o^2 + \frac{3}{2} p_o A L_o = \frac{3}{2} \frac{p_o A L_o}{T_o} T_f \]

\[ \frac{m^2}{2M} \frac{V_o^2}{A M} = \frac{3}{2} \frac{p_o A L_o T_f}{2T_o} - \frac{3}{2} \frac{p_o A L_o}{2} \]

\[ \frac{m^2}{2M} \frac{V_o^2}{A M} = \frac{3}{2} \frac{p_o A L_o T_f}{2T_o} - \frac{3}{2} \frac{p_o A L_o}{2} \]

\[ V_0 = \left( \frac{2M}{A M} \frac{3}{2} \frac{p_o A L_o T_f}{2T_o} - \frac{3}{2} \frac{p_o A L_o}{2} \right)^{1/2} \]

\[ V_0 = \left( \frac{2M}{A M} \frac{3}{2} \frac{p_o A L_o T_f}{2T_o} - \frac{3}{2} \frac{p_o A L_o}{2} \right)^{1/2} \]

\[ V_0 = \left( \frac{3}{2} \frac{p_o A L_o T_f}{2T_o} - \frac{3}{2} \frac{p_o A L_o}{2} \right)^{1/2} \]

\[ V_0 = \left( \frac{3}{2} \frac{p_o A L_o T_f}{2T_o} - \frac{3}{2} \frac{p_o A L_o}{2} \right)^{1/2} \]

\[ \frac{P_o V_o}{T_o} = \frac{P_f V_f}{T_f} \]

\[ \Delta L = L_o - L_f \]

\[ = \frac{L_o - L_o \frac{T_o^{3/2}}{T_f^{3/2}}}{T_f^{3/2}} \]

\[ \Delta L = L_o \left[ 1 - \left( \frac{T_o}{T_f} \right)^{3/2} \right] \]
\[
\frac{P_0 V_0}{T_0} = \frac{P_f V_f}{T_f}
\]

\[
P_0 V_0 T_f = P_f = \frac{P_0 A L_0 T_f}{T_0 A L_f} = \frac{P_0 L_f T_f}{T_0 (L_0 - \Delta L)}
\]

\[
P_f = \frac{P_0 K_0 T_f}{T_0 (L_0 - \Delta L^* (1 - I_0^{-3/2}))} = \frac{P_0 T_f}{T_0} \left[ \frac{1}{\gamma - 1} + \frac{T_0}{T_f^{3/2}} \right]
\]

\[
= \frac{P_0 T_f}{T_0} \cdot T_f^{3/2} = P_0 \left( \frac{T_f}{T_0} \right)^{5/2}
\]

\[
\text{Work} = \int_{V_0}^{V_f} p dV = F \Delta L = P A A L = \Delta E \quad p = \frac{F}{A} \Rightarrow F = PA
\]

\[
\text{Work} = \int_{V_0}^{V_f} p dV \quad \text{but no function } p(v) \ldots \quad ?
\]

Also can't use kinematics b/c \(a\) is not const \(p(t) = \frac{F(t)}{A}\)

\[
\text{So: } \quad p = \frac{E}{A} = \frac{(M+m)g}{A}
\]

\[
P A L^* = P_0 (A L_0)^{5/3}
\]

Assume adiabatic process: \(P_f = P_0 \left( \frac{L_0}{L_0 - \Delta L} \right)^{5/2}\)

To find instantaneous acceleration

\[
\frac{(M+m)g}{A} = P_0 \left( \frac{L_0}{L_0 - \Delta L} \right)^{5/2}
\]

\[
a = \frac{A P_0}{(M+m)} \left( \frac{L_0}{L_0 - \Delta L} \right)^{5/3} \sim \frac{1}{(1 - \frac{\Delta L}{L_0})^{5/3}}
\]

\(\Delta L\) highest deceleration just before stopping too high.
Problem 5 (10 Points):

The Otto cycle is shown in the figure. Stages 2 and 4 are adiabatic, reversible expansion and compression. Stages 3 and 5 are constant volume heating and cooling. Assume this is for an ideal gas.

![Otto cycle diagram]

a. Write down the efficiency, $\eta$, in terms of the work, $W$, and the added heat, $Q$. (2 Points)

b. During which stage or stages is heat added? (1 Points)

c. Calculate the work, $W$, in terms of the heat capacity at constant volume, $C_V$ and the temperature change. (2 Points)

d. Calculate the heat added, $Q$, in terms of $C_V$ and the temperature change. (2 Points)

e. Show that the efficiency is $\eta = \frac{1}{(\frac{V_2}{V_1})^{\gamma-1}}$, where $\gamma = \frac{C_p}{C_V}$. (3 Points)
Isochoric: constant volume

Adiabatic: \( \mathbf{d}Q = 0 \)

(a) In General: efficiency \( \eta = \frac{\text{benefit}}{\text{cost}} = \frac{W_{\text{out}}}{Q_{H}} \)

\[
\frac{Q_{\text{in}}}{Q_{H}} \quad \frac{W_{\text{out}}}{Q_{H}} = \frac{Q_{H} - Q_{L}}{Q_{H}} = 1 - \frac{Q_{L}}{Q_{H}}
\]

(b) Heat added during stage (3) when the temperature is raised. Think this process is reversible by combusting fuel.

(c) \[ W_{\text{out}} = Q_{\text{in}} - Q_{\text{out}} = Q_{H} - Q_{L} \]

\[
= C_{V} \Delta T_{3} - C_{V} \Delta T_{5}
\]

\[
= C_{V} \left[ (T_{C} - T_{b}) - (T_{a} - T_{d}) \right] \quad \text{technically should be mod}
\]

(d) \[ Q_{\text{in}} = h \Delta T_{3} = C_{V} (T_{C} - T_{b}) \]

(e) \[ \eta = 1 - \frac{C_{V} (T_{a} - T_{d})}{C_{V} (T_{C} - T_{b})} = 1 - \frac{T_{a} - T_{d}}{T_{C} - T_{b}} \]

Step 3: \[ P_{b} V_{b} = n R T_{b} \quad P_{c} V_{c} = n R T_{c} \quad V_{b} = V_{c} = V_{i} \]

\[ \frac{P_{b}}{P_{c}} = \frac{T_{b}}{T_{c}} \]

Step 4: \[ P_{c} V_{c} = n R T_{c} \quad P_{d} V_{d} = n R T_{d} \]

\[ \frac{V_{1}}{V_{2}} = \frac{T_{c}}{T_{d}} \]
\( T_c V_c^{x-1} = T_d V_d^{x-1} \)
\( T_c V_1^{x-1} = T_d V_2^{x-1} \) and
\( T_a V_a = T_b V_b^{x-1} \)
\( T_a V_2^{x-1} = T_b V_1^{x-1} \)

\[ T_d = T_c \left( \frac{V_1}{V_2} \right)^{x-1} = T_c x \]
\[ T_a = T_b \left( \frac{V_1}{V_2} \right)^{x-1} = T_b x \]

\[ \eta = 1 - \frac{T_b x - T_c x}{T_c - T_b} = 1 - x \left( \frac{T_b - T_c}{T_c - T_b} \right) \]

\[ = 1 - x \left[ \left( \frac{-1}{T_c - T_b} \right) \frac{\left( T_c - T_b \right)}{\left( T_c - T_b \right)} \right] \]  \& remember to use mod

\[ = 1 - \left( \frac{V_1}{V_2} \right)^{x-1} \]  \( \checkmark \)
Statistical Mechanics

4. The grand free energy or grand potential, $\Xi$, can be obtained from the Helmholtz, $F(T, V, N)$ free energy or the internal energy $U(S, V, N)$ via:

$$\Xi = F - \mu N = U - TS - \mu N$$

(a) What are the normal or proper variables for $\Xi$? (When $\Xi$ is written in terms of its normal or proper variables, it constitutes a complete thermodynamic description, without loss of information). (1 point)

(b) Derive expressions for the conjugate variables in this description. (1 point)

(c) What are the Maxwell relations governing derivatives of $\Xi$? (2 points)

(d) Consider a small system connected to a large thermodynamic reservoir. State under what conditions (e.g. specify what quantities are exchanged between the system and reservoir) $\Xi$ is minimized in equilibrium. Prove that this is the case by showing that $\Xi$ is minimized when the system is in equilibrium. (3 points)

(e) Given the Helmholtz free energy for an ideal gas:

$$F(T, V, N) = -NkT \left( 1 + \log \left( \frac{VT^{3/2}}{N\Phi} \right) \right)$$

where $\Phi$ is an unspecified by fixed constant, calculate the grand free energy for an ideal gas. (3 points)
in GCE $V, T, \mu$ fixed - proper variables

$G(V, T, \mu)$

$G = F - uN = u - TS - \mu N$

$\frac{dG}{dV} = \frac{du}{dV} - SdT - NVd\mu$

$N = -\frac{dG}{d\mu} \quad s = -\frac{dG}{dT} \quad p = -\frac{dG}{dV}$

This is because the temperature is fixed.

these are the desired conjugate variables

find by taking derivatives w.r.t. proper vars

only trick is $U = E \propto \nu$

3 pairs of relations (partial derivatives more than each other)

\[ \frac{\partial}{\partial \mu} \frac{\partial}{\partial T} G = \frac{\partial}{\partial T} \frac{\partial}{\partial \mu} G \]

\[ \frac{\partial}{\partial V} \frac{\partial}{\partial \mu} G = \frac{\partial}{\partial \mu} \frac{\partial}{\partial V} G \]

\[ \frac{\partial}{\partial \mu} (-s) = \frac{\partial}{\partial T} (-N) \quad \frac{\partial}{\partial V} (-p) = \frac{\partial}{\partial T} (-s) \]

\[ \frac{\partial}{\partial \mu} \frac{\partial}{\partial V} G = \frac{\partial}{\partial V} \frac{\partial}{\partial \mu} G \]

\[ \frac{\partial}{\partial \mu} (-p) = \frac{\partial}{\partial V} (-N) \]

In GCE, $N$ and $E$ can change (are exchanged)

\[ G = u - TS - \mu N \frac{du}{dS} + \frac{dTS}{dS} - d(uN) \]

\[ \frac{dG}{dV} = TdS - p dV + \mu dN - TdS - SdT - \mu dN - N d\mu \]

\[ = -p dV - SdT - N d\mu \]

at eqn $T_1 = T_2$, $\mu_1 = \mu_2$

\[ dG = (p_1 + p_2) dV = 0 \]

@ eqn so it is a minimum.
\[ \mathcal{Y} = F - \mu N = \frac{\mathcal{G}(V, T, \mu)}{N} \]

\[ = -NkT \left( 1 + \ln \left( \frac{V T^{3/2}}{N \Phi} \right) \right) - \mu N \]

use \( \mu = \frac{\partial F}{\partial N} \) to replace \( N \)'s

\[ \mu = \frac{\partial}{\partial N} \left[ (-NkT) - NkT \ln \left( \frac{V T^{3/2}}{N \Phi} \right) \right] \]

\[ = -kT - \left[ kT \ln \left( \frac{N^{-1} V T^{3/2}}{N \Phi} \right) + NkT \left( \frac{N \Phi}{V T^{3/2}} \right) \left( -\frac{V T^{3/2}}{N^2 \Phi} \right) \right] \]

\[ \mu = -kT - kT \ln \left( \frac{V T^{3/2}}{N \Phi} \right) + kT \]

\[ -\frac{\mu}{kT} = \frac{V T^{3/2}}{N \Phi} \quad \rightarrow \quad N = \frac{V T^{3/2}}{\Phi} \exp \left( \frac{\mu}{kT} \right) \]

plug in: \( \mathcal{G} = -\frac{V T^{3/2}}{\Phi} e^{\frac{\mu}{kT}} kT \left( 1 + \ln \left( \frac{V T^{3/2}}{\Phi} \right) - \ln \left( \frac{V T^{3/2}}{\Phi} \right) - \frac{\mu}{kT} \right) \)

\[ = -\frac{V T^{3/2}}{\Phi} e^{\frac{\mu}{kT}} \left[ kT \left( 1 - \frac{\mu}{kT} \right) + \mu \right] \]

\[ = -\frac{V T^{3/2}}{\Phi} e^{\frac{\mu}{kT}} \left( kT \right) = -\frac{V}{\Phi} e \]
5. Consider a solid consisting of a lattice of \( N \) atoms. If an atom is knocked slightly out of its proper position in the lattice, this is called a point defect. Assume that the energy of a point defect is \( \epsilon \), and that if there is no defect, the energy at that site is zero. Assume further that the defects are distinguishable, and do not interact with each other.

(a) Determine the number of ways to place \( n \) point defects within the \( N \) lattice sites. From this show that the entropy associated with their configuration is approximately

\[
S(N,n) \sim k [N \ln N - n \ln n - (N-n) \ln (N-n)]
\]

(3 points).

(b) Derive a (simple) expression for the internal energy, \( U \), of the system as a function of \( n \). From this result, write the entropy above, \( S(N,n) \) as function of \( U \) and \( N \), obtaining \( S(U,N) \). (2 points).

(c) Using the expression for \( S(U,N) \) above, calculate the chemical potential for changes in \( N \). (3 points).

(d) In thermal equilibrium we know

\[
T = \left[ \frac{\partial S}{\partial U} \right]^{-1}
\]

Use the approximate results above to calculate this temperature as a function of \( U \) and \( N \). From this result to find the number of defects, \( n(N,T) \). (2 points).

\[
\text{Hasib: } n = \frac{N}{1 + e^{\beta \epsilon}} \quad \text{where } \beta = \frac{1}{kT}
\]
\[ S = \frac{N!}{n!(N-n)!} \]

**Entropy (à la Boltzmann)**

\[ S = k \left[ \ln \frac{N!}{n!(N-n)!} \right] \]

\[ S = k \left\{ \ln N! - \ln n! - \ln (N-n)! \right\} \]

using sterling's approximation: \( \ln N! \approx N(\ln N) - N \)

\[ S \approx k \left\{ N \ln N - N - (n \ln n - n) - [(N-n) \ln (N-n) - (N-n)] \right\} \]

\[ S \approx k \left\{ N \ln N - n \ln n - (N-n) \ln (N-n) + N - n \right\} \]

\[ S \approx k \left[ N \ln N - n \ln n - (N-n) \ln (N-n) \right] \]

\( U = n e \Rightarrow n = \frac{U}{e} \)

\[ S(U, N) = k \left[ N \ln N - \frac{U}{e} \ln \frac{U}{e} - \left( N - \frac{U}{e} \right) \ln \left( N - \frac{U}{e} \right) \right] \]

\( \mu = -T \frac{\partial S}{\partial N} = -\frac{\mu}{T} \ln \left( \frac{N}{U/e} \right) \)

\[ \mu = -kT \left\{ \ln N + 1 - \left[ N(N-\frac{U}{e})^{-1} + \ln(N-\frac{U}{e}) - \frac{U}{e}(N-\frac{U}{e})^{-1} \right] \right\} \]

\[ \mu = -kT \left\{ \ln N + 1 - (N-\frac{U}{e})^{-1} \left[ N \ln \left( \frac{N}{U/e} \right) - \frac{U}{e} \right] - \ln \left( N-\frac{U}{e} \right) \right\} \]
\[ \mu = -kT \left( \ln N - T \right) \neq -kT \left( \ln \left( N - \frac{u}{e} \right) \right) \]

\[ \ln \left( \frac{N^2 - u}{e} \right) = \ln(Ne - u) - \ln e \]

\[ T = \left[ \frac{dS}{du} \right]^{-1} \]

\[ T = \frac{1}{k} \left\{ \frac{1}{e} \ln \frac{u}{e} + \frac{u}{e} \right\} - \frac{1}{\frac{d}{du} \left[ N \ln(N - \frac{u}{e}) - \frac{u}{e} \ln(N - \frac{u}{e}) \right]^{2 \ldots -1}} \]

\[ T = \frac{1}{k} \left\{ \frac{1}{e} \ln \frac{u}{e} + 1 - \left[ \frac{N}{e} - \frac{u}{e} \right] - \left( \frac{1}{e} \ln(N - \frac{u}{e}) + \frac{u}{e} \right) \right\}^{-1} \]

\[ T = \frac{1}{k} \left\{ \frac{1}{e} \ln \frac{u}{e} + 1 + \frac{N}{e} + \frac{1}{e} \ln(N - \frac{u}{e} - \frac{u}{e}) \right\}^{-1} \]

\[ T = \frac{1}{e k} \left\{ \ln u - \ln e + e + N + \ln(Ne - u) - \ln e - \frac{u}{e} \right\}^{-1} \]

\[ (\epsilon T)^{-1} = \ln u - 2\ln e + e + N + \ln(Ne - u) - \ln e - \frac{u}{e} \]

\[ (\epsilon T)^{-1} = \ln u - \epsilon^2 + e + N + Ne - u - \epsilon \]

\[ \tilde{\mu} \]

\[ \tilde{\mu} = \frac{1}{T} = \frac{N}{e} \quad \text{and} \quad E = \frac{Ne}{1 + e^{\tilde{\mu}}} \]
Problem 6 (10 Points):

The following questions refer to a stream of photons in equilibrium at temperature $T$ (thermal light - say from a light bulb) incident on a perfect detector which detects (counts) all the particles that hit it. Your final answers should be in terms of the mean particle number.

a. Given $\bar{n}_s$ photons are counted on average in time $t$, calculate the variance in the photon number $n_s$, $(\Delta n_s)^2$. (2 Points)

b. Calculate the fractional fluctuation of the detector signal defined as the square root of the variance divided by the mean photon number, $\bar{n}_s$, squared, $\sqrt{\frac{(\Delta n_s)^2}{\bar{n}_s^2}}$. This is the inverse of the signal to noise ratio. (2 Points)

The following questions refer to a stream of electrons in equilibrium at temperature $T$ incident on a detector which detects (counts) all the particles that hit it. Again, your final answers should be in terms of the mean particle number.

c. Given $\bar{n}_e$ electrons are counted on average in time $t$, calculate the variance in the electron number $n_e$, $(\Delta n_e)^2$. (2 Points)

d. Calculate the fractional fluctuation of the detector signal defined as the square root of the variance divided by the mean electron number, $\bar{n}_e$, squared, $\sqrt{\frac{(\Delta n_e)^2}{\bar{n}_e^2}}$. (2 Points)

e. Compare the two results. Are the results the same or different? Do the counts detected clump (bunch) or anti-clump (anti-bunch)? Why? (2 Points)
\[
(\Delta n_s)^2 = \langle n_s^2 \rangle - \bar{n_s}^2 \approx \langle n_s \rangle^2
\]

\[
\frac{[\Delta n_s]^2}{\bar{n_s}^2} = \frac{\langle n_s^2 \rangle}{\bar{n_s}^2} - 1
\]

\[
\sqrt{\frac{(\Delta n_s)^2}{\bar{n_s}^2}} = \sqrt{\frac{\langle n_s^2 \rangle}{\bar{n_s}^2} - 1}
\]

but how are bosons and fermions different from each other?
Problem 5 (10 Points):

A system consists of $N$ identical non-interacting particles in equilibrium with a heat bath. The total number of individual states available to each particle is $2N$. Of these states, $N$ are degenerate with energy 0 and $N$ are degenerate with energy $\epsilon$. It is found by observation that the total energy of the system is $N\epsilon/3$.

a. What is the average number of particles in the excited state? (1.5 Points)

Find the temperature of the system under the following three different assumptions.

b. The particles are bosons. (2 Points)

c. The particles are fermions. (2 Points)

d. The particles obey a Boltzmann distribution. (2 Points)

e. Are the temperatures you found in (b.), (c.) and (d.) the same? Why or why not? Explain your answer. (2.5 Points)
Total energy: \( E = \frac{N \varepsilon}{3} \) so \( N \varepsilon / 3 \) or \( \frac{1}{3} \) of particles are excited

Use the canonical ensemble because heat can exchange with a reservoir.

