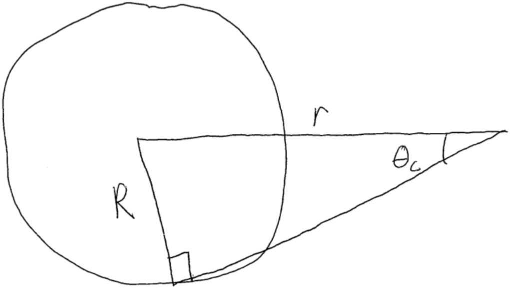
Lecture 6
Sobolev Theory

Dilution Factor I



$$sin\theta_c = R/r$$

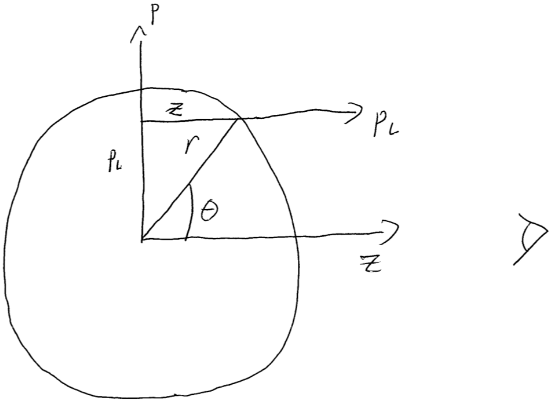
$\omega_* \equiv$ solid angle subtended by stellar disk

$W \equiv \omega_*/(4\pi) \equiv$ dilution factor

Dilution Factor II

$$
\begin{aligned}
\omega_* &= \int d\Omega = 2\pi \int_0^{\theta_c} sin\theta \, d\theta \\
&= 2\pi(-cos\theta)\Big|_0^{\theta_c} \\
&= 2\pi\,[1 - cos\theta_c] \\
&= 2\pi \left[1 - \sqrt{1-(R/r)^2}\right] \\
W &= \omega_*/(4\pi) \\
&= 1/2 \left[1 - \sqrt{1-(R/r)^2}\right] \\
W(R) &= 1/2 \\
W(r >> R) &= 1/4(R/r)^2
\end{aligned}
$$

Flux from a Star I



$$sin(90-\theta) = z/r = cos\theta; \quad sin\theta = p/r$$

$$\mu = cos\theta = \sqrt{r^2 - p^2}/r$$

Flux from a Star II

$$d\mu = -1/2(r^2 - p^2)^{-1/2} 2p/r dp$$

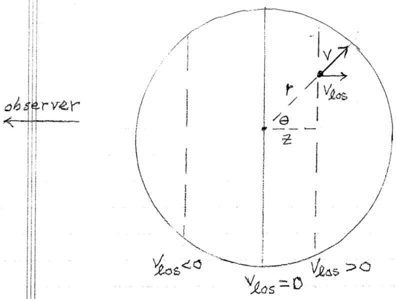
$$d\mu = -(r^2 - p^2)^{-1/2} p/r dp$$

$$\begin{aligned}
\mathcal{F} &= 2\pi \int_0^{\pi} I \, cos\theta d(cos\theta) \\
&= -2\pi \int_r^0 I \, \frac{\sqrt{r^2 - p^2}}{r} \frac{p}{r\sqrt{r^2 - p^2}} dp \\
&= \frac{2\pi}{r^2} \int_0^r I \, p \, dp
\end{aligned}$$

Constant Direction Velocity Surfaces

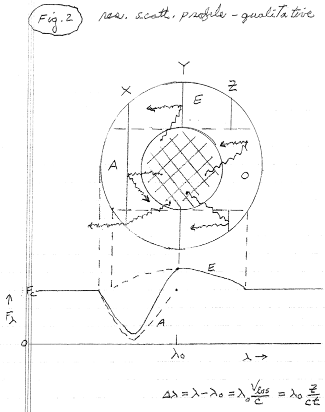
Surfaces of Constant Line-of-sight Velocity

