Lecture 2
Solutions to the Transport Equation

Equation along a ray I

In general we can solve the static transfer equation along a ray in some particular direction. Since photons move in straight lines we want this to be a straight line (geodesic).
Then the RTE is

$$\frac{dI_\nu}{ds} = -\chi_\nu I_\nu + \eta_\nu$$

Now we just have to do the differential geometry to describe s in some coordinate system, which we saw last time.
So let's make our life as simple as possible and assume that we can work in Cartesian coordinates and that our problem is one-dimensional. Then we can let $ds \equiv dz$ and for the moment only look at rays headed directly toward us. Then the RTE becomes:

$$\frac{dI_\nu}{dz} = -\chi_\nu I + \eta_\nu$$

Equation along a ray II

We can write the equation in a somewhat more illuminating form by dividing through by χ_ν

$$\frac{dI_\nu}{d\tau_\nu} = -I + S_\nu$$

where $d\tau_\nu = \chi_\nu dz$ is the "optical depth" and $S_\nu = \eta_\nu / \chi_\nu$ is the "Source function"

Let us also suppress the frequency dependence. Then we have

$$\frac{dI}{d\tau} = S - I$$

or

$$\frac{dI}{d\tau} + I = S$$

We can solve this equation with an integrating factor to obtain

$$I = I_0 e^{-\tau} + S(1 - e^{-\tau})$$

Equation along a ray III

assuming that S is a constant.
Consider the case $I_0 = 0$. Then

$$I_\nu = S_\nu (1 - e^{-\tau_\nu})$$

Then in the case $\tau_\nu << 1$

$$I_\nu = \tau_\nu S_\nu$$

or in LTE

$$I_\nu = \tau_\nu B_\nu = \chi_\nu \, s \, B_\nu$$

so the atmosphere is bright where χ_ν is large and dim where χ_ν is small. Hence you would expect to see an emission line spectrum.
Now consider the thick case $\tau_\nu >> 1$

$$I_\nu = S_\nu = B_\nu$$

independent of χ_ν.

Line Profile

Natural Damping

Power spectrum

$$I(\omega) = \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + (\gamma/2)^2}$$

This is a Lorentzian with width

$$\Delta\lambda = \frac{2\pi c\gamma}{\omega^2} = \frac{4\pi e^2}{3mc^2} = 1.2\times 10^{-4}$$

Pressure Broadening with lead to a bigger width and we also need to convolve this with the effects of Thermal Doppler Broadening. The Doppler Broadening will give a profile

$$\phi_D = \frac{1}{\sqrt{\pi}}\frac{1}{\Delta\lambda_D}e^{-[(\lambda-\lambda_0)^2/\Delta\lambda_D]}$$

where

$$\Delta\lambda_D = \frac{\lambda_0}{c}(\frac{2kT}{m})^{1/2}$$

More on Solution

Now consider $I_0 \neq 0$

$$I = I_0 e^{-\tau} + (1 - e^{-\tau})S$$

expand $e^{-\tau} \approx 1 - \tau$

$$I = I_0 + \tau(S - I_0)$$

$$I = I_0 + \chi\, s\,(S - I_0)$$

so if $I_0 > S$ we see absorption (and it is strongest where χ is strongest, i.e. in lines if $I_0 < S$ we see emission (and it is strongest where χ is strongest, i.e. in lines.
Again in the thick case $I = S$

Line Profiles

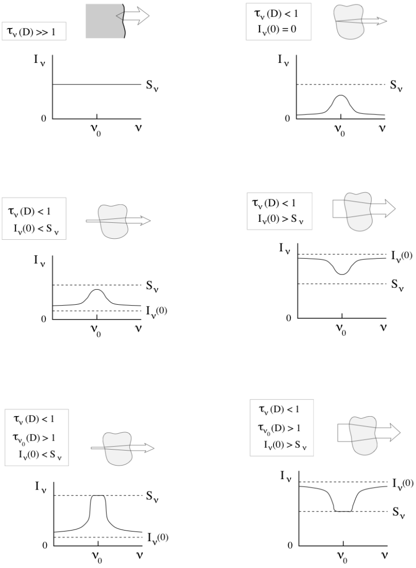


Figure 2.2: Spectral lines from a homogeneous object with $S^l_\nu = S^c_\nu = S_\nu$ everywhere, according to (2.35)–(2.36). No lines emerge when the object is optically thick (top left). When it is optically thin, emission lines emerge when the object is not back-lit ($I_\nu(0) = 0$, top right), or when it is illuminated with $I_\nu(0) < S_\nu$. Absorption lines emerge only when the object is optically thin and $I_\nu(0) > S_\nu$. The emergent lines saturate to $I_\nu \approx S_\nu$ when the object is optically thick at line center.