Bosons: \( Z = \frac{1}{N!} \left( \sum_{n=0}^{\infty} e^{-\beta E_n} \right)^N \)

\( f_B(\varepsilon_i) = \frac{g_i}{e^{\beta(\varepsilon_i - \mu)} - 1} \)

Fermions: \( Z = \left( \sum_{n=0}^{\infty} e^{-\beta E_n} \right)^N \)

\( f_F(\varepsilon_i) = \frac{g_i}{e^{\beta(\varepsilon_i - \mu)} + 1} \)

Boltzmann:

\( f_{Bz}(\varepsilon_i) = \frac{g_i}{e^{\beta(\varepsilon_i - \mu)}} \)

where: \( f \) is occupation number, \( N \) of particles with energy \( \varepsilon_i \)

\( \mu \) is chemical potential

\( g_i \) is degeneracy of state \( i \)

\( Z \) is partition function

\( f_B(\varepsilon) = \frac{N}{e^{\beta(\varepsilon - \mu)} - 1} = \frac{N}{3} \) \( \Rightarrow \) solve for temp using \( \beta = \frac{1}{kT} \)

\( f_B(0) = \frac{N}{e^{\beta(0 - \mu)} - 1} = \frac{2N}{3} \)

\( 3 = e^{\beta(\varepsilon - \mu)} - 1 \quad \Rightarrow \quad 3 = 2(e^{\beta(\varepsilon - \mu)} - 1) \)

\( 4 = e^{\beta(\varepsilon - \mu)} \quad \Rightarrow \quad 5 = 2e^{-\beta \mu} \)

\( \mu + 4 = \beta \varepsilon - \beta \mu \quad \Rightarrow \quad \mu \varepsilon = -\beta \mu \)

\( \ln(\mu + 4) = \beta \varepsilon + \frac{\mu \varepsilon}{2} \)

\( \ln(\mu + 4) = \beta \varepsilon + \frac{\mu \varepsilon}{2} \quad \Rightarrow \quad \frac{1}{kT} = \frac{1}{\xi} \ln(\frac{8}{5}) \)

\( \frac{1}{\xi} \ln(\frac{4 + \frac{2}{5}}{5}) = \beta \) \( \Rightarrow \quad \frac{\xi}{k} \left[ \ln(\frac{8}{5}) \right]^{-1} = T_B \)
Fermions

\[ 3 = e^{\beta (E - \mu)} + 1 \quad \Rightarrow \quad 3 = 2 (e^{\beta (E - \mu)} + 1) \]

\[ 2 = e^{\beta (E - \mu)} \quad \Rightarrow \quad \frac{3}{2} - 1 = e^{\beta \mu} = \frac{1}{2} \]

\[ m_2 = \beta E - \beta \mu \quad \ln \frac{\gamma}{2} = -\beta \mu = -m_2 \]

\[ m_2 = \beta E - \ln 2 \]

\[ 2 \ln 2 = \frac{E}{kT} \]

\[ T_F = \frac{E}{kT} \]

\[ 2 \ln 2 = \frac{E}{kT} \]

\[ = \frac{m_2^2}{E} = m_4 \]

\[ 2 \ln 2 = \frac{E}{kT} \]

\[ = \frac{m_2^2}{E} = m_4 \]

\[ T = \frac{E}{kT} \left[ \ln (2) \right]^{-1} = \frac{E}{kT} \ln 2 \]

so: \( T_B > T_B^2 > T_F \)

Bosons: for bosons it's higher because more particles can occupy the same higher state. For fermions, it's lower because the particles must spread out and occupy lower energy states. Boltzmann is an approximation that falls between these two extremes. The system is at equilibrium with the reservoir in each of the three cases (even though \( T \) differs) b/c no more energy can be transferred into the system (for \( N \frac{3}{2} \) excited fixed)

i.e. it's an energy bath. (energy fixed)

Takes most heat to give bosons a certain energy, least for fermions.
5. A particular solid is made up of \( N \) distinguishable spin 1 atoms each on a fixed position in a lattice. The energy of each atom is given by:

\[
E(\sigma_i) = -V_0 \sigma_i^2 - \mu_0 \sigma_i B
\]

where \( V_0 \) arises from an internal field in the crystal, \( B \) is the applied external magnetic field and \( \mu_0 \) is the Bohr magneton. The \( z \)-component of the spin of an atom can take on values \( \sigma_i \in \{0, \pm 1\} \)

(a) Calculate the free energy, \( F(T, B, N) \). (2 pts.)
(b) Calculate the specific heat. (4 pts.)
(c) Calculate the magnetic susceptibility, \( \chi(T, B, N) \) when \( B = 0 \). (4 pts.)
\[ E(\sigma_i) = -V_0 \sigma_i^2 - \mu_0 \sigma_i B \]

so:
\[ E(-1) = -V_0 + \mu_0 B \]
\[ E(0) = 0 \]
\[ E(1) = -V_0 - \mu_0 B \]

use the canonical ensemble since \( N \) fixed, \( E \) varies
\[
Z = \left( \sum_{n=0}^{\infty} e^{-\beta E_n} \right)^N = \left( e^{-\beta E(-1)} + e^{-\beta E(0)} + e^{-\beta E(1)} \right)^N = \left( e^{-\beta(-V_0 + \mu_0 B)} + e^{-\beta(-V_0 - \mu_0 B)} \right)^N = \left( 1 + e^{\beta V_0 - \beta \mu_0 B} + e^{\beta V_0 + \beta \mu_0 B} \right)^N = \left( 1 + e^{\beta V_0 (e^{-\beta \mu_0 B} + e^{\beta \mu_0 B})} \right)^N
\]

\[
F = -\frac{1}{\beta} \ln Z = -\frac{N}{\beta} \ln \left[ 1 + e^{\beta V_0 \cosh(\beta \mu_0 B)} \right]
\]

\[
E = -\frac{1}{\beta} \ln Z \quad \text{or} \quad C_V = \frac{\partial E}{\partial T} = \frac{\partial E}{\partial \beta} \frac{1}{\partial T} = \frac{1}{\partial x} \frac{\partial}{\partial x}(a + x) = (a + x)^\frac{1}{\partial x}(a + x)
\]
\[
E = -\frac{1}{\beta} N \ln \left[ 1 + e^{\beta V_0 (e^{-\beta \mu_0 B} + e^{\beta \mu_0 B})} \right] = -N \left( 1 + e^{\beta V_0 \cosh(\beta \mu_0 B)} \right)^{-1} \left[ V_0 e^{\beta V_0} \cosh(\beta \mu_0 B) + 2\mu_0 B \sinh(\beta \mu_0 B) e^{\beta V_0} \right]
\]
\[
= -2N e^{\beta V_0} \left[ V_0 \cosh(\beta \mu_0 B) + \mu_0 B \sinh(\beta \mu_0 B) \right] \frac{-1}{1 + e^{\beta V_0 \cosh(\beta \mu_0 B)}}
\]

now find \( C_V \):
\[
\frac{\partial E}{\partial T} = \frac{1}{\partial T} \frac{1}{K} \rightarrow (-1) \frac{T^2}{K} = -\frac{\beta^2}{k^2} = -k \beta^2
\]
\[ \text{Let } x = \beta \mu_0 B, \quad c = y = V_0 + \mu_0 B, \quad \bar{c} = \bar{y} = V_0 - \mu_0 B \]

\[ \Rightarrow V_0 \cosh(x) + \mu_0 B \sinh(x) = \frac{V_0}{2}(e^x + e^{-x}) + \frac{\mu_0 B}{2}(e^x - e^{-x}) \]

\[ = e^x \left( \frac{V_0 + \mu_0 B}{2} \right) + e^{-x} \left( \frac{V_0 - \mu_0 B}{2} \right) = \frac{1}{2}(e^x c + e^{-x} \bar{c}) \]

\[ \Rightarrow 2e^{\beta V_0} \cosh(x) = \frac{2e^{\beta V_0}}{2}(e^x + e^{-x}) = e^{\beta V_0 + x} + e^{\beta V_0 - x} \]

so:

\[ \frac{dE}{d\beta} = -\frac{1}{2} \frac{dN}{d\beta} \left[ e^{\beta \mu_0 B} c + e^{-\beta \mu_0 B} \bar{c} \right]\left[ 1 + e^{\beta(V_0 + \mu_0 B)} + e^{\beta(V_0 - \mu_0 B)} \right]^{-1} \]

\[ = -N\left\{ \mu_0 B e^x c - \mu_0 B e^{-x} \bar{c} \right\}\left[ 1 + e^{\beta(V_0 + \mu_0 B)} + e^{\beta(V_0 - \mu_0 B)} \right]^{-1} \]

\[ + [e^x c + e^{-x} \bar{c}](-1)[1 + e^{\beta y} + e^{\beta \bar{y}}]^{-2} \left\{ y e^{\beta \bar{y}} + \bar{y} e^{\beta y} \right\} \]

\[ \text{but,} \quad c = y, \quad \bar{c} = \bar{y} \]

\[ \frac{dE}{d\beta} = -N\left\{ \mu_0 B [ce^x - ce^{-x}]\left[ 1 + e^{\beta c} + e^{\beta \bar{c}} \right]^{-1} \right. \]

\[ - [ce^x + ce^{-x}] \left[ ce^\beta c + ce^\beta \bar{c} \right]\left[ 1 + e^{\beta c} + e^{\beta \bar{c}} \right]^{-2} \]

using:

\[ (V_0 + \mu_0 B)(V_0 - \mu_0 B) = V_0^2 - \mu_0^2 B^2 \]

\[ x + \beta c = \beta \mu_0 B + \beta V_0 + \beta \mu_0 B = 2 \mu_0 B + \beta V_0 = \beta (2 \mu_0 B + V_0) \]

\[ x + \beta \bar{c} = \beta \mu_0 B + \beta(V_0 - \mu_0 B) = \beta \mu_0 B + \beta V_0 - \beta \mu_0 B = \beta V_0 \]

nothing seems to simplify, \( \text{so}; \) add \( C_V = \frac{dE}{d\beta} \frac{d\beta}{dx} \)

\[ C_V = \frac{\beta^2 N \left\{ \mu_0 B [ce^x - ce^{-x}] - [ce^x + ce^{-x}] \left[ ce^\beta c + ce^\beta \bar{c} \right]\left[ 1 + e^{\beta c} + e^{\beta \bar{c}} \right]^{-2} \right\}}{[1 + e^{\beta c} + e^{\beta \bar{c}}]^2} \]
\[
\chi = \frac{\partial M}{\partial B}
\]

\[
M = \frac{1}{\beta} \frac{\partial}{\partial B} \ln Z = -\frac{\partial F}{\partial B}
\]

So:
\[
\chi = \frac{2}{\beta} \left( -\frac{\partial F}{\partial B} \right) = -\frac{\partial^2 F}{\partial B^2} = -\frac{2}{\beta^2} \left( -\frac{N}{\beta} \right) \ln \left[ 1 + e^{\beta \nu \cos \theta (\beta \mu \cdot B) Z} \right]
\]

\[
= \frac{N}{\beta} \frac{2}{\beta} \left\{ \left[ 1 + 2e^{\beta \nu \cos \theta (\beta \mu \cdot B)} \right]^{-1} \left[ 2e^{\beta \nu \cos \theta (\beta \mu \cdot B)} \right]^2 \right\}
\]

\[
\chi = \frac{N}{\beta} \left\{ (-1)^n \left[ 1 + 2e^{\beta \nu \cos \theta (\beta \mu \cdot B)} \right]^{-n} \left[ 2e^{\beta \nu \cos \theta (\beta \mu \cdot B)} \right]^n \right\}
\]

\[
\chi = \frac{N}{\beta} \left\{ \left[ 2 \beta \mu \cdot e^{\beta \nu \cos \theta (\beta \mu \cdot B)} \right]^2 \right\}
\]

\[
\frac{1}{\left[ 1 + 2e^{\beta \nu \cos \theta (\beta \mu \cdot B)} \right]^2}
\]

\[
+ \frac{2 \beta^2 \mu \cdot e^{\beta \nu \cos \theta (\beta \mu \cdot B)} \left[ 1 + 2e^{\beta \nu \cos \theta (\beta \mu \cdot B)} \right]^{-1}}{\left[ 1 + 2e^{\beta \nu \cos \theta (\beta \mu \cdot B)} \right]^2}
\]

For \( B = 0 \), \( \cos \theta (0) = 1 \), \( \sin \theta (0) = 0 \)
5. By shining an intense laser beam on a semiconductor, one can create a metastable collection of electrons (charge \(-e\) and effective mass \(m_e\)) and holes (charge \(+e\) and effective mass \(m_h\)). These oppositely charged particles may pair up to form an exciton, or they may dissociate into a plasma. This problem considers a simple model of this process. In this problem the densities of electrons and holes are so low that you can ignore their fermionic nature and treat them as classical particles in three dimensions.

(a) Calculate the free energy \(F(T, V, N)\) of a gas of \(N_e\) electrons and \(N_h\) holes at temperature \(T\), treating them as classical, non-interacting, ideal gas particles in a 3D volume \(V\). (2 pts.)

(b) By pairing into an exciton, each electron-hole pair lowers its energy by \(\Delta E\). Calculate the free energy of a gas of \(N_p\) excitons, treating them as classical, non-interacting, ideal gas particles. (2 pts.)

(c) Calculate the chemical potentials \(\mu_e\), \(\mu_h\), and \(\mu_p\) of the electrons, holes, and exciton pairs respectively. What is the condition of equilibrium between excitons and electrons and holes? (3 pts.)

(d) Consider the case where the numbers of electrons and holes are equal, so that \(n_h = n_e = n_0\). Determine the approximate density of excitons as a function of \(n_0\) in the high temperature limit (when the exciton population is low). (3 pts.)
\( Z = Z_0 Z_n \) first find Ze

\[ Ze = \frac{1}{N!} \left[ \int e^{-\beta E} \right]^N \begin{bmatrix} E = \frac{p^2}{2m} \end{bmatrix} \text{ for a free particle} \]

\[ = \frac{1}{N!} \left[ \int d^3x \int d^3p e^{-\frac{\beta p^2}{2m}} \right]^N \]

\[ = \frac{1}{N!} \left[ \frac{V 4\pi}{(2\pi \hbar)^3} \int_0^\infty -\frac{\beta p^2}{2m} \right]^N \]

\[ = \left[ \frac{V 4\pi}{(2\pi \hbar)^3} \right]^N = \left[ \frac{V m^3}{2 \pi^2 \hbar^3} \right]^N \]

using: \( \int_0^\infty x^2 e^{-ax^2} dx = \frac{1}{4a^{3/2}} \)

\[ = \frac{1}{2N} \sqrt{\frac{2\pi m^3}{\beta^3}} \]

using De Broglie wavelength:

\[ \lambda_T = \left( \frac{m}{2\pi \hbar^2 \beta} \right)^{-1/2} \]

\[ Ze = \frac{1}{N!} \left[ V \lambda_T^{-3/2} \right]^N = \frac{1}{N!} \left[ V \lambda_T^{-3} \right]^N \]

are the same calculation

\[ F = \frac{-1}{\beta} \ln Z = \frac{-1}{\beta} \left[ \ln(N!)^2 + \ln(V \lambda_T^{-3})^{2N} \right] = \frac{-1}{\beta} \left[ 2\ln(N!) + 2N \ln(V \lambda_T^{-3}) \right] \]

\[ F = \frac{-2}{\beta} \left( \ln(N!) + N \ln V - 3N \ln \lambda_T^3 \right) \]
\[ Z = Z_e Z_h = \frac{1}{N_e! N_h!} \frac{\sqrt[3]{N_e N_h}}{\lambda^{3N_e} \lambda^{2N_h}} \]

using \( \mu = \left( \frac{1}{N!} \right) = -\ln N \]

\[ \ln Z = \left( -N_e \ln N_e + N_e \right) \left( -N_h \ln N_h + N_h \right) \left( N_e + N_h - 3N_e \ln \lambda - 3N_h \ln \lambda \right) \]

\[ F = \frac{1}{\beta} \left( N_e \ln N_e - N_e \right) \left( -N_h \ln N_h + N_h \right) \left[ N_e \ln (N_e + N_h) - 3 \ln \lambda (N_e + N_h) \right] \]

\[ \ln Z = -\ln N! + N N_e \ln N_e + \frac{N^2}{2} \ln \frac{2 \pi \hbar^2}{m} \]

\[ = -N_e \ln N_e + N_e + N N_e \ln N_e + \frac{N^2}{2} \ln \left( \ln 2 \pi \hbar^2 + \frac{\hbar \beta}{\hbar} \right) \]

\[ E = -\frac{\partial}{\partial \beta} \ln Z = -\frac{2N}{\beta} \frac{1}{2} \sqrt{\mu} \]

\[ \mu = \frac{\partial F}{\partial N} \]

\[ F = \frac{1}{\beta} \ln Z = \frac{1}{\beta} \ln \left( \frac{1}{N!} \frac{V^N}{\lambda^{2N}} \right) = -\frac{1}{\beta} \left\{ -\ln N! + N N_e \ln N_e - 3N_e \ln \lambda^2 \right\} \]

\[ F = \frac{1}{\beta} \left( -N_e \ln N_e + N_e + N N_e \ln N_e - 3N_e \ln \lambda^2 \right) \]

\[ \mu = -\frac{1}{\beta} \left\{ -N_e \ln N_e + N_e + N N_e \ln N_e - 3N_e \ln \lambda^2 \right\} = -\frac{1}{\beta} \left\{ -N_e \ln N_e + N_e + N N_e \ln N_e - 3N_e \ln \lambda^2 \right\} \]

\[ \mu = kT \ln N_e - N_e \ln V_e + 3 N_e \ln \lambda^2 \]

\[ \text{since all 3 } F's \text{ are of the same form then so are } \mu's \]

\[ \mu_e = kT \ln N_e \ln N_e - N_e \ln V_e + 3 N_e \ln \lambda^2 \]

\[ \mu_h = kT \ln N_h \ln N_h - N_h \ln V_h + 3 N_h \ln \lambda^2 \]

\[ \mu_p = kT \ln N_p \ln N_p - N_p \ln V_p + 3 N_p \ln \lambda^2 \]

\[ \text{Eqn happens when } \mu_e + \mu_h = \mu_p \]

\[ \text{Note: this does not mean } N_e + N_h = N_p \]
Use eqn condition: \( \frac{N_p}{V} \)  

\[
\ln \left( \frac{N_e \lambda_e^3}{V_e} \right) + \ln \left( \frac{N_h \lambda_h^3}{V_h} \right) = \ln \left( \frac{N_p \lambda_p^3}{V_p} \right)
\]

\[
\ln \left[ \frac{\lambda_p}{V} \right] = \ln \left[ \frac{\lambda_e \lambda_h}{V} \right]
\]

\[
\frac{N_e^2 \lambda_e^3 \lambda_h^3}{V^2} = \frac{N_p \lambda_p^3}{V} \Rightarrow \frac{N_p}{V} = \frac{N_e^2 \lambda_e \lambda_h}{V^2} \frac{\lambda_p^3}{\lambda_p^3}
\]