Fig. 1 surfaces of constant V_{los}



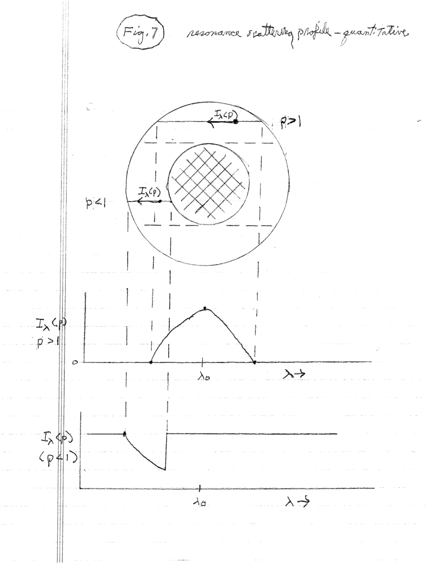
$$V_{los} = V \cos\theta = \frac{r}{t}\frac{z}{r} = \frac{z}{t}$$

Resonant Scattering Qualitative



Calculation of Flux

Resonant Scattering



Sobolev Theory

Castor, 1970

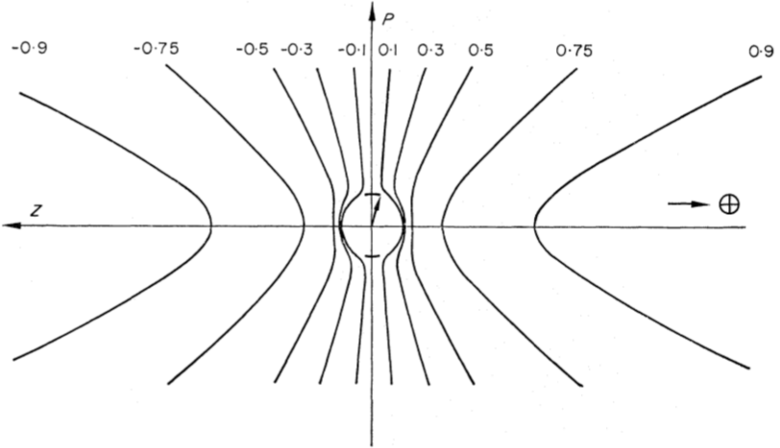


FIG. 1. *The coordinate system used in the transfer equation and several constant velocity surfaces. Only light rays directed toward the Earth are considered. There is symmetry about the z-axis. The radius of the core is indicated by the arrow.*

Mathematics I

Radiation travels from positive to negative z, that is from left to right
Surfaces of constant velocity: $v_z = \mu v$

$$I(r, \mu) = I(p, z)$$

$$p^2 + z^2 = r^2; \quad z = -\mu\, r$$

The transfer equation is:

$$\frac{dI_\nu(p,z)}{dz} = \kappa_\nu(p,z)(I_\nu(p,z) - S(r))$$

Assume that

$$I_\nu(r_c, \mu) = I_c \text{ for all } \nu, \mu$$

$$I_\nu(p,z) = \int_z^\infty S((p^2 + z'^2)^{1/2}) e^{\{\tau_\nu(p,z) - \tau_\nu(p,z')\}}\, d\tau_\nu(p,z')$$

for $p > r_c$ or $z > 0$

Mathematics II

$$
\begin{aligned}
I_\nu(p,z) &= \int_z^{(r_c^2-p^2)^{1/2}} S((p^2+z'^2)^{1/2}) e^{\{\tau_\nu(p,z)-\tau_\nu(p,z')\}}\, d\tau_\nu(p,z') \\
&\quad + I_c e^{\{\tau_\nu(p,z)-\tau_\nu(p,-(r_c^2-p^2)^{1/2}))\}} \text{ for } p < r_c,\ z < 0
\end{aligned}
$$

$$\tau_\nu(p,z) = \int_{-\infty}^{z} \kappa_\nu(p,z')dz'$$

$$\kappa_\nu(p,z) = \kappa_o(r)\phi(\nu + \frac{\nu_0}{c}\frac{v(r)}{r}z - \nu_0$$

$\phi(x)$ is strongly peaked at $x = 0$ so

$$\frac{\nu_0}{c}\frac{v(r)}{r(z_0)}z_0 = -(\nu - \nu_0)$$

$$\frac{\nu_0}{c}\frac{v((p^2+z_o^2)^{1/2})}{(p^2+z_o^2)^{1/2}}z_0 = -(\nu - \nu_0)$$