Simple Stars

Let's consider a star in LTE. Then

$$S_\nu = B_\nu$$

Let's cut the star into two regions bounded by $\tau = 1$. In the upper layer

$$I_0 = B_\nu(\tau > 1)$$

and

$$S_\nu = B_\nu(\tau < 1)$$

so

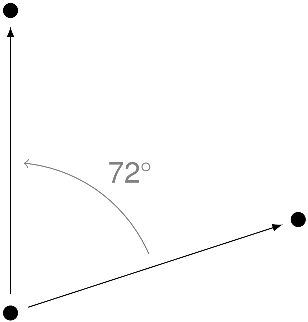
$$I_\nu = I_0 + \chi_\nu\, s\,(B_\nu(\tau < 1) - B_\nu(\tau > 1))$$

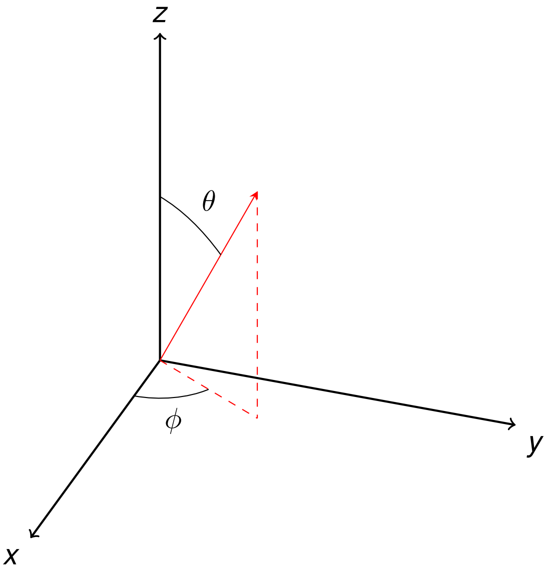
Now in stars we see absorption lines. Thus

$$B_\nu(\tau < 1) \quad < \quad B_\nu(\tau > 1)$$

which implies

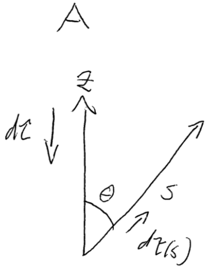
$$T(\tau < 1) \quad < \quad T(\tau > 1)$$


72°


z
θ
ϕ
y
x

General Rays

So far we have considered just one ray directed along the *z* axis. Clearly for the formal solution each ray is independent of all the others. Let us consider a general ray in the direction defined by θ. But we want to measure optical depths: 1) from the outside in; 2) along the *z* axis.



$$cos\theta d\tau(s) = -d\tau_\nu$$

Thus the RTE becomes:

$$cos\theta \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$$

it is customary to define $\mu \equiv cos\theta$

Solution for Plane-Parallel Atmosphere

Let us assume that we have a semi-infinite slab and we wish to know the $I(\tau = 0, \mu)$. Then we find:

$$I(\tau = 0, \mu) = \int_0^\infty S_\nu(\tau_\nu) e^{-\tau_\nu/\mu} d\tau_\nu/\mu$$

Now $\tau_\nu/\mu = \tau(s)$ so this just says that the surface intensity is the sum over the path of the emission (source function).

Eddington Barbier Relation

Let's assume

$$S_\nu(\tau_\nu) = a_\nu + b_\nu \tau_\nu$$

$$I(\tau = 0, \mu) = \int_0^\infty S_\nu(\tau_\nu) e^{-\tau_\nu/\mu} d\tau_\nu/\mu$$

$$I(\tau = 0, \mu) = \int_0^\infty [a_\nu + b_\nu \tau_\nu] e^{-\tau_\nu/\mu} d\tau_\nu/\mu$$

$$I(\tau = 0, \mu) = a_\nu + b_\nu \mu = S_\nu(\tau_\nu = \mu)$$

Thus, the measurement of the emergent intensity as a function of μ gives us information about S as a function of depth.