Expanding:

\[
\lambda^3 = \left( \frac{m_e m_h}{2\pi \hbar^2} \right)^{3/2} \frac{\lambda_p^3 \lambda_e \lambda_h}{\lambda_p^3} = \left( \frac{m_e m_h}{2\pi \hbar^2} \right)^{3/2} \left( \frac{m_e}{m_p} \right)^{3/2} \frac{2\pi \hbar^2}{m_p} = \left( \frac{m_e m_h}{2\pi \hbar^2} \right)^{3/2}
\]

So, density:

\[
\frac{N_p}{V} = \frac{N_e^2}{V^2} \left( \frac{m_p 2\pi \hbar^2}{m_e m_h kT} \right)^{2/3}
\]

As \( T \to \infty \), density \( \to 0 \) molecules are too excited to combine - they "ionise".
5. Consider a classical ideal gas in 3D that feels a linear gravitational potential,
\[ V(z) = mgz \]
where \( m \) is the mass of a single gas atom and \( 0 < z < \infty \). This is not an interaction between gas atoms, it is simply their gravitational potential energy near the surface of the Earth.
The gas is in a box of dimensions \( L_x, L_y, \) and \( L_z \), so that:
\[
0 < z < L_z \\
0 < x < L_x \\
0 < y < L_y
\]

(a) Calculate the partition function in the canonical ensemble. (3 points)

(b) Determine the internal energy of the gas. (3 points)

(c) Calculate the specific heat \( c_v \). (3 points)

(d) Explain the behavior of the specific heat when \( \beta mgL_z >> 1 \) and when \( \beta mgL_z << 1 \). (The approximation for the gravitational potential may or may not be valid for large \( L_z \). Don’t worry about that.) (1 point)
a) \[ Z = \frac{1}{N!} \left( \int e^{-\frac{\beta E}{h}} \frac{d^3x d^3p}{(2\pi)^6} \right)^N \]

\[ Z = \frac{1}{N!} \left( \int e^{-\frac{\beta E}{h}} \frac{d^3x d^3p}{(2\pi)^6} \right)^N \]

To the integrals:

\[ \int d\mathbf{x} \int d\mathbf{y} \int d\mathbf{z} e^{-\beta m g z} = L x_L g (\frac{1}{\beta m g} e^{-\beta m g z} + \frac{1}{\beta m g}) \]

\[ \int_{0}^{\infty} \rho^2 dp \int_{0}^{\pi} \sin \theta d\theta \int_{0}^{\infty} dp \ e^{-\frac{\beta E}{h}} = 4\pi \int_{0}^{\infty} \rho^2 e^{-\rho^2 \beta m g} \ dp \]

\[ = 4\pi \frac{1}{4} \sqrt{\pi \left( \frac{2m}{\beta} \right)^3} = \pi \left[ \frac{1}{2} \left( \frac{2m}{\beta} \right)^3 \right]^{1/2} \]

\[ Z = \frac{1}{N!} \left[ \frac{1}{(2\pi)^3} L x_L g \frac{1}{\beta m g} \left( 1 - e^{-\beta m g z} \right) \left( \frac{1}{\beta m g} \right) \right]^{N/2} \]

\[ Z = \frac{1}{N!} \left[ \frac{L x_L g (1 - e^{-\beta m g z})}{(2\pi)^{3/2} \beta^{3/2} \beta m g} \right]^N \]

\[ Z = \frac{1}{N!} \left[ \frac{L x_L g (1 - e^{-\beta m g z})}{\beta^{3/2} \beta^{3/2} \beta^{3/2} \beta} \right]^N \]

b) \[ E = -\frac{\partial \ln Z}{\partial \beta} = -\frac{\partial}{\partial \beta} N \left\{ -ln N! + ln L x_L g \frac{1}{2^3 \beta} + ln (1 - e^{-\beta m g z}) \right\} \]

\[ E = -N \left\{ e^{-\beta m g z} \right\} \left( -m g z e^{-\beta m g z} \right) - \frac{N}{2} \frac{1}{\beta} \]

\[ E = \frac{SN}{2\beta} + \frac{-m g z e^{-\beta m g z}}{1 - e^{-\beta m g z}} = \frac{SN}{2\beta} + \frac{N m g z e^{-\beta m g z}}{e^{\frac{1}{\beta}} (e^{-\frac{1}{\beta}} - 1)} \]
\[ C_V = \frac{\delta E}{\delta T} = \frac{d}{dT} \left[ \frac{SNkT}{2} - NmgLz \left( e^{\frac{mglz}{kT}} - 1 \right) \right] \]

\[ = \frac{SNk}{2} - KN(mgLz)^2 \frac{e^x}{x^2} \left( \frac{mglz}{kT} e^{\frac{mglz}{kT}} \right) \frac{1}{t^2K} = \beta^2 k \]

\[ x \gg 1 \text{ (low T)} \text{ then } e^x - 1 \approx e^x \text{ this is gravitational freeze out, gravity > thermal } \]

so: \[ \frac{e^x}{e^{x^2}} = \frac{1}{e^x} = e^{-x} \]

and: \[ C_V \approx \frac{SNk}{2} - KN(mgLz)^2 e^{-\beta mglz} \]

for \( x << 1 \text{ (high T)} \text{ then: } e^x \approx 1 + x \]

so: \[ x^2(1+x) = \frac{(1+x)x^2}{(1-4x-1)^2} = 1 + \beta mglz = 1 + x \]

and: \[ C_V \approx \frac{SNk}{2} - KN(1 + \beta mglz) \]

\[ = \frac{SNk}{2} - NK\beta mglz \text{ gravity has little effect (free gas)} \]
4. Consider a certain hard sphere model of a gas of $N$ particles in which we have an “excluded volume term.” The entropy in this case can be given as:

$$S(U, V, N) = Nk \ln \left( \frac{V}{N} \frac{U}{\epsilon_0 N^2} \frac{1}{N-1} \right)$$

Here $b$ represents the volume of one gas particle, $V$ is the volume of the container, $U$ is the internal energy of the gas, and $k$ is Boltzmann’s constant. The constant $\epsilon_0$ has dimensions of energy $\times$ volume, and is included to keep the argument of the logarithm dimensionless.

(a) Does this system satisfy the third law of thermodynamics (i.e. does the entropy of the system go to zero as the temperature goes to zero)? Prove your answer. (3 points)

(b) What is the specific heat at constant volume for this gas? (2 points)

(c) A gas with $N_1$ particles with total energy $U_1$ in a volume $V_1$ has an excluded volume/particle of $b_1$. It is separated by a moveable, insulating piston from a second gas of $N_2$ particles with total energy $U_2$ in a volume $V_2$ and an excluded volume/particle of $b_2$. The piston is allowed to move so that $V_{\text{tot}} = V_1 + V_2$ is a constant, but $V_1$ and $V_2$ can change. What is the value of $V_1$ in equilibrium? (5 points)
The Second Law is intrinsically a mechanical law, little marbles don't obey QM.

\[ \text{d}E = T \text{d}s - P \text{d}V + \mu \text{d}N \]

\[ \rightarrow \text{holding } V, N \text{ const } \rightarrow \text{d}E = T \text{d}s \rightarrow \frac{\text{d}s}{\text{d}E} = \frac{1}{T} \]

\[ \frac{\text{d}}{\text{d}u} \left[ \text{NKln} \left( \frac{(V-Nb)}{60N^2} \right) \right] = \text{NK} \left( \frac{\text{d}(\text{ln}u)}{\text{d}u} \right) \frac{1}{\text{d}u} = \frac{1}{T} \rightarrow T = \frac{u}{\text{NK}} \]

Plug this back into the entropy:

\[ S = \text{NKln} \left[ \left( V-Nb \right) \frac{\text{NK}T}{60N^2} \right] \text{ so } S \rightarrow 0 \text{ as } T \rightarrow 0 \]

Entropy does not converge as \( T \rightarrow 0 \)

\[ C_V = \frac{\text{d}E}{\text{d}T} = \frac{\text{d}T}{\text{d}E} \text{Nk} = \text{NK} \]

\[ \text{holding } N \text{ and } E \text{ const } \rightarrow P = T \frac{\text{d}s}{\text{d}V} \]

At eqn: \[ P_1 = P_2 \rightarrow T_1 \frac{\text{d}s_1}{\text{d}V_1} = T_2 \frac{\text{d}s_2}{\text{d}V_2} \]

\[ T_1 \frac{\text{NK}}{(V_1-N_1b)} = T_2 \frac{N_2K}{(V_2-N_2b_2)} \quad \text{solved for } V_1 \]

\[ \frac{T_1N_1K}{T_2N_2} (V_2-N_2b_2) = V_1 - N_1b_1 \quad \text{or} \quad V = V_1 + V_2 \]

\[ V-V_1 = V_2 \]

\[ \frac{T_1N_1}{T_2N_2} (V-N_2b_2) + N_1b_1 = V_1 \]

\[ \frac{T_1N_1}{T_2N_2} (V-N_2b_2) + N_1b_1 = V_1 + V_1 \frac{T_1N_1}{T_2N_2} = \frac{T_1N_1V - T_1N_1N_2b_2 + N_1b_1T_2N_2}{T_2N_2} \]

\[ V_1 = \left( \frac{T_2N_2}{T_2N_2 + T_1N_1} \right) \frac{T_1N_1V - T_1N_1N_2b_2 + N_1b_1T_2N_2}{T_2N_2} \]
5. A crystalline solid contains $N$ similar, immobile, statistically independent defects. Each defect has 5 possible states $s_1, s_2, \ldots, s_5$. The energies of the states are given by $E_1 = E_2 = 0$, and $E_3 = E_4 = E_5 = \Delta$.

(a) Find the partition function for the defects as a function of their number, and the temperature $T$. (3 points)

(b) Find the defect contribution to the entropy of the crystal as a function of $\Delta$ and the temperature $T$. (4 points)

(c) Without doing a detailed calculation state the contribution to the internal energy due to the defects in the limit $kT >> \Delta$. Explain your reasoning. (3 points)
\[ Z = \left( \frac{N}{\beta} e^{-\beta E_0} \right)^N = \left( e^{\beta E_0} + e^{\beta E_0} + e^{-\beta E_0} + e^{-\beta E_0} \right)^N = (2 + 3e^{-\beta E_0})^N \]

\[ S = -\frac{\partial F}{\partial T} = -\frac{\partial F}{\partial \beta} \frac{\partial \beta}{\partial T} = \frac{\partial \beta}{\partial T} \frac{1}{K} = -\frac{T^{-1}}{K} = -\frac{1}{kT^2} = -k \beta^2 \]

\[ F = -\frac{1}{\beta} \ln Z = -\frac{N}{\beta} \ln (2 + 3e^{-\beta E_0}) \]

\[ S = -\frac{\partial F}{\partial \beta} \frac{\partial \beta}{\partial T} = -(k \beta^2) \frac{\partial}{\partial \beta} (-N \beta \ln (2 + 3e^{-\beta E_0})) \]

\[ = -k \beta^2 \left[ -N \beta^2 \ln (2 + 3e^{-\beta E_0}) + N \beta (2 + 3e^{-\beta E_0})^{-1} (-3 \Delta e^{-\beta E_0}) \right] \]

\[ = KN \ln (2 + 3e^{-\beta E_0}) + \frac{KN \beta \Delta e^{-\beta E_0}}{2 + 3e^{-\beta E_0}} \]

\[ \frac{e^{-x}}{1 + e^{-x}} = \frac{1}{e^x + 1} \]

\[ S = KN \ln (2 + 3e^{-\beta E_0}) + \frac{KN \beta \Delta}{kT (3 + 3e^{-\beta E_0})} \]

\[ E = -\frac{\partial}{\partial \beta} \ln Z = -\frac{\partial}{\partial \beta} N \ln (2 + 3e^{-\beta E_0}) = N \Delta \beta \]

\[ = \frac{3N \Delta E_0}{(2e^{-\beta E_0} + 3)} \]

For \( KT >> \Delta \Rightarrow \beta << \Delta \) and \( \Delta \beta \) is small

\[ \text{so } e^{\beta E_0} \approx 1 + \beta E_0 \text{ and } (1 + \Delta \beta)^{-1} \approx 1 - \Delta \beta \]

\[ \frac{3N \Delta}{2(1 + \beta E_0) + 3} = \frac{3N \Delta}{5 + 2\beta E_0} = \frac{3N \Delta}{5} (1 + \frac{\Delta}{2 \beta E_0})^{-1} \]
\[ \approx \frac{3N_A}{5} (1 - \frac{3}{5} \beta \Delta) \]

as temperature increases the energy effects due to the defects is less than that due to the high temperature.
Statistical Mechanics

4. A certain system can be modelled as an ideal gas of point particles, but the point particles have two internal states, with energies 0 and $\Delta$.

(a) Show that in the canonical ensemble the partition function $Z(T, V, N)$ for the gas can be written as

$$Z(T, V, N) = Z_0 \left(1 + e^{-\Delta/kT}\right)^N \frac{(VT^{3/2})^N}{N!}$$

where $Z_0$ is a multiplicative constant that has no effect on the equation of state. (2 points)

(b) Calculate the specific heat at constant volume for the gas. (2 points)

(c) Assume further that we have two such gases, $A$ and $B$, and that each has an internal state, but that $\Delta_A \neq \Delta_B$. Determine $Z(T, V, N_A, N_B)$, where $N_A$ and $N_B$ are the number of gas atoms of type $A$ and $B$, respectively. (1 point)

(d) Finally, if gas particles of type $A$ can convert into type $B$ and vice versa, calculate $N_A/N_{\text{tot}}$ in equilibrium, where $N_{\text{tot}} = N_A + N_B$. (5 points)
\[ Z_{\text{int}} = \frac{S}{n} e^{-\beta E_n} = (e^0 + e^{-\beta \Delta E}) = (1 + e^{-\beta/kT}) \]

\[ Z_{\text{trans}} = \int \frac{d^3 x d^3 p}{(2\pi h)^3} e^{-\beta \frac{p^2}{2m}} = \frac{4\pi V}{(2\pi h)^3} \int_0^{\infty} p^2 e^{-\beta p^2/2m} dp \]

\[ \frac{4\pi V}{(2\pi h)^3} \frac{1}{\sqrt{\pi}} \left( \frac{2m}{\beta^3} \right)^{3/2} = \frac{\pi V}{2\sqrt{\pi}} \frac{T^{3/2}}{m^{3/2} h^3} \]

\[ Z = \frac{1}{N!} (\beta N) = \frac{1}{N!} (Z_0 V T^{3/2})^N (1 + e^{-\Delta/kT})^N \]

\[ Z = Z_0 (1 + e^{-\Delta/kT})^N (V T^{3/2})^N \]

\[ E = -\frac{2}{\beta} \ln Z = -\frac{2}{\beta} \left( \ln \frac{Z_0}{N!} + N \ln \left( 1 + e^{-\Delta E} \right) (V \frac{R}{k^{3/2}}) \right) \]

\[ E = -\frac{2}{\beta} \left( \ln \frac{Z_0}{N!} + N \left[ \ln \left( 1 + e^{-\Delta E} \right) + \ln \left( V \frac{R}{k^{3/2}} \right) \right] \right) \]

\[ C_V = \frac{\partial E}{\partial T} = \frac{\partial E}{\partial \beta} \frac{\partial \beta}{\partial T} \]

\[ C_V = -\beta^2 \frac{d}{dt} \left( \frac{1}{e^{\beta kT}} \right)^{-1} \]

\[ C_V = -\beta^2 \frac{d}{dt} \left( \frac{1}{e^{\beta kT}} \right)^{-1} \]

\[ -\frac{3}{2} V V^{-\frac{1}{2}} (\beta k)^{-\frac{3}{2}} (\beta k)^{-\frac{5}{2}} = -\frac{3}{2} \beta^{-3/2} \]
\[
C_V = -\beta^2 k \frac{d}{d\beta} \left[ -N \left\{ -\Delta (1+e^{\Delta \beta})^{-1} - \frac{3}{2} \beta^{-3/2} \right\} \right] \\
= N\beta^2 k (-1) \frac{d}{d\beta} \left[ \Delta (1+e^{\Delta \beta})^{-1} + \frac{3}{2} \beta^{-3/2} \right] \\
= -N\beta^2 k \left( -\Delta (1+e^{\Delta \beta})^{-2} \left( \Delta e^{\Delta \beta} \right) - \frac{3}{2} \beta^{-5/2} \right) \\
= N\beta^2 k \left( \frac{\Delta^2 e^{\Delta \beta}}{(1+e^{\Delta \beta})^2} + \frac{9}{4} \beta^{-5/2} \right)
\]

\[
\mathcal{Z} = \frac{z_A z_B}{N_A! N_B!} \\
= z_A z_B \left( 1 + e^{-\Delta \beta/kT} \right)^{N_A} \left( 1 + e^{-\Delta \beta/kT} \right)^{N_B} \left( VT^{3/2} \right)^{N_A+N_B}
\]

\[
F = -\frac{1}{\beta} \ln \mathcal{Z} = -\frac{1}{\beta} \left\{ \ln \frac{z_A z_B}{N_A! N_B!} + N_A \ln \left( 1 + e^{-\Delta \beta/kT} \right) + N_B \ln \left( 1 + e^{-\Delta \beta/kT} \right) \right\}
\]

\[
\mu_A = \frac{\partial F}{\partial N_A} = -\frac{1}{\beta} \left\{ \frac{\partial}{\partial N_A} \left[ -\ln(N_A!) + N_A \ln \left( 1 + e^{-\Delta \beta/kT} \right) + N_A \ln \left( VT^{3/2} \right) \right] \right\}
\]

\[
\mu_A = -\frac{1}{\beta} \left\{ -\frac{N_A}{N_A} \ln N_A(-1) + 1 + \ln \left( 1 + e^{-\Delta \beta/kT} \right) + \ln \left( VT^{3/2} \right) \right\}
\]

\[
= -kT \left[ \ln N_A + \ln \left( 1 + e^{-\Delta \beta/kT} \right) + \ln \left( VT^{3/2} \right) \right] = -\frac{1}{\beta} \ln \left[ N_A \left( 1 + e^{-\Delta \beta/kT} \right) \left( VT^{3/2} \right) \right]
\]

\[
\mu_B \text{ is of the same form.}
\]

In eqn \( \mu_A = \mu_B \), solve for \( \frac{N_A}{N_A+N_B} \).
\[ e^{\mu_A} = e^{\mu_B} \]

\[ N_A \left( 1 + e^{-\Delta_B \beta} \right) \left( \frac{1}{V} + \frac{3}{2} \right) = N_B \left( 1 + e^{-\Delta_A \beta} \right) \left( \frac{1}{V} + \frac{3}{2} \right) \]

\[ \frac{N_B}{N_A} = \frac{\left( 1 + e^{-\Delta_A \beta} \right)}{\left( 1 + e^{-\Delta_B \beta} \right)} \]

**solve:**

\[ \frac{N_A}{N_A + N_B} = \frac{N_A}{N_A \left( 1 + \frac{N_B}{N_A} \right)} = \left( 1 + \frac{N_B}{N_A} \right)^{-1} \]

\[ = \left[ 1 + \frac{1 + e^{-\Delta_A \beta}}{1 + e^{-\Delta_B \beta}} \right]^{-1} = \left[ \frac{1 + e^{-\Delta_B \beta} + 1 + e^{-\Delta_A \beta}}{1 + e^{-\Delta_B \beta}} \right]^{-1} \]

\[ = \left[ \frac{1 + e^{-\Delta_B \beta}}{2 + e^{-\Delta_B \beta} + e^{-\Delta_A \beta}} \right] \]
5. A gas of $N$ distinguishable classical non-interacting atoms is held in a neutral atom trap by a potential of the form $V(\mathbf{r}) = a \, r$ where $r = \sqrt{x^2 + y^2 + z^2}$. The gas is in thermal equilibrium at a temperature $T$.