Mathematics III

Then let x be the argument of ϕ

$$x = \nu - \nu_0 + \nu_0 \frac{v_0(r')}{r'} z'$$

$$\begin{aligned} \int_{-\infty}^{z} \phi(\nu - \nu_0 + \nu_0 \frac{v_0(r')}{r'} z')\, dz' &= \int_{-\infty}^{x} \phi(x') \frac{dz'}{dx} dx \\ &\approx \left[\int_{-\infty}^{x} \phi(x')\, dx' \right] 1 / \frac{dx}{dz'} |_{z_0} \end{aligned}$$

$$\begin{aligned} \frac{dx}{dz'} &= \frac{\partial}{\partial z} \left(\frac{\nu_0}{c} \frac{v(r)}{r} z \right) \\ &= \frac{\nu_0}{c} \left[v(r) \frac{\partial z' / (z'^2 + p^2)^{1/2}}{\partial z'} + \frac{z'}{r} \frac{dv}{dr} \frac{dr}{dz'} \right] \end{aligned}$$

$$\frac{dr}{dz} = \frac{z}{r}$$

More Math

$$
\begin{aligned}
\frac{dx}{dz'} &= \frac{\nu_0}{c}\left[\frac{v(r)}{r} + z\frac{d(v/r)}{dz'}\right] \\
&= \frac{\nu_0}{c}\frac{v(r)}{r}\left[1 + z(\frac{v(r)}{r})^{-1}\frac{d(v/r)}{dr}\frac{dr}{dz'}\right] \\
&= \frac{\nu_0}{c}\frac{v(r)}{r}\left[1 + \frac{z^2}{r}(\frac{v(r)}{r})^{-1}\frac{d(v/r)}{dr}\right] \\
&= \frac{\nu_0}{c}\frac{v(r)}{r}\left[1 + \frac{z^2}{r}(\frac{v(r)}{r})^{-1}\frac{d(v/r)}{dr}\right] \\
&= \frac{\nu_0}{c}\frac{v(r)}{r}\left[1 + \frac{z^2}{r}(\frac{v(r)}{r})^{-1}[1/r\frac{d(v)}{dr} - \frac{v}{r^2}]\right] \\
&= \frac{\nu_0}{c}\frac{v(r)}{r}\left[1 + \frac{z^2}{r}\frac{r}{v}[1/r\frac{d(v)}{dr} - \frac{v}{r^2}]\right]
\end{aligned}
$$

More Math I

$$
\begin{aligned}
\frac{dx}{dz'} &= \frac{\nu_0}{c}\frac{v(r)}{r}\left[1 + \frac{z^2}{r^2}[r/v\frac{d(v)}{dr} - 1]\right] \\
&= \frac{\nu_0}{c}\frac{v(r)}{r}\left[1 + \frac{z^2}{r^2}[\frac{d\ln v}{d\ln r} - 1]\right]
\end{aligned}
$$

So Finally

$$
\tau_\nu(p,z) = \kappa_0((p^2+z^2)^{1/2})\frac{y(\nu + \frac{\nu_0}{c}\frac{v}{r} - \nu_0)}{\frac{\nu_0}{c}\frac{v}{r}}\left[1 + \frac{z^2}{r^2}[\frac{d\ln v}{d\ln r} - 1]\right]^{-1}
$$

$$
y(x) = \int_{-\infty}^{x}\phi(x')\,dx' \left\{ \begin{array}{rl} y(+\infty) = & 1 \\ y(-\infty) = & 0 \end{array} \right.
$$

$$
\tau_\nu(p,z) = \tau_\nu(p,\infty)y
$$

More Math II

In the limit:

$$\phi \rightarrow \delta$$

$$y \rightarrow \Theta(z - z_0)$$

$$\tau_\nu(p, \infty) = \frac{\frac{\pi e^2}{mc}(gf)_{lu}(\frac{N_l}{g_l} - \frac{N_u}{g_u})}{\frac{\nu_o}{c}\frac{v}{r}} \left[1 + \frac{z^2}{r^2}[\frac{d\ln v}{d\ln r} - 1]\right]^{-1}$$

$$\begin{aligned} I_\nu(p, z) &= \int_{y(z)}^{1} S((p^2 + z^2)^{1/2}) e^{\{\tau_\nu(p,\infty)(y(z) - y(z'))\}} \\ & \tau_\nu(p, \infty) dy(z') \begin{cases} p > r_c \\ \text{or} \\ z > 0 \end{cases} \end{aligned} \quad (1)$$