Eddington-Barbier and Limb Darkening

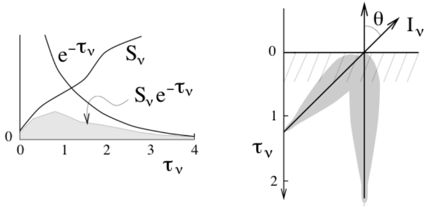


Figure 2.3: The Eddington-Barbier approximation. Left: the integrand $S_\nu \exp(-\tau_\nu)$ measures the contribution to the radially emergent intensity $I_\nu(\tau_\nu\!=\!0, \mu\!=\!1)$ from layers with different optical depth τ_ν. The value of S_ν at $\tau_\nu = 1$ is a good estimator of the area under the integrand curve, *i.e.*, the total contribution. Right: for a slanted beam the characteristic Eddington-Barbier depth is shallower than for a radial beam; it lies at $\tau_\nu = \mu$.

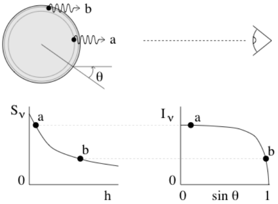


Figure 2.4: Solar limb darkening. The viewing angle θ increases with the fractional radius $r/R_\odot = \sin\theta$ of the apparent solar disk. The emergent intensity samples shallower layers towards the limb, with smaller source function. The final drop at $r/R_\odot = 1$ marks the viewing angle at which the sun becomes optically thin. Note that substantial decrease of $\mu = \cos\theta$ is reached only close to the limb, for $r/R_\odot = \sin\theta = (1-\mu^2)^{1/2}$ close to unity (Table 7.2 on page 159). The off-limb extension to this sketch is given in Figure 7.2 on page 148.

General Formal Solution I

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$$

$$I(\tau,\mu) = -\frac{1}{\mu}\int_C^\tau e^{(\tau-t)/\mu} S(t)\,dt + e^{(\tau-C)/\mu} I(C,\mu)$$

Clearly the Formal Solution is a boundary value problem. But the boundary depends on the sign of μ. For out-going rays, $\mu \geq 0$ I is specified at the base of the atmosphere. For in-going rays, $\mu < 0$ I is specified at the surface of the atmosphere. We will generally assume that there is no external illumination and the $I(\tau = 0, \mu < 0) = 0$. In that case, unless $I(\tau_{\max}, \mu)$ increases exponentially with $\tau_{\max}$ the boundary term $\rightarrow 0$ as $\tau_{\max} \rightarrow \infty$
Then for large $\tau_{\max}$ we have:

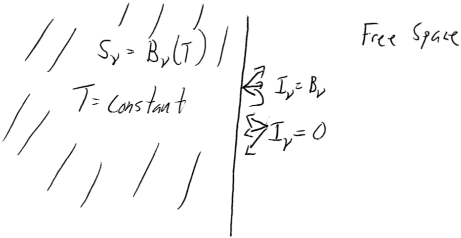
$$I(\tau,\mu) = \frac{1}{\mu}\int_\tau^\infty e^{-(t-\tau)/\mu} S(t)\,dt \; ; \quad \mu \geq 0$$

General Formal Solution II

and the emergent intensity in the semi-infinite case is:

$$I(0, \mu) = \frac{1}{\mu} \int_0^\infty e^{-t/\mu} S(t)\, dt \; ; \quad \mu \geq 0$$

More on the Source Function I



In strict *thermal equilibrium* the RTE is:

$$\frac{dB_\nu}{ds} = -\chi_\nu (B_\nu - S_\nu) = 0 \rightarrow S_\nu = B_\nu(T)$$

While $I_\nu = B_\nu$ in all directions implies $S_\nu = B_\nu$, the converse is NOT true.

$$S_\nu = B_\nu \not\rightarrow I_\nu = B_\nu$$

More on the Source Function II

At the surface $I_\nu(\mu \geq 0) = B_\nu(T)$ and $I_\nu(\mu < 0) = 0$. Thus

$$J = 1/2 \int_{-1}^{1} I_\nu \, d\mu = 1/2 B_\nu(T)$$

$$H = 1/2 \int_{-1}^{1} I_\nu \, \mu \, d\mu = 1/4 B_\nu(T)$$

$$\mathcal{F}_\nu = 4\pi H_\nu = \pi B_\nu$$

$$B = \int_0^\infty B_\nu \, d\nu = \frac{\sigma}{\pi} T^4 \quad \sigma = \frac{2\pi^5 k^4}{15 h^3 c^2} = \text{Stephan's constant}$$