(a) Find the single particle partition function $Z_1$ for a trapped atom. Express your answer in the form $Z_1 = A \, T^\alpha a^{-\eta}$. Find the prefactor $A$ and the exponents $\alpha$ and $\eta$. (3 points)

(b) Find the entropy of the gas in terms of $N$, $k$, and $Z_1(T, a)$. Do not leave any derivatives in your answer. (4 points)

(c) The gas can be cooled if the potential is lowered reversibly (by decreasing $a$) while no heat is allowed to be exchanged with the surroundings, $dQ = 0$. Under these conditions, find $T$ as a function of $a$ and the initial values $T_0$ and $a_0$. (4 points)
9. 

\[ \mathcal{H}_1 = \sum_{n=1}^{\infty} \frac{E_n}{n^2} e^{-\beta E_n} \]

\[ e^{-\beta (E_n^2 / 2m + \omega)} \approx e^{-\beta E_n} e^{-\beta \omega} \]

\[ n_x = \int e^{-\frac{\beta}{2} E_n} d^3x d^3p = \frac{4\pi}{(2\pi \hbar)^3} \int_0^\infty \frac{e^{-\frac{\beta}{2} p^2}}{e^{-\beta \omega}} p^2 dp \int e^{-\beta x^2} dx \]

\[ \mathcal{J}_0 = \frac{1}{4} \sqrt{\frac{\pi}{\nu \beta}} \left( \frac{2m}{\beta} \right) \]

\[ \mathcal{J}_2 = 4\pi \int_0^\infty r^2 e^{-\beta m} dr = 4\pi \frac{2}{(\beta \omega)^{2+1}} = \frac{8\pi}{\beta^2 a^2} \]

so:

\[ A = \frac{4\pi \sqrt{\pi}}{2 \sqrt{\pi} t^6} \frac{\beta}{\nu \beta} \frac{\pi^{1/2} 2^{3/2}}{4} \frac{T^{3/2}}{k^2} \frac{2^{3/2}}{4} \frac{\pi^{1/2}}{t^6} \frac{3}{a^3} \frac{T^3}{k^3} \]

so:

\[ \alpha = \frac{9}{2}, \quad \eta = \frac{3}{2} \]

6. 

\[ \frac{\partial}{\partial t} F = \frac{\partial}{\partial t} \frac{1}{\beta} \ln Z = \frac{1}{\beta} kT \ln \left( \frac{1}{N!} \right) \]

\[ = \frac{\partial}{\partial t} kT \left[ \ln A + \frac{9}{2} \ln T - 3 \ln \alpha \right] \]

\[ = \frac{1}{\beta} kT \left[ \ln A \right] - \frac{9}{2} kT \ln \alpha \]

\[ = kT \left[ \frac{1}{\beta} \ln A - \frac{9}{2} \ln \alpha \right] \]

\[ = \frac{1}{\beta} kT \ln A - \frac{9}{2} kT \ln \alpha \]

\[ = \frac{1}{N} \ln \left( \frac{1}{N!} \right) \]

\[ S = \frac{1}{\beta} \ln \left[ \frac{1}{N!} \left( \frac{1}{N!} \right)^N \right] = \frac{1}{\beta} kT N \ln (\ln Z_1, \ldots, Z_N) \]

\[ = kT N \left[ \frac{9}{2} \ln T + (\ln Z_1, \ldots, Z_N) \right] \]

\[ = kT N \left( \beta \omega + \left( \ln Z_1, \ldots, Z_N \right) \right) \]
for a reversible process $\frac{dS}{dt} = 0$

Note: $S$ changes wrt $A(t)$ and $T(t)$

$\frac{dS}{dt} = \frac{d}{dt}(KN\frac{a}{2} + KN \ln AT^{q/2} a^{-3})$

$0 = \frac{d}{dt} KN\frac{a}{2} + KN \frac{d}{dt}(\ln A + \frac{q}{2} \ln T - 3 \ln a)$

$0 = 0 + KN \left( \frac{q}{2} \frac{dT}{dt} \ln T - 3 \frac{da}{dt} \ln a \right)$

$0 = KN \left( \frac{q}{2} \frac{dT}{dt} - 3 \frac{da}{dt} \right)$

\[
\frac{3}{a} \frac{da}{dT} = \frac{q}{2} \frac{dT}{dt}
\]

\[
\int_{a_0}^a a^{-3} \, da = \frac{q}{2} \int_{T_0}^T T^{-1} \, dT
\]

\[
3 \ln a \bigg|_{a_0}^a = \frac{q}{2} \ln T \bigg|_{T_0}^T
\]

\[
3 \ln a \bigg/ a_0 = \frac{q}{2} \ln \frac{T}{T_0} \div 3 = \frac{q}{6} = \frac{3}{2}
\]

\[
\ln \frac{a}{a_0} = \frac{3}{2} \ln \frac{T}{T_0}
\]

\[
\frac{a}{a_0} = \left( \frac{T}{T_0} \right)^{3/2}
\]

\[
\left( \frac{a}{a_0} \right)^{2/3} = \frac{T}{T_0}
\]

\[
T_0 \left( \frac{a}{a_0} \right)^{2/3} = T
\]
Problem 6 (10 Points):

The partition function for an ideal gas of molecules in a volume $V$ can be written as,

$$Z = \frac{1}{N!}(V\zeta)^N,$$

where $V\zeta$ is the partition function for a single molecule (involving its kinetic energy plus internal energy if it is not monotonic) and $\zeta$ is a function that depends only on the absolute temperature.

When these molecules are condensed to form a liquid, the crudest approximation consists of treating the liquid as if the molecules still formed a gas of molecules moving independently provided that,

1. each molecule is assumed to have a constant potential energy $-\eta$ due to its average interaction with the rest of the molecules.

2. each molecule is assumed to move throughout a volume $N\nu_0$, where $\nu_0$ is the constant volume available per molecule in the liquid phase.

a. With these assumptions and the equation given in the above text, write down the partition function for a liquid consisting of $N_L$ molecules. (2 Points)

b. Using the equation in the above text, write down the chemical potential $\mu_G$ for $N_G$ molecules of the vapor in a volume $V_G$ at temperature $T$. Treat the system as an ideal gas. (1 Points)

c. Write down the chemical potential $\mu_L$ for $N_L$ of the molecules of the liquid at the temperature $T$ using the result from (a.). (1 Points)

d. Using your results in b and c, find an expression relating the vapor pressure to the temperature $T$ where the gas is in equilibrium with the liquid. (2 Points)

e. Use the Clausius-Clayperon equation and the fact that the gas can be considered ideal to show that $P = P_0 e^{-L/RT}$, where $L$ is the latent heat of vaporization per mole. (2 Points)

f. Calculate the molar entropy difference between the gas and liquid in equilibrium at the same temperature and pressure. (2 Points)
\[ Z = \frac{1}{N_L!} \left( N_{V_0} \bar{z} \right)^{N_L} \]

\[ \mu = \frac{\partial F}{\partial N} \quad F = -\frac{1}{\beta} \ln Z \]

so:
\[ \mu_\varepsilon = -\frac{1}{\beta} \frac{\partial}{\partial N_G} \ln \left( \frac{1}{N_G} \left( \frac{V \bar{z}}{N_G} \right)^{N_G} \right) \]
\[ = -\frac{1}{\beta} \sum \left\{ -N_G \ln N_G + N_G + N_G \ln V \bar{z} \right\} \]
\[ = -\frac{1}{\beta} \sum \left\{ -\ln N_G + 1 + \ln V \bar{z} \right\} \]
\[ = -\frac{1}{\beta} \sum \left\{ -\ln N_G + 1 + \ln V \bar{z} \right\} \]
\[ \mu_c = -\frac{1}{\beta} \ln \left( \frac{V \bar{z}}{N_G} \right) = -kT \ln \left( \frac{V \bar{z}}{N_G} \right) \]

\[ \mu_L = -kT \mu_c \left( \frac{N_{V_0} \bar{z}}{N_L} \right) \]

but \( N \) is probably meant to be \( N_L \) (in part 2). If so, then there is an extra term in \( \mu_L \).

\[ \mu_L = -\frac{1}{\beta} \sum \left\{ -\ln N_L + \frac{\partial}{\partial N_L} N_L \ln N_{V_0} \bar{z} \right\} \]
\[ = -\frac{1}{\beta} \sum \left\{ \ln \left( \frac{N_{V_0} \bar{z}}{N_L} \right) + 1 \right\} \]
\[ = -kT \sum \left( \ln \left( \frac{N_{V_0} \bar{z}}{N_L} \right) + 1 \right) \]
\[ \alpha + \text{eqm} \quad \mu_G = \mu_L \]

\[ \mu \left( \frac{V \xi}{N_G} \right) = \mu \left( v_0 \xi \right) + 1 = \mu \left( v_0 \xi \right) + \ln(e) = \mu \left( v_0 \xi e \right) \]

\[ \frac{V \xi}{N_G} = v_0 \xi e = v_0 \xi e^{\beta n + 1} \]

\[ \frac{V}{N_G} = v_0 e^{\beta n + 1} \]

Now find the pressure: \[ P_G = -\frac{\partial F_G}{\partial V} \]

\[ = -\frac{\partial}{\partial V} \left( -\frac{1}{\beta} \ln \left( \frac{1}{N! (V \xi)^N} \right) \right) \]

\[ = kT \frac{1}{\beta} \left[ -N \ln(N! + N \ln(V \xi)) \right] = kT \frac{N}{V \xi} \frac{1}{\beta} = kT \frac{1}{V/N_G} \]

\[ P_G = \frac{kT}{v_0 e^{\beta n + 1}} \]

\( C-C \) equation: \[ \frac{dP}{dT} = \frac{L}{T (RT^2/P)} \]

\( \int_{P_0}^{P} \frac{1}{P} dP = \int_{0}^{T} \frac{L T^{-2}}{R} dT \]

\[ \frac{p}{P_0} = \frac{LT^{-1}}{R} = -\frac{1}{RT} \]

\[ \exp: \quad \frac{P}{P_0} = e^{-L/RRT} \]

\[ P = P_0 e^{-L/RRT} \]

Can use \[ \frac{dP}{dT} = \frac{S_G - S_L}{V_G - V_L} \] from Schröder.
\[ -\frac{\partial F}{\partial T} = -\frac{1}{T} \left( -\beta \ln Z \right) = \frac{\partial}{\partial \beta} \frac{\partial}{\partial T} \beta^{-1} \ln Z \]

\[ = \frac{\partial}{\partial \beta} \left( -k\beta^2 \right) \beta^{-1} \ln Z = -k\beta^2 \frac{\partial}{\partial \beta} \beta^{-1} \ln Z \]

\[ \therefore \text{true but lets use } \frac{1}{T} \]

\[ S_G = -\frac{1}{T} \left( -\beta^{-1} \ln Z \right) = \frac{\partial}{\partial T} kT \ln \frac{1}{N_0^1} \left( V S \right)^{N_0^1} \]

\[ = \frac{\partial}{\partial T} kT \left( -\ln N_0^1 + N_0 \ln V S \right) \]

\[ S_G = k \left( -\ln N_0^1 + N_0 \ln V S \right) + kT \left( \frac{N_0}{V S} \frac{d}{d t} \frac{1}{T} \right) \]

\[ S_L = \frac{\partial}{\partial t} kT \left( -\ln N_0^1 + N_0 \ln V S e^{\frac{N_0}{T \text{e}^{\frac{N_0}{T}}} \text{e}^{\frac{N_0}{T}}} \right) \]

\[ = k \left( \frac{N_0}{V S} \text{e}^{\frac{N_0}{T \text{e}^{\frac{N_0}{T}}} \text{e}^{\frac{N_0}{T}}} \right) \left( N_0 \left[S \left( \frac{N_0}{V S} \text{e}^{\frac{N_0}{T \text{e}^{\frac{N_0}{T}}} \text{e}^{\frac{N_0}{T}}} \right) + \text{e}^{\frac{N_0}{T \text{e}^{\frac{N_0}{T}}} \text{e}^{\frac{N_0}{T}}} \right] \right) \]

Per mole: \( N_K \rightarrow n_K \)

Eqn: \( \frac{V_S}{N_0} = N_0 \text{e}^{\beta \mu + 1} \)

\[ \Delta S = S_G - S_L \]
5. Consider the quantum mechanical linear rotator. It has energy levels

\[ E_J = \frac{\hbar^2}{2I} J (J + 1) \]

where \( I \) is the moment of inertia and \( J \) is the angular momentum quantum number, \( J = 0, 1, 2, \ldots \). Each energy level is \((2J + 1)\)-fold degenerate.

(a) In the low temperature limit \((\hbar^2/2I \gg kT)\) determine approximate expressions for:
   i. The rotation partition function. (2pts)
   ii. The internal energy. (1pt)
   iii. The specific heat. (1pt)

(b) In the high temperature limit \((\hbar^2/2I \ll kT)\) determine approximate expressions for:
   i. The rotation partition function. (2pt)
   ii. The internal energy. (1pt)
   iii. The specific heat. (1pt)

(c) How do the quantum results compare with the equipartition theorem for a classical rotator with two transverse degrees of freedom? (2pts)
\( Z = \sum_{J=0}^{\infty} (2J+1) e^{-\frac{\beta J^2}{2}} = \sum_{J=0}^{\infty} e^{-\beta J^2 / 2} (2J+1) \)

(a) For \( T \to 0, \beta \to \infty \) so \( e^{-\beta \cdot} \) is small so keep only a few terms

\( Z = e^0 + 3 e^{-\frac{\beta_1^2}{2}} + 5 e^{-\frac{\beta_2^2}{2}} + \ldots \)

Using \( N = 1 \) for one rotator, \( Z = \beta \)

(ii) \( \hat{E} = \hat{N} \cdot \beta \)

\( E = -\frac{\beta}{\beta} \ln Z = \frac{\beta}{\beta} \ln (1 + 3 e^{-\beta_1^2 / 2} + \ldots) \)

\( E = -\left(1 + 3 e^{-\beta_1^2 / 2}\right)^{-1} \left(-\frac{\beta_1^2}{2} e^{-\beta_1^2 / 2}\right) \)

(iii) \( C_V = \frac{\beta}{\beta} \frac{\delta E}{\delta \beta} = -k \beta^2 \frac{\delta}{\delta \beta} \left(1 + 3 e^{-\beta_1^2 / 2}\right) \left(\frac{\beta_1^2}{2} e^{-\beta_1^2 / 2}\right) \)

\( = -k \beta^2 \frac{\partial}{\partial \beta} \left(\frac{\beta_1^2}{2} e^{-\beta_1^2 / 2} (1 + 3 e^{-\beta_1^2 / 2}) \right) \)

\( = -8k \beta^2 \frac{\beta_1^2}{2} e^{-\beta_1^2 / 2} \)

\( = -2k \beta^2 \beta_1^2 e^{-\beta_1^2 / 2} \)
\( \beta \) limits from 0 to \( \infty \) where \( \alpha = \frac{\beta^2}{2} \)

\[ T = -\alpha^{-1} e^{-\alpha [x(x+1)]} \bigg|_0^\infty = -\alpha^{-1}(e^{-\infty} - e^0) = -\alpha^{-1}(0 - 1) \]

i) \( T = \alpha^{-1} = \frac{2T}{\beta^2} \)

ii) \[ E = \frac{1}{\beta} \ln \frac{\beta}{2\pi} - \frac{3\beta x^2}{2 T} + \frac{\beta^2}{2}(\frac{x}{T}) - \frac{3T^2}{4}\beta^2 \]  

\[ \text{Classical: } E = \frac{2}{3} N k T = kT \]

b) classical dominates @ \( T \)

Quantum @ \( T \)

\[ C_V = \frac{\partial E}{\partial T} = k \]
6. Consider the "bogon," a spin 5/2 fermion with the charge of an electron but with a dispersion relationship
\[ E = cp^3. \]
where \( p \equiv |\vec{p}| \) Assume that your bogons are confined in a three dimensional sample and are non-interacting.

(a) Working in the grand canonical ensemble, determine the density, \( \rho = \langle N \rangle / V \), as a function of the chemical potential, \( \mu \) (or the fugacity, \( z = e^{\beta \mu} \)), \( T \), and \( V \). (3pts)

(b) What is the bogonic Fermi energy (\( \mu \) at \( T = 0 \)) as a function of their density? (3pts) (Hint: This should not involve any complicated integrals).

(c) Derive a series expansion in \( z \) for the grand canonical free entropy, \( \Xi = \frac{PV}{kT} = \log Z \), where \( Z \) is the grand canonical partition function. (4pts)
Find \( Q \) by summing over states and spins

\[
Q = \sum_{N=0}^{\infty} \sum_{\{\mathbf{n}, \sigma\}} e^{-N\beta\mu - \beta E_i, \sigma} \sum_{\mathbf{n}, \sigma} \rho_{\mathbf{n}, \sigma} \beta (\mu - E_i, \sigma) = \prod_i \left( \sum_{n_i=0}^{\infty} e^{-\beta E_i - \mu} \right)^{2s_i+1} \\
= \prod_i \left( \sum_{n_i=0}^{\infty} e^{-\beta E_i - \mu} \right)^{n_i} \quad n_i = 0,1 \text{ for fermions}
\]

\[
Q = \prod_i \left( 1 + e^{-\beta E_i - \mu} \right)^{n_i} \quad G = -\frac{1}{\beta} \ln Q = -\frac{1}{\beta} \sum \ln \left( 1 + e^{-\beta E_i - \mu} \right)
\]

\[
N = -\frac{\partial G}{\partial \mu} = \sum_i \left( 1 + e^{-\beta E_i - \mu} \right)^{-1} \left( \beta \right) e^{-\beta E_i - \mu}
\]

\[
N = \sum_i \frac{6}{e^{\beta E_i - \mu} + 1} = \sum_i f(E_i)
\]

Now \( \sum \) sums over a really big box and becomes an integral

\[
N = \int \frac{dx \: dp}{(2\pi \hbar)^3} \frac{6}{e^{\beta E} + 1} = \frac{4\pi \sqrt{6}}{(2\pi \hbar)^3} \int_0^{p^2} \frac{p^2}{e^{\beta c p^2 - \mu} + 1} \: dp
\]

\[
p = \frac{\langle N \rangle}{V} = \frac{24\pi}{(2\pi \hbar)^3} \int_0^{p^2} \frac{p^2}{e^{\beta c p^2 - \mu} + 1} \: dp = \frac{3}{\pi^2 \hbar^3} \int_0^{\infty} p^2 f(E) \: dp
\]

(b) at \( T=0 \), \( \mu = E_F \), \( f(E) = \delta(E_F - E) \)

\[
E = c p^2 \quad \frac{dE}{dp} = 2 c p \quad \frac{dE}{3c} = p^2 dp
\]

\[
p = \frac{3}{\pi^2 \hbar^3} \int_0^{\infty} \frac{dE}{3c} \delta(E_F - E) = \frac{3 E_F}{\pi^2 \hbar^3 8c} \quad E_F = p \pi^2 \hbar^2 c
\]
\[ \sum_{i} \ln(1 + e^{-\beta(E_i - m)}) = \sum_{i} \ln(1 + e^{-\beta \frac{E_i}{m}}) \]

Again, \( \sum \) sums over a really big box and becomes an integral:

\[ \sum_{i} \longrightarrow \int_{0}^{\infty} d\rho \left( 1 + \rho e^{-\beta \rho} \right) \rho^2 \ln(1 + \rho e^{-\beta \rho}) \]

Since \( \rho \) is small, \( \ln(1 + \rho x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \)

\[ \sum_{i} \longrightarrow \int_{0}^{\infty} d\rho \left[ \rho e^{-\beta \rho} - \frac{\rho^2 e^{-\beta \rho}}{2} + \frac{\rho^3 e^{-\beta \rho}}{3} - \frac{\rho^4 e^{-\beta \rho}}{4} + \ldots \right] \]
6. A black body may be thought of as a system of harmonic oscillators possessing all possible frequencies—equivalently, it is a system of photons governed by the Bose-Einstein distribution.