More Math III

$$
\begin{aligned}
I_\nu(p,z) \quad = \quad & \int_{y(z)}^{y(-(r_c^2-p^2)^{1/2})} S((p^2+z^2)^{1/2}) e^{\{\tau_\nu(p,\infty)(y(z)-y(z'))\}} \\
& \tau_\nu(p,\infty) dy(z') \\
& + I_c e^{\left\{\tau_\nu(p,\infty)(y(z)-y(-(r_c^2-p^2)^{1/2}))\right\}} \left\{ \; p < r_c, z < 0 \right.
\end{aligned}
$$

Recall,

$$
y(z) = y(\nu + \frac{\nu_0}{c}\frac{v}{r} z - \nu_0)
$$

and that if $z > z_0$ the integrals nearly vanish. If $z < z_0$ then the lower limit $\rightarrow 0$. In this case most of the contribution to I_ν comes at $z = z_0$. Thus we can pull S out of the integral and evaluate it at $z = z_0$

$$
I_\nu(p,z) = S((p^2+z_0^2)^{1/2}) \left(1 - e^{\{\tau_\nu(p,\infty)(y(z)-1))\}}\right) \left\{ \begin{array}{l} p > r_c \\ \text{or} \\ z > 0 \end{array} \right.
$$

More Math IV

$$
\begin{aligned}
I_\nu(p,z) &= S((p^2+z_0^2)^{1/2})\left(1-e^{\{\tau_\nu(p,\infty)(y(z)-y(z'))\}}\right) \\
&\quad + I_c e^{\left\{\tau_\nu(p,\infty)(y(z)-y(-(r_c^2-p^2)^{1/2}))\right\}} \left\{\ p < r_c, z < 0\right.
\end{aligned}
$$

Flux I

$$
\begin{aligned}
F &= 4\pi \int_0^{r_c} \Big[S((p^2 + z_0^2)^{1/2}) \big(1 - e^{-\tau(p,\infty) y(-(r_c^2 - p^2)^{1/2})}\big) \\
&\quad + I_c e^{-\tau(p,\infty) y(-(r_c^2 - p^2)^{1/2})} \Big] 2\pi p \, dp \\
&\quad + 4\pi \int_{r_c}^{\infty} \Big[S((p^2 + z_0^2)^{1/2} \big(1 - e^{-\tau(p,\infty)}\big) \Big] 2\pi p \, dp
\end{aligned}
$$

Define

$$
F_c = 4\pi \int_0^{r_c} 2\pi p I_c \, dp = 4\pi^2 r_c^2 I_c
$$

Flux II

$$
\begin{aligned}
\frac{F_\nu - F_c}{F_c} \quad = \quad & \frac{1}{r_c^2}\int_0^\infty \frac{S((p^2+z_0^2)^{1/2})}{I_c}\left(1-e^{-\tau(p,\infty)}\right)2p\,dp \\
& -\frac{1}{r_c^2}\int_0^{r_c}\left[1-e^{-\tau(p,\infty)y(-(r_c^2-p^2)^{1/2})}\right]2p\,dp \\
& -\frac{1}{r_c^2}\int_0^{r_c}\frac{S((p^2+z_0^2)^{1/2}}{I_c}\left[e^{-\tau(p,\infty)y(-(r_c^2-p^2)^{1/2})}\right. \\
& \quad \left. -e^{-\tau(p,\infty)}\right]2p\,dp
\end{aligned}
$$

- Term 1: Emission from entire core as if it were transparent
- Term 2: Radiation removed from continuum by material in front of the core
- Radiation that would have come from the part of the envelope that is occulted by the core