Effective Temperature of a star defined by

$$\mathcal{F} = \sigma T_{\text{eff}}^4 \text{ or } L = 4\pi R_*^2 \sigma T_{\text{eff}}^4$$

In LTE $S_\nu = B_\nu(T(r))$, then we get I_ν from the formal solution. Another standard case is $S_\nu = J_\nu$: pure, monochromatic (coherent), isotropic scattering. This follows from integrating the RTE over all

More on the Source Function III

directions and equating power absorbed to power emitted in a given volume.
If we combine thermal emission and scattering we get that the RTE is given by

$$\begin{aligned} \frac{dI_\nu}{ds} &= -\kappa_\nu(I_\nu - B_\nu) - \sigma_\nu(I_\nu - J_\nu) \\ &= -(\kappa_\nu + \sigma_\nu)(I_\nu - S_\nu) \\ \rightarrow S_\nu &= (1 - \epsilon_\nu)J_\nu + \epsilon_\nu B_\nu \end{aligned} \tag{1}$$

$$\epsilon_\nu \equiv \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu}$$

$$\chi_\nu \equiv \kappa_\nu + \sigma_\nu$$

Moment Equations I

For a plane-parallel atmosphere the RTE is:

$$\mu \frac{dI_\nu}{dz} = -\chi_\nu (I_\nu - S_\nu)$$

Let S_ν and χ_ν be independent of μ
Recall

$$J_\nu = 1/2 \int_{-1}^{1} I_\nu \, d\mu$$

$$H_\nu = 1/2 \int_{-1}^{1} I_\nu \, \mu \, d\mu$$

$$K_\nu = 1/2 \int_{-1}^{1} I_\nu \, \mu^2 \, d\mu$$

Integrating the RTE over μ gives

$$\frac{dH_\nu}{dz} = -\chi_\nu (J_\nu - S_\nu)$$

Moment Equations II

or multiplying the RTE by μ and integrating

$$\frac{dK_\nu}{dz} = -\chi_\nu H_\nu$$

The net flux integrated over all frequencies is

$$H = \int_0^\infty H_\nu \, d\nu$$

In Radiative equilibrium (only radiation carries energy)

$$\frac{dH}{dz} = \int_0^\infty \chi_\nu (J_\nu - S_\nu) \, d\nu = 0$$

For combined thermal emission and scattering

$$S_\nu = \epsilon_\nu B_\nu + (1 - \epsilon_\nu) J_\nu$$

$$J_\nu - S_\nu = \epsilon_\nu (J_\nu - B_\nu)$$

Moment Equations III

Thus, Radiative Equilibrium $\rightarrow$

$$\int_0^\infty \kappa_\nu (J_\nu - B_\nu)\, d\nu = 0$$

scattering drops out.

Two-Stream Eddington Approximation I

Assume

$$I(\mu) = \begin{cases} I^+ & \mu \geq 0 \\ I^- & \mu < 0 \end{cases}$$

Where $I^\pm = \mathrm{constant}$ Then

$$J_\nu = 1/2 \int_{-1}^{1} I_\nu \, d\mu = 1/2(I^+ + I^-)$$

$$H_\nu = 1/2 \int_{-1}^{1} I_\nu \, \mu \, d\mu = 1/4(I^+ - I^-)$$

$$K_\nu = 1/2 \int_{-1}^{1} I_\nu \, \mu^2 \, d\mu = 1/6(I^+ + I^-) = 1/3 J_\nu$$