(a) Calculate the average energy \( u(\nu) \) of a quantum harmonic oscillator of frequency \( \nu \) at temperature \( T \) where the allowed energies of the oscillator are:

\[
E(n) = h\nu n
\]

and we have ignored the zero-point energy. (3 points)

(b) The number of oscillators per unit phase space is \( 2 d^3q d^3p/h^3 \), where the factor of 2 comes from the two transverse polarization states of the photon. Calculate the total energy of the black body

\[
U = 2 \int \frac{d^3q d^3p}{h^3} u(\nu)
\]

in terms of a single dimensionless integral. This is the famous Planck formula. [Use the relativistic relation between frequency and momentum for photons, \( h\nu = pc \).] (3 points)

(c) Derive the Stefan-Boltzmann law, \( u = \sigma T^4 \), and compute the constant \( \sigma \) using the formula

\[
\int_0^\infty dx \frac{x^{n-1}}{e^x - 1} = \zeta(n) \Gamma(n),
\]

where \( \zeta(n) \) is the Riemann zeta function, and \( \Gamma(n) \) is the gamma function. Your answer will be in terms of mathematical and physical constants. (4 points)
\[ E_{\text{avg}} = \sum_{k} E_k(n) f_{BE}(E_k) \]

\[ = \sum_{k} \frac{E_k(n)}{\beta(E_k(n) - \mu)} = \sum_{n} \frac{h\nu n}{e^{\beta(h\nu - \mu)} - 1} \]

Consider only one oscillator, \( n = 1 \), \( \mu = 0 \) for boson \( E = \frac{h\nu}{e^{\beta h\nu} - 1} \)

Use the Canonical Ensemble for one particle \( Z = \sum_{n=0}^{\infty} e^{-\beta E_n} = \sum_{n=0}^{\infty} e^{-\beta h\nu n} = \sum_{n=0}^{\infty} e^{-\beta h\nu} = \frac{1}{1 - e^{-\beta h\nu}} = Z \)

\[ E = \frac{\partial}{\partial \beta} \ln Z = \frac{\partial}{\partial \beta} \ln(1 - e^{-\beta h\nu}) = (1 - e^{-\beta h\nu})^{-1}(h\nu e^{-\beta h\nu}) \]

\[ E = \frac{h\nu}{(e^{\beta h\nu} - 1)} \text{ same answer } \checkmark \]

\[ U = 2 \int \frac{d^3 q d^3 p}{h^3} \ u(q) = 2 \cdot 4\pi V \int_{0}^{\infty} \frac{p^2 h\nu}{e^{\beta h\nu} - 1} \ dp \]

\[ = \frac{8\pi V}{h^3} \int_{0}^{\infty} \frac{p^3 c}{e^{\beta p c} - 1} \ dp = \frac{8\pi V}{h^3 c^2} \int_{0}^{\infty} \frac{p^3 c^3}{e^{\beta p c} - 1} \ dp \]

Let: \( E = pc \rightarrow \frac{dE}{dp} = c \rightarrow dE = c dp \rightarrow \frac{dE}{c} = dp \)

\[ U = \frac{8\pi V}{h^3 c^3} \int_{0}^{\infty} \frac{E^3}{e^{\beta E} - 1} \ dE \]

Use: \( E = \frac{x}{\beta} \rightarrow \beta x \rightarrow dE = \frac{1}{\beta} dx \)

\[ U = \frac{8\pi V}{h^3 c^3} \int_{0}^{\infty} \frac{x^3}{e^{\beta x} - 1} \ dx \]
\( \int_0^\infty x^{n-1} e^{-x} \, dx = \xi(n) \Pi(n) \)

for \( n = 2 \):
\( \xi(2) \Pi(2) = \frac{\pi^2}{6} (1!) = \frac{\pi^2}{6} \)

so:
\( U = \frac{8\pi V}{h^3 c^3} B^* \frac{\pi^2}{6} \)

\( u = \frac{U}{V} = \frac{8\pi^3}{h^3 c^3} k^4 T^4 \)

so:
\( a = \frac{3k^4 \pi^3}{4h^3 c^3} \)
Problem 5 (10 Points):

Consider a one dimensional ideal gas of electrons as a model for the conduction electrons in a one dimensional wire.

a. Determine the density of states $g(E)$ for the one dimensional non-interacting electron system confined to a length, $L$. (3 Points)

b. What is the Fermi energy for this system? (2 Points)

c. What is the root mean square velocity of the electrons at $T = 0^\circ K$? (3 Points)

d. What is the entropy of the electrons at $T = 0^\circ K$? Justify your answer. (2 Points)
\[ g(E) = \int dk \, D_k \delta(E - E_k) \]
\[ = \int \frac{dk}{2\pi} \frac{2L}{2\pi} \delta(E - E_k) \]
\[ = \int \frac{L}{\pi} \delta(E - E_k) \sqrt{\frac{m}{2E_k \hbar^2}} \frac{1}{2} dE_k \]
\[ = \frac{L}{\pi \hbar^2} \sqrt{\frac{m}{2E}} = \frac{mL}{\pi \hbar^2 \sqrt{2mE}} \]

As in Schroed eq 7.52

\[ N = \int_0^{E_f} g(E) dE @ T = 0 \text{K} \]
\[ = \int_0^{E_f} \frac{mL}{\pi \hbar^2} (2mE)^{-\frac{1}{2}} dE = \frac{\sqrt{mL}}{\pi \hbar^2} \sqrt{\pi m} \frac{1}{2} \int_0^{E_f} E^{-\frac{1}{2}} dE \]
\[ = \frac{L\sqrt{m}}{\pi \hbar^2 \sqrt{2}} \frac{E_f^{1/2}}{1/2} \bigg|_0^{E_f} = \frac{L\sqrt{m}}{\pi \hbar^2 \sqrt{2}} \frac{2}{E_f^{1/2}} \bigg|_0^{E_f} \]
\[ = \frac{N \sqrt{2 \pi \hbar^2}}{2L \sqrt{m}} \]
\[ E_f = \frac{1}{2m} \left( \frac{N \pi \hbar^2}{L} \right)^2 \]

At \( T = 0 \text{K} \), \( \sqrt{V_{avg}^2} \approx V_{rms} \)

\[ E_T = N E = \int_0^{E_f} g(E) E dE = \frac{1}{2} m v^2 \]
\[ = \int_0^{E_f} \frac{mL}{\pi \hbar^2 \sqrt{2mE}} E^{1/2} dE = \frac{mL}{\pi \hbar^2 \sqrt{2m}} \frac{E_f^{3/2}}{3/2} \bigg|_0^{E_f} \]
\[ E_T = \frac{mL}{\pi \hbar^2 \sqrt{2m}} \frac{2}{3} \left( \frac{1}{2m} \left( \frac{N \pi \hbar^2}{L} \right)^2 \right)^{3/2} \]

(Using result for \( E_f \) above)

\[ V^2 = \frac{2}{m} \frac{mL^2}{\pi \hbar^2 \sqrt{2m} \left( \frac{1}{3/2} \frac{N \pi \hbar^2}{3L^2} \right)^{3/2}} \]

\[ \approx \frac{8N^3 \pi^{3/2} \frac{2L^2}{2m \pi^{3/4} \frac{3}{3L^2}}}{\frac{3}{2} \frac{N \pi \hbar^2}{L^2}} \]

\[ \hat{S} = 0 \] 3rd law of thermodynamics. Fermions are in the most ordered state possible (ground state)
\[ E_T = \int g(E)E f(E) dE \]

but \( f(E) = 1 \) for \( T = 0 \)!

So answer is same

\[
V^2 = \frac{2^{1/4} N^3 \pi^{1/4} L^2}{2^{1/4} m^{3/4} \pi L^2} \quad \Rightarrow \quad V = \sqrt{\frac{2^{1/4} N^3}{m^{3/4} \pi} \frac{\pi^{1/4} L^2}{L}}
\]

\[
\sqrt{\langle V^2 \rangle} = V_{\text{rms}}
\]

\[
\langle V \rangle^2 = \langle V \rangle^2_{\text{avg}}
\]

\[
\langle V^2 \rangle = \frac{L}{\pi} \int_{-k_f}^{k_f} V^2 dk
\]
6. Consider a free, non-interacting spin zero Bose gas in two dimensions. The energy of each particle is given by:

\[ \mathcal{E}(\vec{k}) = \frac{\hbar^2 k^2}{2m} \]

where \( m \) is the mass of the boson. Assume your system is confined to a square region of length \( L \) on a side.

(a) Write down an expression for the grand canonical free energy \( \mathcal{G}(T, V, \mu) \) as a sum over \( \vec{k} \) states. Do not evaluate the sum. (1 pt.)

(b) Calculate the number of particles in the system as a function of \( T, V \) and \( \mu \). (3 pts.)

(c) Analyze your expression for \( N(T, V, \mu) \) in the limit \( T \to 0 \). What does it imply about the possibility of a Bose-Einstein transition in this system? (3 pts.)

(d) Prove that the pressure is equal to the energy density, so that \( PV = U \). (Hint: you do not have to do any sums over states - you need only prove that this holds using analytic expressions for \( P \) and \( U \) in this particular system). (3 pts.)
\[ Q = \sum_{N=0}^{\infty} e^{-\beta E_N} \sum_{\text{states}} \frac{\beta}{\exp(\beta E_N) - 1} \]

\[ = \sum_{N=0}^{\infty} \sum_{\eta_i} e^{-\beta E_i} \frac{\beta}{\exp(\beta E_i) - 1} = \prod_{n=0}^{\infty} e^{-\eta_1 \beta (E_i - \mu)} \]

\[ = \prod_{k} \left( e^{-\beta \left( E_k - \mu \right)} \right)^{\eta_k} \]

Bose: \[ \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \forall \quad Q = \prod_{k} \frac{1}{1-e^{-\beta (E_k - \mu)}} \]

\[ \eta_k = 0, 1 \quad \forall \quad Q = \prod_{k} \left( 1 + e^{-\beta (E_k - \mu)} \right) \]

\[ N = -\frac{\partial \mathcal{Z}}{\partial \mu} = \frac{2}{\beta} \ln \left( 1 - e^{-\beta (E_k - \mu)} \right)^{-1} \]

\[ = \frac{1}{\beta} \ln \left( 1 - e^{-\beta (E_k - \mu)} \right)^{-1} \]

\[ = -\frac{1}{\beta} \ln \left( 1 - e^{-\beta (E_k - \mu)} \right) \left( -e^{-\beta (E_k - \mu)} \right)^{-1} \]

\[ = + \sum_{k} \frac{1}{e^{\beta (E_k - \mu)} - 1} \]

Consider a big box so

\[ \mathcal{A} = \int d^3k = \int \frac{2\pi}{(2\pi)^2} kdkd\theta \]

\[ N = \frac{A m k T}{2 \pi \hbar^2} \ln \left( 1 + \frac{1}{1 - e^{\mu/kT}} \right) \]

Using \( \eta = 1 - e^{-\beta \left( \frac{\hbar^2 k^2}{2m} - \mu \right)} \)

Set \( \mu = 0 \), \( T = 0 \) see if \( N \) diverges, or converges (condensate)

\[ N = 0 \ln \left( \frac{1}{1 - e^{-\beta}} \right) = 0 \ln \left( \frac{1}{1 - 1} \right) = 0 \ln \left( \frac{1}{0} \right) \]

but put all terms together to actually compare

\[ N = \frac{A m}{2 \pi \hbar^2} \ln e^{kT} \ln \left( \frac{1}{1 - e^{\mu/kT}} \right) \]

Can't really solve using this method.
but if $\mu$ can be solved for there is no condensate

\[ \frac{N 2\pi \hbar^2}{k} = \frac{1}{1 - e^{-\mu/kT}} \Rightarrow \mu (1 - e^{\mu/kT}) kT = \mu \]

\[ 1 - e^{\mu/kT} = \frac{1}{e^{\mu/kT}} \]

\[ 1 - \frac{1}{e^{\mu/kT}} = e^{-\mu/kT} \]

so no condensate
5. Consider a gas of \( N \) non-interacting \textbf{one dimensional} diatomic molecules enclosed in a box of "volume" \( L \) (actually, just a length) at temperature \( T \).

(a) The classical energy for a single molecule is:

\[
E(p_1, p_2, x_1, x_2) = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}K(x_1 - x_2)^2
\]

where \( p_1 \) and \( p_2 \) are the classical momenta of the atoms in one diatomic molecule, \( x_1 \) and \( x_2 \) are their classical positions, and \( K \) is the spring constant. Calculate the specific heat for the gas. (You should assume that \( KL^2/2 >> k_B T \), where \( k_B \) is Boltzmann's constant.) (4 points).

(b) In the quantum limit the energy levels of the molecule are discrete. In a semiclassical approach we can write the energy of one molecule as:

\[
E(P, n) = \frac{P^2}{4m} + \hbar\omega(n + \frac{1}{2})
\]

where \( P \) is the momentum of the diatomic molecule (of mass \( 2m \)), and \( \omega \) is the natural frequency of the oscillator, and \( n \) is a non-negative integer \( (n \geq 0) \). Calculate the specific heat. (4 points).

(c) Calculate the high and low temperature limits of your result in (b), and explain how they relate to the result of (a). (2 points)
\[ Z = \sum_n e^{-\beta E_n} = \int e^{-\beta \left( \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2} k (x_1 - x_2)^2 \right)} \frac{dx_1 dp_1}{2\pi}\]

but how to integrate? and what limits?

for \( \text{Dirac M.} \quad E_{\text{cm}} = \frac{p_1^2}{2m}, \quad E_{\text{sep}} = \frac{p_2^2}{2m} + \frac{k}{2} x^2 \)

cm coords are bounded by box \( 0 \rightarrow L \)

sep coords are "free" (they will never stretch out and hit the walls)

\[ Z = Z_{\text{cm}} Z_{\text{free}} = \frac{1}{h^2} \int e^{-\beta E_{\text{cm}}} d\mathbf{x}_{\text{cm}} \frac{1}{h^2} \int e^{-\beta E_{\text{sep}}} dx_5 dp_5 \]

\[ = \frac{1}{h^2} \int_0^L dx_{\text{cm}} \int_{-\infty}^{\infty} e^{-\beta \frac{p_{\text{cm}}^2}{4m}} dp_{\text{cm}} \int_{-\infty}^{\infty} e^{-\beta \frac{p_5^2}{m}} dp_5 \int_{-\infty}^{\infty} e^{-\frac{\beta k}{2} x^2} dx_5 \]

\[ = \frac{L}{h^2} \sqrt{\frac{4\pi}{\beta}} \sqrt{\frac{m}{\beta}} \sqrt{\frac{\pi}{\beta}} \frac{2}{\sqrt{\pi}} \frac{m L^2}{\hbar^2} \left( \frac{m L^2}{\beta} \right)^{3/2} \]

\[ C = \frac{\partial E}{\partial T} = \frac{\partial}{\partial T} \left( \frac{\partial^2 E}{\partial \beta^2} \right) = \frac{\partial^3 E}{\partial \beta^3} = -k \beta^2 \frac{\partial}{\partial \beta} \frac{\partial}{\partial \beta} \ln Z \]

\[ = k \beta^2 \frac{\partial}{\partial \beta} \ln \left( \frac{1}{N!} \frac{e^N}{N^N} \right) = k \beta^2 \frac{\partial^2}{\partial \beta^2} \left( -\ln N + N \ln \frac{1}{N} \right) \]

\[ = k \beta^2 \frac{\partial}{\partial \beta} \left( \ln \left( \frac{N}{2} \frac{1}{\beta} \right) \right) = k \beta^2 \frac{\partial}{\partial \beta} \left( -N \frac{\beta}{2} \frac{1}{\beta} \right) \]

\[ = -k \beta \frac{N}{2} \beta^2 (-1) = k N \frac{3}{2} \frac{1}{\beta^3} = \frac{3}{2} \frac{N}{\beta} k \]
\[ Z = \frac{1}{N!} E^n \]

\[ C = \frac{\partial E}{\partial T} = \frac{\partial E}{\partial T} = -k\beta^2 \frac{\partial}{\partial \beta} \left( -\frac{1}{\beta} \ln Z \right) = k\beta^2 \frac{\partial}{\partial \beta} \ln \left( \frac{1}{N!} \right) \]

\[ C = k\beta^2 \left( -\ln N! + N\ln \frac{1}{N} \right) \]

\[ C = k\beta^2 \left( -\ln N! -\beta^2 \omega^2 \frac{1}{2} - \ln \left( 1 - e^{-\beta \omega} \right) + \frac{1}{2\pi^2} \sqrt{4\pi m} \right) \]

\[ C = k\beta^2 \left( -\frac{1}{2} - \left( 1 - e^{-\beta \omega} \right)^{-1} \right) \left( \frac{1}{\beta \omega} e^{-\beta \omega} \right) + \beta \left( -\beta^2 \right) \]

\[ C = k\beta^2 \left( -\omega e^{-\beta \omega} -1 \right) \left( e^{\beta \omega} -1 \right)^{-2} \left( \frac{1}{\beta \omega} e^{\beta \omega} \right) - \left( -1 \right) \beta^2 \]

\[ C = Nk\beta^2 \left( \frac{\omega^2 e^{\beta \omega}}{e^{\beta \omega} -1} + 2 - 2 e^{\beta \omega} + \frac{1}{\beta^2} \right) \]

\[ C = \frac{Nk\beta^2 \omega^2 e^{\beta \omega}}{e^{\beta \omega} -1} + \frac{Nk\beta^2}{\beta^2} \]
\( T \to 0, \beta \to \infty, e^{\beta T_0} \gg 1 \)

i) \( C \underset{T \to 0}{=} \frac{Nk \beta T^2 \omega_0^2 e^{\beta T}}{e^{\beta T_0}} + NK = NK + NK \beta^2 T^2 \omega^2 e^{-\beta T_0} \quad \text{Q.M. dominates} \)

ii) \( T \to \infty, \beta \to 0, e^{\beta T_0} \approx 1 + \beta T_0 \)

\[ C \underset{T \to \infty}{=} \frac{Nk \beta^2 T^2 \omega^2 (1 + \beta T_0)}{(1 + \beta T_0 - 1)^2} + NK \]

\[ \approx NK/\beta^2 T_0 + NK + NK = NK(\beta T_0 + 2) \quad \text{classical effects dominate} \]
6. Fermions:

(a) Show that for any non-interacting spin 1/2 fermionic system with chemical potential \( \mu \), the probability of occupying a single particle state with energy \( \mu + \delta \) is the same as finding a state vacant at an energy \( \mu - \delta \). (2 points)

(b) Consider non-interacting fermions that come in two types of energy states:

\[
E_{\pm}(\vec{k}) = \pm \sqrt{m^2c^4 + \hbar^2k^2c^2}
\]

At zero temperature all the states with negative energy (all states with energy \( E_- (\vec{k}) \)) are occupied\(^1\) and all positive energy states are empty, and that \( \mu (T = 0) = 0 \). Show that the result of part (a) above means that the chemical potential must remain at zero for all temperatures if particle number is to be conserved. (2 points)

(c) Using the results of (a) and (b) above, show that the average excitation energy, the change in the energy of the system from it’s energy at \( T = 0 \) in three dimensions is given by:

\[
\Delta E \equiv E(T) - E(0) = 4V \int \frac{d\vec{k}}{(2\pi)^3} E_+(\vec{k}) \frac{1}{1 + e^{\beta E_+(\vec{k})}}
\]

(2 points)

(d) Evaluate the integral above for massless \( (m = 0) \) particles. (2 points)

(e) Calculate the heat capacity of such particles. (2 points)

---

\(^1\)Technically this means the total energy of the system diverges. If this bothers you, you can assume some large cut-off to the wavevectors, \( \hbar k_{\text{max}} c >> kT \), which will have no effect on your final answers.
Distribution function for fermions:
\[ f(E) = \frac{1}{e^{\beta(E - \mu)} + 1} = \bar{n}(e) \]

This is also the average number of particles having energy \( e \).