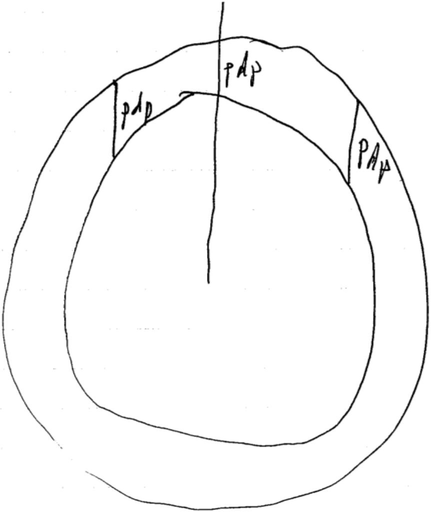
Flux III

$$y(-((r_c^2 - p^2)^{1/2}) = \begin{cases} 0 \text{ if } \Delta\nu < 0 \\ 1 \text{ if } \Delta\nu > 0 \end{cases}$$

So absorption and occultation terms are images of each other around the line center modulo the factor S/I_c in the occultation term.

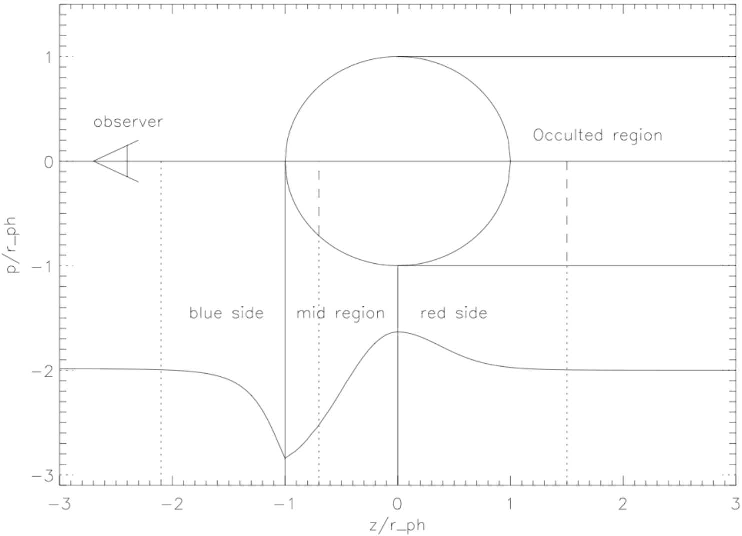
Shell

If we have a shell



Emmssion will be flat topped

Homologous Lines



SYNOW Assumptions I

$$S = W I_c$$

$$W = \frac{1}{2}\left[1 - \sqrt{1 - \frac{r_c}{r}^2}\right]$$

$$v \propto r$$

so photons come into resonance with lines i.e., $y = 0, 1$ and we assume $\tau \equiv \tau(r)$ which is specified to be for example a power-law or exponential in SYNOW. Then the flux for $z \geq 0$ is

$$\begin{aligned} F(z) &= 2\pi \int_0^{r_c} I_c p\, dp + \int_{r_c}^{\infty} S(r)\left[1 - e^{-\tau(r)}\right] p\, dp \\ &= \pi r_c^2 I_c + 2\pi \int_{r_c}^{\infty} S(r)\left[1 - e^{-\tau(r)}\right] p\, dp \end{aligned}$$

SYNOW Assumptions II

Then the flux for $z < 0$ is

$$
\begin{aligned}
F(z) &= 2\pi \int_0^{p_0} I_c p \, dp + 2\pi \int_{p_0}^{\infty} S(r) \left[1 - e^{-\tau(r)}\right] p \, dp \\
&\quad + 2\pi \int_{p_0}^{r_c} I_c e^{-\tau(r)} p \, dp \\
&= \pi p_0^2 I_c + \int_{r_c}^{\infty} S(r) \left[1 - e^{-\tau(r)}\right] p \, dp
\end{aligned}
$$

$$
p_0 = \begin{cases} \sqrt{r_c^2 - z^2} & \text{for} \quad -r_c < z < 0 \\ 0 & \text{for} \quad z \leq -r_c \end{cases}
$$