From the moment equations we have

$$\frac{dK}{d\tau} = H \rightarrow 1/3 \frac{dJ}{d\tau} = H$$

Two-Stream Eddington Approximation II

Then

$$J = 3H\tau + \text{constant}$$

at $\tau = 0$

$$J = 1/2 I_0^+ \quad H = 1/4 I_0^+$$

$$\rightarrow J = 2H$$

$$J(\tau) = 3H(\tau + 2/3)$$

In RE $J = B$

$$B = 3H(\tau + 2/3)$$

$$\frac{\sigma T^4}{\pi} = 3\frac{\sigma T_{\text{eff}}^4}{4\pi}(\tau + 2/3)$$

$\rightarrow$

$$T = T_{\text{eff}} \text{ at } \tau = 2/3$$

Exponential Integral Solution I

The solution to the RTE is really not I_ν, but rather J_ν, since once we have J_ν we know S_ν and we *just* have to perform a formal solution. From the general expression for $I_\nu(\tau, \mu)$ we can write

$$
\begin{aligned}
J_\nu &= 1/2 \int_{-1}^{0} - \left[\frac{1}{\mu} \int_0^{\tau} e^{(\tau - t)/\mu} S(t)\, dt + e^{\tau/\mu} I(0, \mu) \right] d\mu \\
&+ 1/2 \int_0^1 \left[\frac{1}{\mu} \int_{\tau}^{\tau_{\max}} e^{-(t-\tau)/\mu} S(t)\, dt \right. \\
&+ \left. e^{-(\tau_{\max} - \tau)/\mu} I(\tau_{\max}, \mu) \right] d\mu
\end{aligned}
$$

For simplicity consider zero incident intensity at both boundaries. Define:

$$
E_1(x) = \int_0^1 \frac{e^{-x/\mu}}{\mu}\, d\mu
$$

Exponential Integral Solution II

where E_1 is the first exponential integral. Let $y = 1/\mu$, then

$$E_1(x) = \int_0^\infty \frac{e^{-xy}}{y}\, dy = \int_x^\infty \frac{e^{-z}}{z}\, dz$$

where $z = xy$. For large x

$$E_1(x) \approx e^{-x}/x$$

Then we can write

$$J(\tau) = 1/2 \int_0^{\tau_{\rm max}} E_1(|t-\tau|) S(t)\, dt + \text{ incident terms}$$

$$J(\tau) = \Lambda_\tau\{S\}$$

Exponential Integral Solution III

Similarly

$$
\begin{aligned}
H_\nu(\tau) &= 1/2 \int_{-1}^{1} I(\tau,\mu)\mu \, d\mu \\
&+ 1/2 \int_{-1}^{0} - \left[\int_0^{\tau} e^{(\tau - t)/\mu} S(t)\, dt + e^{\tau/\mu} I(0,\mu) \right] d\mu \\
&+ 1/2 \int_{0}^{1} \left[\int_{\tau}^{\tau_{\max}} e^{-(t-\tau)/\mu} S(t)\, dt + e^{-(\tau_{\max}-\tau)/\mu} I(\tau_{\max},\mu) \right] d\mu
\end{aligned}
$$

For simplicity again consider zero incident intensity at both boundaries.
Define:

$$E_n(x) = x^{n-1} \int_x^{\infty} \frac{e^{-z}}{z^n}\, dz$$

Then

$$E_n'(x) = -E_{n-1}(x)$$

Exponential Integral Solution IV

and

$$E_n(x) = \left[e^{-x} - xE_{n-1}(x)\right] /(n-1)$$

Then we can write

$$\begin{aligned} H(\tau) &= -1/2\int_0^{\tau} E_2(\tau - t)S(t)\,dt + 1/2\int_{\tau}^{\tau_{\max}} E_2(t-\tau)S(t)\,dt \\ &+ \text{incident terms} \end{aligned}$$

$$H(\tau) = 1/2\int_0^{\tau_{\max}} E_2(|t-\tau|)sgn(t-\tau)S(t)\,dt$$

$$H(\tau) = \Phi_\tau\{S\}$$

Similarly,

$$K(\tau) = 1/2\int_0^{\tau_{\max}} E_3(|t-\tau|)S(t)\,dt$$

$$K(\tau) = X_\tau\{S\}$$

Exponential Integral Solution V

Can show

$$\Lambda_\tau(a+bt) = a + b\tau + 1/2\,[bE_3(\tau) - aE_2(\tau)]$$

$$\Phi_\tau(a+bt) = 4/3b + 2\,[aE_3(\tau) - bE_4(\tau)]$$

$$X_\tau(a+bt) = 4/3(a+b\tau) + 2\,[bE_5(\tau) - aE_4(\tau)]$$