Probability of occupying a single particle state
\[ P(E) = \frac{\bar{n}}{N} \]
\( \# \) of particles \( \text{found} \) in a single state \( N = 1 \) for fermions.

So \( P(E) = f(E) \)

- **Filled**
  \[ P(1) = \frac{1}{e^{\beta(\mu + \sigma - \mu)} + 1} = \frac{1}{e^{\beta \sigma} + 1} \]

- **Vacant**
  \[ P(0) = \frac{1}{e^{\beta(\mu - \sigma) - \mu} + 1} = 1 - f(\mu - \sigma) = 1 - \frac{1}{e^{\beta(\mu - \sigma - \mu)} + 1} \]

  \[ = 1 - \frac{1}{e^{-\beta \delta} + 1} = \frac{e^{-\beta \delta} + 1 - 1}{e^{-\beta \delta} + 1} = \frac{1}{1 + e^{\beta \delta}} \]

\( \# \) of particles \( \text{in} \) \( E_+ \) = \( \# \) of particles \( \text{not in} \) \( E_- \)
\[ n(P(E_+)) = n(1 - P(E_-)) \]

\[ \frac{1}{1 + e^{\beta(E_+ - \mu)}} = 1 - \frac{1}{1 + e^{\beta(E_- - \mu)}} \]

\[ = \frac{1 + e^{\beta(E_+ - \mu)} - 1}{1 + e^{\beta(E_- - \mu)}} \]

\[ \frac{1}{1 + e^{\beta(E_+ - \mu)}} = \frac{1}{1 + e^{\beta(E_- - \mu)}} = \frac{1}{1 + e^{\beta(E_+ + \mu)}} \]

For this to be true then \( \mu = 0 \) for all \( T \).
Average energy \[ \langle E \rangle = \int d^3x d^3k \ g(k) f(k) \langle E \rangle , \]
where \[ g(k) = \frac{1}{(2\pi)^3} \]

\[ \Delta E = \langle E(\tau) \rangle - \langle E(0) \rangle \]

\[ = \int d^3x d^3k \ \frac{2}{(2\pi)^3} \left( P(E_+) E_+ + P(E_-) E_- \right) - \int d^3x d^3k \ \frac{1}{(2\pi)^3} \ P(E_-) E_- \]

\[ - (1 - P(E_+) E_+) \]

\[ = \frac{2}{(2\pi)^3} \int d^3x d^3k \left( P(E_+) E_+ - E_+ + P(E_+) E_+ + E_+ \right) \]

\[ = \frac{4V}{(2\pi)^3} \int d^k P(E_+) E_+ \]

\[ \frac{1}{e^{\beta E_+} + 1} \]

for \[ m = 0, \ E = \hbar k \]

\[ \Delta E = \frac{4V}{(2\pi)^3} \int \frac{1}{4\pi^2 k c (1 + e^{\beta E_+})} \] 

\[ \Delta E = \frac{2V t c}{\pi^2 (\beta^2 c)^4} \int_0^{\infty} u^3 (1 + e^u)^{-1} du \]

\[ u = \beta^2 c k \]
\[ du = \beta^2 c \]
\[ \frac{du}{dk} = \beta^2 c \]

\[ = \frac{\pi V}{6 \beta^2 c^2} \]

\[ \frac{\pi^2}{12} \]

For integral table

\[ C = \frac{\partial E}{\partial \tau} \]
\[ \frac{\partial}{\partial \tau} \frac{\pi V}{6 h^2 c^2} k^2 T^3 = \frac{3\pi V k^2 T^2}{2 \hbar^2 c^2} = \frac{\pi V k^2 T^2}{2 \hbar^2 c^2} \]
6. Fermions:

(a) Show that for any non-interacting spin 1/2 fermionic system with chemical potential \( \mu \), the probability of occupying a single particle state with energy \( \mu + \delta \) is the same as finding a state vacant at an energy \( \mu - \delta \). (2 points)

(b) Consider non-interacting fermions that come in two types of energy states:

\[
E_{\pm}(\vec{k}) = \pm \sqrt{m^2 c^4 + \hbar^2 k^2 c^2}
\]

At zero temperature all the states with negative energy (all states with energy \( E_{-}(\vec{k}) \)) are occupied\(^1\) and all positive energy states are empty, and that \( \mu(T = 0) = 0 \). Show that the result of part (a) above means that the chemical potential must remain at zero for all temperatures if particle number is to be conserved. (2 points)

(c) Using the results of (a) and (b) above, show that the average excitation energy, the change in the energy of the system from its energy at \( T = 0 \) in three dimensions is given by:

\[
\Delta E \equiv E(T) - E(0) = 4V \int \frac{d\vec{k}}{(2\pi)^3} E_{+}(\vec{k}) \frac{1}{1 + e^{\beta E_{+}(\vec{k})}}
\]

(2 points)

(d) Evaluate the integral above for massless (\( m = 0 \)) particles. (2 points)

(e) Calculate the heat capacity of such particles. (2 points)

---

\(^1\)Technically this means the total energy of the system diverges. If this bothers you, you can assume some large cut-off to the wavevectors, \( \hbar k_{\text{max}} c >> kT \), which will have no effect on your final answers.
b) The non-interacting fermions have energy

\[ E_+ = + \sqrt{m^2c^4 + \hbar^2k^2c^2} \]
\[ E_- = - \sqrt{m^2c^4 + \hbar^2k^2c^2} = -E_+ \]

Now, using part a), we know, the probability

of occupying a single particle state \( E_+ \) is the

same as finding \( E_- \) state vacant. So, if \( n \) is the no.

of particles in \( E_+ \) then \( n \) is also the no. of particles not in \( E_- \)

\[ n \ P(E_+) = n(1 - P(E_-)) \]

\[ \Rightarrow \frac{n}{1 + e^{(E_+ - \mu)/kT}} = n\left(1 - \frac{1}{1 + e^{(E_- - \mu)/kT}}\right) \]

\[ \Rightarrow \frac{1}{1 + e^{(E_+ - \mu)/kT}} = 1 - \frac{1}{1 + e^{-(E_+ - \mu)/kT}} \]

\[ \Rightarrow \frac{1}{1 + e^{(E_- - \mu)/kT}} = \frac{- (E_+ - \mu)/kT}{1 + e^{-(E_+ - \mu)/kT}} \]

\[ \Rightarrow \frac{1}{1 + e^{(E_- - \mu)/kT}} = \frac{e^{-(E_+ - \mu)/kT}}{1 + e^{-(E_+ - \mu)/kT}} \]

\[ \Rightarrow \mu = 0 \]
\[ E_+(k) = \frac{\hbar k}{1 + e^{\beta k/c}} \]

Thus,

\[
\Delta E = \frac{4\pi\hbar c}{(2\pi)^3} \int d^3k \frac{\hbar k}{1 + e^{\beta k/c}} \sin^2(\theta_k) d\phi_k
\]

\[
= \frac{4\pi\hbar c}{(2\pi)^3} \int_0^\infty dk \frac{k^3}{1 + e^{\beta k/c}}
\]

\[
= \frac{2\pi\hbar c V}{\pi} \int_0^\infty du \frac{u^3}{1 + e^u} = \frac{2\pi^2}{12} \text{ using Schuam's}
\]

\[
= \frac{\pi V}{6 \beta^3 \hbar c^2}
\]
6. Consider a set \((N \gg 1)\) of spinless bosons confined in a harmonic oscillator potential. The characteristic frequency of the harmonic potential is \(\omega_0\), and \(\hbar \omega_0 \ll kT\), where \(T\) is the temperature and \(k\) is Boltzmann’s constant.

(a) Assuming the system is one dimensional, so that the energy of the system is given by \(E = \hbar \omega_0 (n + 1/2)\), calculate \(N(T, V, \mu)\), in the above limit, where \(\mu\) is the chemical potential. (3 points)

(b) Show that there is no Bose-Einstein transition for this system in 1D. (1 point)

(c) Assuming the system is two dimensional, calculate \(N(T, V, \mu)\), again in the limit \(\hbar \omega_0 \ll kT\). (3 points)

(d) Show that there is a Bose-Einstein transition and calculate the critical temperature as a function of the number of particles. (Do not simply quote a result.) (3 points)
Sum over all possible energies $\sum_{n} n \epsilon_{n; i}, \epsilon_{n; i} = \text{tanh} (\frac{i + \frac{1}{2}}{\beta})$

\[ Q = \lim_{N \to \infty} \prod_{n=0}^{N} \frac{1}{1 - e^{-\beta(E_{i} - \mu)}} = \prod_{n=0}^{\infty} \frac{1}{1 - e^{-\beta(E_{i} - \mu)}} \]

\[ N = -\frac{\partial Q}{\partial \mu} = \frac{1}{\beta} \sum_{i} \frac{1}{e^{\beta(E_{i} - \mu)}} \left(1 - e^{-\beta(E_{i} - \mu)}\right) \]

Turn the sum into an integral over $n$ since $t_{i}w_{o} < kT$

\[ N \approx \frac{1}{e^{\beta(t_{i}w_{o} - \frac{1}{2}\mu) - \mu}} \ln \left(1 - e^{-\beta(t_{i}w_{o} - \mu)}\right) \]

\[ N = \frac{1}{\beta t_{i}w_{o}} \ln \left(1 - e^{-\beta(t_{i}w_{o} - \mu)}\right) \]

\[ e^{-N\beta t_{i}w_{o}} = \left(1 - e^{-\beta(t_{i}w_{o} - \mu)}\right) \]

\[ -\beta(t_{i}w_{o} - \mu) = \ln(1 - e^{-N\beta t_{i}w_{o}}) \]

\[ \mu = \ln \left(1 - e^{-N\beta t_{i}w_{o}}\right) \right|_{\beta t_{i}w_{o} < 0} \mu \text{ is solvable} \]

\[ \beta t_{i}w_{o} > 0 \text{ no condensate} \]

...should never be BEC in 1D anyway.
\[ N = \sum_{n_x, n_y} \frac{1}{\beta (\omega (n_x + n_y + 1) - \mu)} \]

\[ N = \frac{N+1}{N} \frac{1}{e^{\beta \omega (N+1) - \mu}} \]

\[ = \int_0^\infty \frac{N+1}{e^{\beta (\omega (N+1) - \mu)}} \frac{dn}{n} \]

\[ = \int_0^\infty \frac{u}{\beta^2 \omega u - \beta \mu} du \]

\[ = \frac{1}{(\beta \omega)^2} \int_0^\infty \frac{x \, dx}{e^{x \omega / \beta} - 1} \]

**Bohr integral**

\[ \int_0^\infty \frac{x^{\gamma - 1}}{e^x - 1} \, dx = \Gamma(\gamma) \]

\[ \Gamma(2) = (2-1)! = 1! = 1 \]

\[ g_2(z)(z^2) \]

\[ \frac{1}{e^{\beta \mu}} \]

\[ e^{-\beta \mu} = \frac{1}{e^{\beta \mu}} = \frac{1}{z} = z^{-1} \]

When we have grounded state b/c it does not contribute.

\[ N = \frac{g_2(z)}{(\beta \omega)^2} \]

\[ \frac{N}{\omega} \rightarrow \frac{g_2(e^0)}{(\frac{1}{e^{kT_c}})^2} = \frac{g_2(1)}{\infty} = 0 \quad \text{condensate!} \]

To find \( T_c \), set \( \mu = 0 \)

\[ N = \frac{g_2(e^0)}{(\frac{1}{kT_c})^2} \]

\[ N^{\frac{1}{2}} \omega^2 = g_2(1) k^2 T_c \]

\[ \frac{N^{\frac{1}{2}} \omega^2}{g_2(1) k^2} = T_c \]
6. Consider a fictitious spin 5/2 fermion with the charge of an electron but with a dispersion relationship

\[ E = v_0 p. \]

where \( p \equiv |\vec{p}| \). We will call this particle the “offon.” Assume that your offons are confined in a three dimensional sample and are non-interacting. We will work in the Grand Canonical Ensemble.

(a) Determine the density, \( \rho = \langle N \rangle / V \), as a function of the chemical potential \( \mu \) (or the fugacity, \( z \equiv e^{\beta \mu} \)), \( T \), and \( V \). (3 points)

(b) What is the offonic Fermi energy (\( \mu \) at \( T = 0 \)) as a function of their density? (Hint: This should not involve any complicated integrals). (3 points)

(c) Derive a series expansion in \( z \) for the grand canonical free entropy, \( \Xi = \frac{PV}{kT} = \log Z \), where \( Z \) is the grand canonical partition function. (4 points)
\( S = \frac{d}{d} \) \( E = \nu \rho \)

(1) Find \( Q \) for one particle

\[ Q = \sum_{N=0}^{\infty} \frac{\beta}{\nu} \left( \frac{\beta}{\nu} \right)^{\frac{\nu}{2}} e^{-\beta \nu} \sum_{n_i} e^{-\beta (E_i - \mu)} \]

\[ Q = \sum_{N=0}^{\infty} \frac{\beta}{\nu} \left( \frac{\beta}{\nu} \right)^{\frac{\nu}{2}} e^{-\beta (E_i - \mu)} \sum_{n_i} e^{-\beta (E_i - \mu)} \]

\[ Q = \prod_{i} \left( 1 + e^{-\beta (E_i - \mu)} \right) \]

\[ N = \frac{\beta}{\nu} \sum_{i} \left( 1 + e^{-\beta (E_i - \mu)} \right)^{-1} \left( e^{-\beta (E_i - \mu)} \right) \]

\[ N \to \int \sum_{i} \frac{\beta}{\nu} \left( 1 + e^{-\beta (E_i - \mu)} \right)^{-1} \left( e^{-\beta (E_i - \mu)} \right) dE_i \]

\[ \rho = \frac{N}{V} = \frac{\beta}{\nu^2 \hbar^3} \int \frac{E^2 dE}{1 + e^{\beta (E - \mu)}} \]

\[ P(E_i) = \delta (E_i - \mu) \]

\[ \int \frac{E^2 dE}{\mu^2 \nu^2} = \frac{3 \mu^2}{\hbar^2 \nu^2} \]

\[ \frac{\sqrt{\rho \pi^2 \hbar^2 \nu^2}}{1 \over 3} = E_f = \mu \]
\[ \Xi' = \ln \prod_i (1 + e^{-\beta(E_i - \mu)})^6 \]

\[ = 6 \sum_i \ln (1 + e^{-\beta(E_i - \mu)}) \]

\[ = \frac{6 \cdot 4 \pi N}{(2\pi \hbar)^3} \int_0^\infty \rho^2 d\rho \ln (1 + e^{-\beta(\nu_0 \rho - \mu)}) \]

\[ \ln \left(1 + e^{-\beta \frac{\nu_0 \rho}{\hbar}} \right) \]

\[ \ln \left(1 + e^{-\beta \nu_0 \rho} \right) \]

\[ \frac{2 e^{-\beta \nu_0 \rho}}{2} - \frac{3 e^{-3 \beta \nu_0 \rho}}{3} - \frac{4 e^{-4 \beta \nu_0 \rho}}{4} + \ldots \]
Problem 5 (10 Points):

The distribution function for an ideal Bose gas is given by,

\[ f(\vec{x}, \vec{p}) = g \left[ e^{\left(\frac{\epsilon - \mu}{kT}\right)} - 1 \right]^{-1} \]

a. Define all the quantities found in \( f(\vec{x}, \vec{p}) \). (1 Points)

b. What is the value of \( g \) for photons? (1 Points)

c. What is the meaning of the distribution function? Sketch the distribution as a function of energy. Make sure to label your sketch with the parameters. (1 Points)

d. For photons to be in thermal equilibrium there must be at least a small amount of matter present, since the interaction between photons is negligible. What processes bring the photons into equilibrium with the matter? (1 Points)

e. Use the information in part d. and the definition of chemical potential, \( \mu = \frac{\partial F}{\partial N} \bigg|_{T,V} \) to explain why the chemical potential of photons must be zero. (2 Points)

f. Find the mean energy density of a photon gas in thermal equilibrium at temperature \( T \). (4 Points)
4. degeneracy of each state \((2S+1)\) if spin,\n\[ \varepsilon = \text{energy}, \quad K = \text{Boltzmann's const.} \]
\[ \mu = \text{chemical potential} \]
\[ T = \text{temperature} \]

5. \(2S+1 \neq 0\) but photons have no spin, and no +1 \(s\) so: \(g_s = 2\)

6. \(\frac{f_B(\varepsilon)}{\varepsilon} - \frac{f_F(\varepsilon)}{\varepsilon}\)

7. Photoelectric Effect, Compton scattering—these don't conserve \(N\).
Blackbody radiation.

8. Matter can absorb and release photons \(\varepsilon \leq \varepsilon + \varepsilon\)
At equilibrium \(\mu_i = \mu_f\) so \(\mu_e = \mu_e + \mu_s\)
for this to be true then \(\mu_s\) must be zero.
Also at eqm \(F\) is minimized even if the number of photons changes.
so: \(\frac{\partial F}{\partial N} = 0\) (which \(= \mu\))

9. \[
\langle E \rangle = \int E g_F(\varepsilon) d^3E = \frac{2\sqrt{\frac{\pi}{3}}} {\hbar^3} \int \frac{pc}{c^3} \frac{p^2 dp}{e^\beta(c-pc-\mu) - 1} \]
\[
\mu_s = 0
\]

10. \[
\langle E \rangle = \int E g_F(\varepsilon) d^3E = \frac{2\sqrt{\frac{\pi}{3}}} {\hbar^3} \int \frac{E^2} {c^3} \frac{E dE}{e^\beta E - 1}
\]
\[
\langle u \rangle = \text{integrand of } \langle E \rangle \]
\[
\langle u \rangle = \frac{8\sqrt{\pi}} {\hbar^3} \frac{1} {C^3} \frac{E^3} {e^{\beta E} - 1}
\]

Energy density: \(\frac{\langle u \rangle} {V}\)

Using \(x = \beta E\) as integration

\(\varepsilon\) gives \(E \sim T^4\), as expected for bb radiation.
**Problem 6 (10 Points):**

A sample consists of N independent electric dipoles. Each dipole has two possible quantum states with energies $\pm \mu E$ where $E$ is the magnitude of an externally applied electric field. The lower energy state has dipole moment $\mu$ and the higher energy state has dipole moment $-\mu$.

a. Find the total electric dipole moment of the sample in an electric field $E$ at temperature $T$. (2 Points)

b. What is the entropy of the sample? (2 Points)

c. Without using your result in b. explain physically what the entropy should be in the limits of $E \to 0$ and $E \to \infty$. (2 Points)

Entropy versus temperature curves for two values of electric field are shown below. Imagine that the sample is initially at state $A$, with temperature $T_1$ and field $E_1$.

![Entropy vs Temperature Graph]

d. How much heat must be extracted from the sample to move it from state $A$ to state $B$, maintaining its temperature at $T_1$ while the field is raised from $E_1$ to $E_2$? (2 Points)

e. Once the sample is in state $B$, it is thermally isolated and the field is slowly reduced from $E_2$ to $E_1$, bringing the system from state $B$ to state $C$. What is the temperature of the sample once it reaches state $C$ in terms of the other variables given in the problem? (2 Points)
Problem 6 (10 Points):

Consider a white dwarf star that is composed of fully ionized $^{12}\text{C}$ and $^{16}\text{O}$ (a neutral plasma). The particle density of the star is uniform, and the electrons must be treated relativistically, $E=pc$.

a. Derive a relation between the Fermi energy of the electrons and the electron density. (2 Points)

b. Derive a relation between the average kinetic energy of the electrons and the Fermi energy. (1 Points)

c. The mass density is $10^{12}\text{kg/m}^3$. Calculate the average kinetic energy of an electron, in MeV. (One MeV = $1.6 \times 10^{-13}\text{J}$.) (1 Points)

d. The temperature is $10^9\text{K}$. Calculate the average kinetic energy of the nuclei. (1 Points)

e. According to the virial theorem the internal energy of a system is approximately equal to its gravitational potential energy. For a sphere of uniform density, the gravitational potential energy is $3GM^2/5R$. Derive an expression for the mass of the white dwarf in terms of fundamental constants only. (3 Points)

f. Calculate the mass of the white dwarf in solar masses. (1 solar mass = $2 \times 10^{30}\text{kg}$). A white dwarf in which the electrons are relativistic is unstable with respect to collapse, so the quantity that you have calculated is approximately the maximum mass of a white dwarf, a quantity called the Chandrasekhar mass (1.4 solar masses). Does your numerical result look reasonable? Why or why not? (2 Points)
\[ N = 2 \left( \frac{4}{3} \pi \left( \frac{p_f}{c} \right)^3 \right) \frac{L^3}{h^3} \]

DensIf \( e = \frac{N}{V} \]

\[ N = \frac{8}{3} \pi \frac{E_f}{c^3} \frac{V}{h^2} \]

\[ E_f = p_f c \quad \Rightarrow \quad p_f = \frac{E_f}{c} \]

\[ p_f = \frac{h}{2\pi} \]

\[ E_f = p_f c = \frac{h}{2\pi} c \]

\[ 2\pi \frac{h}{2\pi} = h \]

\[ \frac{N}{8} = \frac{3}{2} \frac{C^3}{\pi} \frac{L^3}{V} \]

\[ E_f = \frac{\frac{1}{2}h}{2} \left( \frac{N}{2\pi V} \right)^{\frac{1}{3}} \]

\[ E_{\text{total}} = \int_0^N E_f \, dN = \left( \frac{3}{2\pi} \right) \frac{1}{4} \int_0^N N^{\frac{1}{3}} \, dN = \frac{N^{\frac{1}{3}+1}}{\frac{1}{3}+1} \]

\[ E_{+} = \left( \frac{3}{2\pi} \right) \frac{1}{4} \frac{3}{2} N^{\frac{4}{3}} \]

\[ \frac{E_{\text{avg}}}{e} = \frac{E_{\text{total}}}{N} = \frac{\frac{1}{2}h}{2} \left( \frac{3N}{2\pi} \right)^{\frac{1}{3}} \frac{3}{4} \]

\[ E_{\text{avg}} = \frac{E_{\text{avg}}}{e} \cdot E_f \frac{3}{4} = \overline{E} \]

\[ N = \frac{V \text{ Vol} \text{ of phase space}}{h^3} \]

\[ \text{Density} \ e = D_e \]

MeV = 1.6 \times 10^{-13} J = 1.6 \times 10^{-13} \frac{\text{kg} \cdot \text{m}^2}{\text{s}^2} \\

\[ c = 3 \times 10^8 \frac{\text{m}}{\text{s}} \]

\[ h = \frac{\hbar}{2\pi} \]

\[ \hbar = 6.6 \times 10^{-34} \frac{\text{J} \cdot \text{s}}{\text{m}^2} = 6.6 \times 10^{-34} \frac{\text{kg} \cdot \text{m}^2}{\text{s}^2} \]

\[ K = 8.6 \times 10^{-5} \frac{\text{eV}}{\text{K}} \]
\[ E_e = \frac{c^2}{2} \left( \frac{3}{\pi} D_e \right)^{\frac{3}{2}} \frac{3}{4} \]

\[ D_e = \text{(number of ions x 14 electrons per ion)} = N_i \times 14 \]

\[ N_i = \frac{\text{mass density} \times V}{14 \text{ ions/mole}} = 10^{45} \frac{\text{g}}{\text{m}^3} \times \frac{\text{m}^3}{6.2 \times 10^{27} \text{ ions}} = 10^{14} \text{ ions/m}^3 \]

\[ N_i = \frac{\text{mass density}}{0.014 \text{ kg/mole}} \times 6.2 \times 10^{23} \text{ ions/mole} = 62 \times 10^{37} \frac{1}{\text{m}^3} \]

\[ D_e = 62 \times 10^{37} \frac{1}{\text{m}^3} \]

\[ E_e = \frac{6m}{2} \frac{3 \times 10^2 \text{ m}^3}{2} \frac{6.6 \times 10^{-28} \text{ kg m}^2}{5} \left( \frac{3}{\pi} \right)^{\frac{3}{2}} \frac{62 \times 10^{37}}{m} \frac{3}{4} \]

\[ \approx 9 \times 10^{-12} \text{ J} = 56 \text{ MeV} \]

\( \text{d) } \quad E_i = \frac{3}{2} kT = \frac{3}{2} 8.6 \times 10^{-5} \text{ eV x } 10^9 K = 12.9 \times 10^4 \text{ eV} = 1.8 \times 10^2 \text{ keV} \]

\( \text{e) } \quad U = \frac{3GM^2}{5R} = E_e n_e + E_i n_i \quad \text{negligible} \quad \approx \left( \frac{4}{3} \pi R^3 \right) D_e \times 9 \times 10^8 \text{ 56 MeV} \]

\[ M = \frac{\sqrt{5R \frac{E_e}{3G}}} \]

\[ \text{Volume x density} \]

\[ E_e n_e \]
Solution:

(a) \[ N = \frac{2V}{\hbar^3} \int_{p \leq p_F} \int \int dp, \]
giving \[ n = \frac{N}{V} = \frac{8\pi}{3} \left( \frac{p_F}{\hbar} \right)^3. \]

With \[ p_F = \frac{m_e c}{10}, \]
we have \[ n = \frac{8\pi}{3} \left( \frac{m_e c}{10\hbar} \right)^3 = 5.8 \times 10^{32} \text{ /m}^3. \]

(b) For a strong degenerate Fermi gas (under the approximation of zero valence), we get \[ \overline{E} = \frac{3}{5} N \mu_0, \]
and \[ p = \frac{2 \overline{E}}{3V} = \frac{2}{5} n \mu_0 = \frac{2}{5} n \cdot \frac{p_F^2}{2m} = 9.5 \times 10^{16} \text{ N/m}^2. \]

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A white dwarf is a star supported by the pressure of degenerate electrons. As a simplified model for such an object, consider a sphere of an ideal gas consisting of electrons and completely ionized Si_{28}, and of constant density throughout the star. (Note that the assumption of a constant density is inconsistent with hydrostatic equilibrium, since the pressure is then also constant. The assumption that the gas is ideal is also not really tenable. These shortcomings of the model are, however, not crucial for the issues which we wish to consider.) Let \( n_i \) denote the density of the silicon ions, and let \( n_e = 14n_i \) denote the electron density. (The atomic number of silicon is 14).

(a) Find the relation between the mean kinetic energy \( \overline{E}_e \) of the electrons and the density \( n_e \), assuming that the densities are such that the electrons are "extremely relativistic," i.e., such that the rest energy is negligible compared with the total energy.

(b) Compute \( \overline{E}_e \) (in MeV) in the case that the (rest mass) density of the gas equals \( \rho = 10^9 \text{ g/cm}^3 \). Also compute the mean kinetic energy \( \overline{E}_i \) of the silicon ions in the central region of the dwarf, assuming that the
temperature is $10^8$ K and assuming that the "ion gas" can be regarded as a Maxwell-Boltzmann gas, and hence convince yourself that $\bar{E}_e \gg \bar{E}_i$.

(c) If $M$ is the mass of the star, and if $R$ is its radius, then the gravitational potential energy is given by

$$U_G = \frac{3GM^2}{5R}.$$ 

In the case in which the internal energy is dominated by extremely relativistic electrons (as in part (b) above), the virial theorem implies that the total internal energy is approximately equal to the gravitational potential energy. Assuming equality, and assuming that the electrons do not contribute significantly to the mass of the star, show that the stellar mass can be expressed in terms of fundamental physical constants alone. Evaluate your answer numerically and compare it with the mass of the sun, $2 \times 10^{30}$ kg. (It can be shown that this is approximately the maximum possible mass of a white dwarf.)

*UC, Berkeley*

**Solution:**

(a) Use the approximation of strong degenerate electron gas and $\varepsilon = pc$. From the quantum state density of electrons, it follows

$$\frac{2}{\hbar^3}dp = \frac{8\pi}{h^3c^3} \varepsilon^2 d\varepsilon,$$

then

$$n_e = \int_{0}^{*\varepsilon} \frac{8\pi}{h^3c^3} \varepsilon^2 d\varepsilon$$

$$= \frac{8\pi}{3h^3c^3} \varepsilon_F^3.$$

Therefore

$$\bar{E}_e = \frac{\int_{0}^{*\varepsilon} \varepsilon \cdot \varepsilon^2 d\varepsilon}{\int_{0}^{*\varepsilon} \varepsilon^2 d\varepsilon} = \frac{3}{4} \varepsilon_F = \frac{3}{4} \frac{hc}{8\pi} \left(\frac{3n_e}{8\pi}\right)^{1/3}$$

(b) When $\rho = 10^9$ g/cm$^3$,

$$n_e = 14n_i = 3 \times 10^{32} \text{cm}^{-3} = 3 \times 10^{38} \text{ m}^{-3},$$

$$\bar{E}_e = 5 \times 10^{-13} \text{ J} = 3 \text{ MeV},$$

$$\bar{E}_i = \frac{3}{2}kT = 2 \times 10^{-15} \text{ J} = 1.3 \times 10^{-2} \text{ MeV}.$$ 

Obviously, $\bar{E}_i \ll \bar{E}_e$. 
(c) From the virial theorem, we have

\[
\left(\frac{4\pi}{3} R^3 n_e\right) \cdot \frac{3}{4} \frac{\hbar c}{8\pi} \left(\frac{3n_e}{8\pi}\right)^{1/2} = \frac{3}{5} \frac{GM^2}{R}
\]

Noting that

\[
M = \frac{4\pi}{3} R^3 n_e \frac{m_i}{14} = \frac{8\pi}{3} R^3 n_e m_p,
\]

we obtain

\[
M = \frac{15}{128\pi} \frac{\hbar c}{GM_p^2} \sqrt{\frac{5\hbar c}{2G}} = 8.5 \times 10^{30} \text{ kg} = 4.1 M_\odot,
\]

where \( M_\odot \) is the mass of the sun.

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(a) Given that the mass of the sun is \( 2 \times 10^{33} \) g, estimate the number of electrons in the sun. Assume the sun is largely composed of atomic hydrogen.

(b) In a white dwarf star of one solar mass the atoms are all ionized and contained in a sphere of radius \( 2 \times 10^9 \) cm. Find the Fermi energy of the electrons in eV.

(c) If the temperature of the white dwarf is \( 10^7 \) K, discuss whether the electrons and/or nucleons in the star are degenerate.

(d) If the above number of electrons were contained in a pulsar of one solar mass and of radius 10 km, find the order of magnitude of their Fermi energy.

\( \text{(Columbia)} \)

Solution:

(a) The number of electrons is

\[
N = \frac{2 \times 10^{33}}{1.67 \times 10^{-24}} \approx 1.2 \times 10^{57}.
\]

(b) The Fermi energy of the electrons is

\[
E_{Fe} = \frac{\hbar^2}{2m_e} \left(\frac{3}{8\pi V} N\right)^{2/3} = \frac{\hbar^2}{2m_e} \left(\frac{9}{32\pi^2 R^3} \frac{N}{R^3}\right)^{2/3} \approx 4 \times 10^4 \text{ eV}.
\]
Problem 6 (10 Points): Like HW 6 Q 3

A large flat surface is in contact with a mono-atomic gas above it. The volume of gas above the surface acts as an infinite reservoir of gas atoms, but does not otherwise enter into the problem. The surface consists of a square lattice of sites that gas atoms can occupy; denote the number of gas atoms on site \( i \) by \( n_i \), where \( n_i \in \{0,1\} \), and the total number of lattice sites by \( N_s \). The energy of the system is given by:

\[
E(\{n_i\}) = -\left[ \sum_i n_i \epsilon + v_0 \sum_{i \neq j} n_i n_j \right]
\]

(1)

where \( \epsilon \) is a binding energy of atom to the substrate, \( v_0 \) is an interaction between adjacent atoms, and the sum over \( j \) is restricted to the nearest neighbors of \( i \).

a. Write down an expression for the grand canonical partition function \( Z(T, \mu) \). Your answer should be in the form of a sum over states. (2 Points)

b. Calculate the grand canonical free energy, \( \Omega(T, \mu, N_s) \) when \( v_0 = 0 \). (2 Points)

c. Calculate \( N_s \), the number of gas atoms adsorbed to the surface, as a function of \( T, \mu \) and \( N_s \) when \( v_0 = 0 \). (2 Points)

d. When \( v_0 \neq 0 \) the problem is in general more difficult. To simplify it, replace \( n_j \) in the above sum by \( \bar{n} \), a constant that will be set equal to the average occupation of any site. Calculate the number of gas atoms adsorbed to the surface, \( N_s \), as a function of \( T, \mu, N_s \) and \( \bar{n} \). (2 Points)

e. Discuss the possibility of a phase transition in \( \bar{n} \) as a function of \( \beta \). This can be done by graphically investigating the requirement that \( N(T, \mu, \bar{n})/N_s \equiv \bar{n} \), or by returning to the expression for the energy given in equation (1) and mapping it on to other well known problems in statistical mechanics. (2 Points)
\[ Q = \sum_{N} e^{\beta \mu N} Z = \sum_{N} e^{\beta \mu N} \prod_{n=1}^{N} e^{-\beta E_{n}N} \]
\[ = \sum_{N} e^{\beta \mu N} \sum_{n_{i}} e^{\beta \left[ \sum_{i} n_{i} \epsilon + \nu_{o} \sum_{i,j \in \text{neighbours}, j \neq i} n_{i} n_{j} \right] N} \]
\[ = \sum_{N} e^{\beta \mu N} \sum_{\{n_{i}\}} e^{\beta \left[ \sum_{i} n_{i} \epsilon + \nu_{o} \sum_{i,j} n_{i} n_{j} \right]} \]
\[ N = \sum_{i} n_{i} \]
\[ = \sum_{N} \sum_{\{n_{i}\}} e^{\beta \left[ \mu + \epsilon + \nu_{o} \sum_{j} n_{j} \right]} \]

"sum over states"

\[ = \prod_{n_{i}=0}^{1} e^{\beta \left[ \mu + \epsilon + \nu_{o} \sum_{j} n_{j} \right] n_{i}} \]
\[ = \prod_{N} \left( 1 + e^{\beta \left[ \mu + \epsilon + \nu_{o} \frac{N_{s}}{N} \right]} \right) \]
\[ Q = \left( 1 + e^{\beta \left[ \mu + \epsilon + \nu_{o} \frac{N_{s}}{N} \right]} \right)^{N_{s}} \]

\[ G = -\frac{1}{\beta} \ln Q = -\frac{1}{\beta} N_{s} \ln \left( 1 + e^{\beta \left[ \mu + \epsilon + \nu_{o} \frac{N_{s}}{N} \right]} \right) \]
\[ = -\frac{N_{s}}{\beta} \ln \left( 1 + e^{\beta \left( \mu + \epsilon \right)} \right) \]

\[ N = -\frac{\partial G}{\partial \mu} = N_{s} \frac{2}{\beta} \frac{\ln \left( 1 + e^{\beta \left( \mu + \epsilon \right)} \right)}{\partial \mu} \]
\[ N = \frac{N_{s}}{\beta} \left( 1 + e^{\beta \left( \mu + \epsilon \right)} \right)^{-1} e^{\beta \left( \mu + \epsilon \right)} \]
\[ = \frac{N_{s}}{e^{\beta \left( \mu + \epsilon + \nu_{o} \frac{N_{s}}{N} \right)} + 1} \]

\[ N = \frac{N_{s}}{e^{-\beta \left( \mu + \epsilon + \nu_{o} \frac{N_{s}}{N} \right)} + 1} \]

\[ \beta = \frac{1}{kT} \]

\[ Q = \left( 1 + e^{\beta \left( \mu + \epsilon + \nu_{o} \frac{N_{s}}{N} \right)} \right)^{N_{s}} \]

\[ N = \frac{N_{s}}{e^{-\beta \left( \mu + \epsilon + \nu_{o} \frac{N_{s}}{N} \right)} + 1} \]
\[
\frac{N}{N_s} = \bar{n} \quad \text{defined, so from (4)}
\]

\[
\bar{n} = \frac{1}{e^{-\beta (\mu + e + u_0 \bar{n})} + 1}
\]

\[
e^{-\beta (\mu + e + u_0 \bar{n})} + 1 = \frac{1}{\bar{n}} \quad \text{or} \quad \frac{1}{\bar{n}} - 1 = \frac{1 - \bar{n}}{\bar{n}} = \bar{n} \left( \frac{1}{\bar{n}} - 1 \right)
\]

\[-\beta (\mu + e + u_0 \bar{n}) = \ln \left( \frac{1}{\bar{n}} - 1 \right)
\]

\[
\mu + e + u_0 \bar{n} = -\frac{1}{\beta} \ln \left( \frac{1}{\bar{n}} - 1 \right)
\]

\[
\bar{n} - n_0 = \frac{kT}{c v_0} \ln \left( \frac{\bar{n}}{1 - \bar{n}} \right)
\]

Graph this: depending on the number of intersections it tells you what number of values for \( \bar{n} \)
multiple solutions for \( \bar{n} \) = density differ in different places = phase transitions are possible

The problem has graph:

\[\text{1 intersection} \leftrightarrow \text{3 intersections}\]
6. **Boson Magnetism** Consider a gas of non-interacting spin-1 bosons in 3D, each subject to the Hamiltonian

$$H(\vec{p}, s_z) = \frac{\vec{p}^2}{2m} - \mu_0 s B$$

$$= \frac{h^2 k^2}{2m} - \mu_0 s B$$

where $s$ takes on one of three possible states, $s \in \{-1, 0, +1\}$, and $k \equiv \vec{p}/\hbar$. In this Hamiltonian $B$ is the $z$-component of the magnetic field, $m$ is the mass of a particle, and $\mu_0$ is the Bohr magneton. (We will ignore the orbital effect (or Lorentz force) where the momentum $\vec{p}$ would have been replaced, $\vec{p} \rightarrow \vec{p} + e\vec{A}/c$).

(a) In a grand canonical ensemble of chemical potential $\mu$ (which is **not** to be confused with the Bohr magneton, $\mu_0$, above) and temperature $T$, write down $n_s(k)$, the average occupation number of the state with wave vector $k$ and spin $s$. (1 point).

(b) Show that the total number of particles in a given spin state $s$ is given by

$$N_s = \frac{V}{\lambda^3} \pi^{3/2} g_{3/2}(z e^{\beta \mu_0 s B})$$

where $z$ is the fugacity, $z = e^{\beta \mu}$, $\lambda$ is the thermal de Broglie wavelength,

$$\lambda = \frac{\hbar}{\sqrt{2\pi mk_B T}}$$

and $g_\lambda(z)$ is defined on the formula section on page 2 above. (4 points)

(c) The magnetization for fixed $\mu$ and $T$ is given by

$$M(T, \mu) = \mu_0 (N_+ - N_-)$$

Show that the zero field susceptibility, $\chi$, is given by:

$$\chi = \frac{\partial M}{\partial B} \bigg|_{B=0} = \frac{2\mu_0^2}{k_B T} \pi^{3/2} \frac{V}{\lambda^3} g_{3/2}(z).$$

(5 points).
\( f(E) = \frac{1}{e^{\beta(E-\mu)} - 1} \) (bosons)

\[ N_s(K) = \frac{1}{\beta\left(\frac{k^2}{2m} - \mu_{SB} - \mu\right) - 1} \]

for a given spin \( n \)

\[ Q = \sum_{\{n_i\}} e^{\beta E_i} \leq n \leq \sum_{\{n_i\}} e^{\beta E_i} = \sum_{\{n_i\}} e^{\beta \sum_{i} n_i E_i} \]

using: \( E_i = \sum_n E_i \) and \( N = \sum_n n_i \)

\[ Q = \sum_{\{n_i\}} e^{\beta \sum_{i} n_i E_i} = \sum_{\{n_i\}} e^{\beta \sum_{i} n_i (\mu - E_i)} \]

using: \( e^{\alpha} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \)

\[ Q = \prod_{\{n_i\}} e^{\beta (\mu - E_i) n_i} \]

\[ Q = \prod_{\{n_i\}} \frac{1}{1 - e^{-\beta (\mu - E_i)}} = \prod_{\{n_i\}} \frac{1}{1 - e^{-\beta (E_i - \mu)}} \]

\( \frac{\delta Q}{\delta \mu} = -\frac{1}{\beta} \frac{1}{1 - e^{-\beta (E_i - \mu)}} \sum_{i} n_i \ln(1 - e^{-\beta (E_i - \mu)}) \)

\[ -\frac{1}{\beta} \frac{\delta}{\delta \mu} \ln(1 - e^{-\beta (E_i - \mu)}) = -\frac{1}{\beta} \sum_{i} (1 - e^{-\beta (E_i - \mu)}) (-e^{-\beta (E_i - \mu)}') \]

\[ N_s = \prod_{\{n_i\}} \frac{1}{e^{\beta (E_i - \mu)} - 1} = \prod_{\{n_i\}} \frac{1}{e^{\beta\left(\frac{k^2}{2m} - \mu_{SB} - \mu\right)} - 1} = \text{turn into an integral} \]
you need this but it's not given…

\[ g_0(z) = \frac{1}{\frac{2\pi^2}{3} k^{3/2} e^{-\frac{1}{2} k^2}} \int_0^\infty \frac{u^{3/2-1}}{e^{u/2} - 1} \, du \]

\[ N_s = \int \frac{d^3x d^3k}{(2\pi)^3} \frac{1}{e^{\frac{Bk^2}{2m} - \beta M_0 S_B - \beta \mu} - 1} \]

\[ = V \frac{4\pi}{2\sqrt{\pi}} \int_0^\infty \frac{k^2 \, dk}{e^{\frac{1}{2} k^2} - 1} \]

let \( u = \frac{1}{2} k^2 \)

\[ \frac{du}{dk} = k \quad \Rightarrow \quad dk = \frac{du}{2\sqrt{u}} \]

\[ = \frac{V}{2\pi^2} \int_0^\infty \frac{k^2}{(e^{u/2} - 1) 2\sqrt{u}} \, du \]

\[ = \frac{V}{2\pi^2} \int_0^\infty \frac{u^{1/2}}{\sqrt{u} e^{u/2} - 1} \, du \]

now use \( u = \frac{3}{2} \)

\[ e^{-1} = e^{\frac{1}{2}} \Rightarrow z = \sqrt{u} \]

\[ \nabla(\frac{3}{2}) = \nabla(\frac{1}{2} + 1) = \sqrt{\frac{3}{2}} \]

\[ \sqrt{\frac{3}{2}} = e^{\frac{1}{2} \beta M_0 S_B} \]

\[ \frac{z}{\sqrt{u}} = \frac{e^{\frac{1}{2} \beta M_0 S_B}}{e^{\frac{1}{2} \beta M_0 S_B}} \]

factor of \( \pi^{3/2} \) is a typo.
\[ M = \mu_0 \left( N_+ - N_- \right) \text{ using } N_{CS} \text{ alone} \]

\[ M = \frac{\nu \mu_0}{\lambda^3} \left[ g_{3/2}(Z e^{\beta \mu_0 B}) - g_{3/2}(Z e^{-\beta \mu_0 B}) \right] \]

\[ \chi = \left. \frac{\partial M}{\partial \beta} \right|_{\beta=0} \]

\[ \frac{\partial}{\partial \alpha} g_{p}(Z) = aZ + aZ \]

\[ g_{3/2}^+ = g_{3/2}^0 \left( Z e^{\beta \mu_0 B} \right) = \sum_{n=1}^{\infty} \frac{Z^n}{n^{3/2}} = \sum_{n=1}^{\infty} \frac{Z^n}{n^{3/2}} \]

\[ g_{3/2}^- = g_{3/2}^0 \left( Z e^{-\beta \mu_0 B} \right) = \sum_{n=1}^{\infty} \frac{\beta \mu_0 Z^n}{n^{3/2}} \]

\[ \frac{\partial}{\partial \beta} g_{3/2}^+ = \sum_{n=1}^{\infty} n \beta \mu_0 \frac{Z e^{\beta \mu_0 B}}{n^{3/2}} = \sum_{n=1}^{\infty} \frac{\beta \mu_0 (Z e^{\beta \mu_0 B})^n}{r^{3/2-1}} \]

so similarly \[ \frac{\partial}{\partial \beta} g_{3/2}^- = -\frac{\beta \mu_0}{\lambda^3} \]

\[ \chi = \frac{\nu \mu_0}{\lambda^3} \left[ g_{1/2}(Z e^{\beta \mu_0 B}) - g_{1/2}(Z e^{-\beta \mu_0 B}) \right] \]

\[ = \frac{\nu \beta \mu_0}{\lambda^3} \left[ g_{1/2}(Z) + g_{1/2}(-Z) \right] \]

\[ = \frac{\nu \beta \mu_0^2}{\lambda^3} \alpha_{1/2}(Z) = \frac{2 \mu_0^2 V}{\lambda^3 KT} g_{1/2}(Z) \]
Problem 4 (10 Points):

This problem involves the mean field Ising model. Consider a solid containing N electrons localized at lattice sites. Each electron has a magnetic moment $\mu$. In a magnetic field $H$ each electron can exist in one of two states, with energies $\pm\mu H$.

a. Show that for non-interacting electrons the total magnetic moment is given by $M = N\mu \tanh\left(\frac{\mu H}{kT}\right)$. (2 Points)

b. In order to add interactions between the electrons, assume that each electron sees an effective magnetic field equal to the applied field plus a local field arising from its neighbors. In this case, $H_{eff} = H + \frac{\alpha}{N} M$, where $\alpha$ is a positive constant. Write down a self consistency equation that determines $M$. (2 Points)

c. Show that there is a spontaneous magnetization (e.g. when $H = 0$) below some critical temperature, $T_c$, and determine its value. (3 Points)

d. Show that the magnetic susceptibility $\chi$, diverges at $T \rightarrow T_c$ from the high T side. (Hint: be careful because you will have to take a derivative of a transcendental equation.) (3 Points)
\[ \mathcal{Z} = \sum_{n} \alpha_{n} e^{-\beta E_{n}} \]
\[ E_{1} = +\mu H \]
\[ E_{2} = -\mu H \]
\[ \mathcal{Z} = e^{-\beta H} + e^{-\beta H} = 2 \cosh(\beta \mu H) \]
\[ M = \frac{1}{\beta} \frac{\partial}{\partial H} \ln \mathcal{Z} \]
\[ M = \frac{1}{\beta} \frac{\partial}{\partial H} \ln \left[ 2 \cosh(\beta \mu H) \right] = \frac{1}{\beta} \frac{\partial}{\partial H} \ln \cosh(\beta \mu H) \]
\[ M = \frac{N}{\beta} \left( \cosh(\beta \mu H) \right)^{-1} \frac{\partial}{\partial H} \sinh(\beta \mu H) = N \mu \tanh \left( \frac{\mu H}{kT} \right) \]

now
\[ E_{1} = \mu H + \frac{\alpha}{N} M \]
\[ E_{2} = -\mu H + \frac{\alpha}{N} M \]
\[ \mathcal{Z} = e^{-\beta \mu H} + e^{-\beta \mu H} = e^{-\beta \mu^{2} H N} - \beta \mu \frac{\alpha}{N} M + e^{-\beta \mu^{2} H N} - \beta \mu \frac{\alpha}{N} M \]
\[ \mathcal{Z} = e^{-\beta \mu \alpha M} (e^{-\beta \mu^{2} H N} + e^{-\beta \mu^{2} H N}) \]
\[ M = \frac{1}{\beta} \frac{\partial}{\partial H} \left( -\beta \mu \alpha M + \ln 2 + \ln \cosh(\beta \mu^{2} H N) \right) \]
\[ M = N \mu \tanh \left( \frac{\mu H_{\text{eff}}}{kT} \right) = N \mu \tanh \left( \frac{N - H + \frac{\alpha}{N} M}{kT} \right) \]
graphing the function from (b), one curve for each side

As $T$ decreases, slope gets steeper. At $T \geq T_c$ it intersects with $M$.

Slope near $M=0$ must be $\geq$ slope of $f(M)$ for $T_c$.

For $H=0 \rightarrow \frac{M}{N\mu} = \tanh\left(\frac{\mu}{kT} \frac{\alpha M}{N}\right)$

Find slopes: $\frac{1}{N\mu}$

They must be at least equal for $T_c$.

$\frac{1}{N\mu} = \frac{\alpha x}{kTN}$

$T_c = \frac{\mu^2 \alpha}{K}$

Apply $x = \frac{dM}{dH}$ to (b)

$$\frac{d}{dH} \left[ M = N\mu \tanh\left(\frac{\mu}{kT} \left(H + \frac{\alpha x}{N} M^2 H^2\right)\right) \right]$$

by definition: $x = N\mu \text{sech}^2\left(\frac{\mu}{kT} \left(1 + \frac{\alpha x}{N} \left(\frac{dM}{dH} \right)^2\right)\right)$

$$x = N\mu \text{sech}^2\left(\frac{\mu}{kT} \left(\frac{dM}{dH} \right)^2\right)$$

$$x = N\mu \text{sech}^2\left(\frac{\mu}{kT} + \frac{\alpha x}{kTN} x\right)$$

$$x = N\mu \text{sech}^2\left(\frac{\mu}{kT} + N\mu \text{sech}^2\left(\frac{\mu}{kT} \left(1 - N\mu \text{sech}^2\left(\frac{\mu}{kTN}\right)\right)\right)\right)$$

$$x = \frac{N\mu \beta}{\cosh^2\left(\frac{\mu}{kT}\right) - \mu^2 \beta x}$$
\[ \chi \propto \left( \frac{M^2}{M^*} \right)^{\frac{1}{2}} \left( H + \frac{\alpha}{N} \mu \right)^{\frac{1}{2}} = \frac{H}{M^*} + \frac{M}{\mu N} \]

not really diverging....
Problem 5 (10 Points):

A crystal lattice consists of $N$ atoms. Each atom is in a quantum state in which the total orbital angular momentum is zero and the total spin angular momentum is $S = \frac{1}{2}$. The crystal is in an external magnetic field $B_0 = \mu_0 H$ of magnetic field intensity $H$, where $\mu_0$ is the permeability of free space. Choosing the $z$-axis to lie along the field, we can specify a microstate in terms of the site indices $\sigma_i$ for each lattice site $j$, which are defined as $\sigma_j = \pm 1$ if $(M_s)_j = \pm \frac{1}{2}$, respectively. In the Ising model, the energy $E_p$ for a microstate $\psi_p$ of a one-dimensional crystal in this field is,

$$E_p = -J \sum_{(i,j)_{nn}}^N \sigma_i \sigma_j - B_0 \sum_{j=1}^N \sigma_j,$$

where $J > 0$ is a constant, and the subscript $(i,j)_{nn}$ means to sum once over each nearest neighbor pair of sites. Now, define

- $N_+ = \text{number of atoms with } \sigma_j = 1$
- $N_- = \text{number of atoms with } \sigma_j = -1$
- $N_{++} = \text{number of nearest neighbor pairs } (i,j) \text{ with } \sigma_i = 1 \text{ and } \sigma_j = 1$
- $N_{+-} = \text{number of nearest neighbor pairs } (i,j) \text{ with } \sigma_i = 1 \text{ and } \sigma_j = -1$

In terms of these quantities, the microstate energy for $E_p$ can be written

$$E_p = -4JN_{++} + 2(fJ - B_0)N_+ - \frac{1}{2}(fJ - 2B_0)N,$$

where $f$ is defined so that the number of nearest neighbor pairs with at least one $\sigma_i = 1$ is $fN_+ = 2N_{++} + N_{+-}$.

a. Write down expressions for the Helmholtz potential $F(T,B_0,N)$ in terms of the canonical partition function $Z(T,B_0,N)$, and for the partition function in terms of the microstate energies $E_p$. Do not try to evaluate or simplify your expression. (3 Points)
b. Let $m_0$ denote the difference between the fraction of atoms with $M_s = \frac{1}{2}$ and the fraction with $M_s = -\frac{1}{2}$; $m_0 = (N_+ - N_-)/N$. Derive the following approximate implicit equation for $m_0$ in the limit of zero field strength ($H \rightarrow 0$):

$$B_0 + f J m_0 = k_B T \tanh^{-1}(m_0)$$

(2 Points)

c. From the expression in b., derive an expression for the critical temperature $T_c$ for spontaneous magnetization. Express your answer in terms of $f$, $J$, and Boltzmann’s constant. (2 Points)

d. Derive the value of the critical exponent $\beta$ (the degree of the coexistence curve) that describes how the order parameter $M_0(T)$ behaves as the temperature $T$ approaches the critical temperature $T_c$ from below:

$$M_0 \sim \left( \frac{T - T_c}{T_c} \right) \text{ for } T \leq T_c$$

(3 Points)
Problem 4 (10 Points):

A closed system consists of two distinguishable spin 1 magnets. Each magnet can have one of three orientations, $\uparrow$, $\leftrightarrow$, and $\downarrow$, with respect to the z axis. The respective magnetic moments are $+m$, $0$, and $-m$. There is no applied field. The Hamiltonian, $H = B \sum m_i$.

a. List all the possible microstates of the system. What is the total number of states? (1 Points)

b. For $B=0$ what is the probability that the total magnetic moment, $M$, of the system is zero? (1 Points)

c. For $B=0$ compute average value of the total magnetic moment, $\langle M \rangle$, using the list in part (a.). (1 Points)

d. If $\Delta M = M - \langle M \rangle$, show that $(\Delta M)^2 = \langle M^2 \rangle - \langle M \rangle^2$, and compute $(\Delta M)^2$ for $B=0$. (2 Points)

e. If the spins were indistinguishable, what would be the total number of microstates of the system? (1 Points)

For the last two parts of this problem consider $N$ of the spins described in the initial part of the problem. These $N$ spins are now in contact with a heat bath at temperature, $T$, and $B\neq0$.

f. Find the partition function of the $N$ spins. (2 Points)

g. What is the Helmholtz free energy of the $N$ spins? (2 Points)
microstates \( 3 \times 3 = q_{\text{total}} \)

\[
M = m + n
\]

\[ \begin{array}{c}
\uparrow & \uparrow & \uparrow \\
\downarrow & \downarrow & \downarrow \\
\uparrow & \downarrow & \downarrow \\
\end{array}
\]

macrostates or states

are 2m, m, 0, \( -m \), \( -2m \)

They would have to cancel, by facing the opposite direction (or having \( m = 0 \))

\[
P(M = 0) = \frac{3}{q} = \frac{1}{3}
\]

\[
\langle M \rangle = 2 + 1 + 0 + 1 + 0 - 1 + 0 - 1 - 2 = 0 \quad q = 0
\]

\[
(\Delta M)^2 = \langle (M - \langle M \rangle)^2 \rangle
\]

\[
= \langle M^2 + \langle M \rangle^2 - 2M\langle M \rangle \rangle
\]

average each term independently

\[ = \langle M^2 \rangle + \langle M \rangle^2 - 2\langle M \rangle \langle M \rangle \]

average of average is average

\[ = \langle M^2 \rangle + \langle M \rangle^2 - 2\langle M \rangle^2
\]

\[ = \langle M^2 \rangle - \langle M \rangle^2
\]

For \( B = 0 \), \( \langle M \rangle = 0 \) as \( n \to \infty \)

So: \( (\Delta M)^2 = \langle M^2 \rangle = \frac{(2^2 + 1^2 + 1^2 + (-1)^2 + (-1)^2 + (-2)^2)}{q} \)

\[ = m^2(4 + 1 + 1 + 1 + 1 + 4) = \frac{12m^2}{q} = \frac{4m^2}{3}
\]

If indistinguishable then microstates = macrostates

\[ Z = \sum_{n=1}^{\infty} e^{-\beta E_n} \]

\[ n \to \text{possible energies}
\]

\[ Z = Z^N = \left( 1 + e^{-\beta E_m} + e^{-\beta E_m} \right)^N \]

now distinguishable again
\[ F = -\frac{1}{\beta} \ln Z = -\frac{1}{\beta} \ln \mathcal{Z}^N = -\frac{1}{\beta} N \ln (1 + e^{\beta m} + e^{-\beta m}) \]

could use \( \frac{e^x + e^{-x}}{2} = \cosh x \)