Lecture 1

Introduction and Course Overview

- Introduction
- Formal Solution
- Spherical Geometry
- Basic Moving Atmospheres
- Rate Equations, Temperature Correction
- Sobolev Theory
- 3-D Radiative Transfer

Credits

I have learned Radiative Transfer directly from 20 years of talking to Peter Hauschildt and David Branch. Formally I have learned it through reading and re-reading Mihalas (1978) as well as the lecture notes of Pomraning. Many of the illustrations I will present were liberated from the lecture notes of Rutten and from an unpublished copy of Mihalas & Hubeny. I also have benefited from the notes, books, and papers of Castor, Avrett, Höflich, Rybicki & Lightman as well as my students and postdocs and many others. Stealing from Mihalas and Binney "Any errors that remain after the conscientious tutelage of my colleagues are solely my own"

Why Study Radiative Transfer?

Edwin Salpeter to Dimitri Mihalas:

> *Why in the world would anyone want to study stellar atmospheres? They contain only* 10^{-10} *of the mass of a typical star; surely such a negliable fraction of a star's mass cannot affect the overall structure and evolution.*

Answer

- The atmosphere is what we see
- The atmosphere *does* affect the evolution!
  - In extended atmospheres R is not well defined, but that occurs at sensitive evolutionary stages, i.e., giant and supergiant branch
  - Mass loss
- Interesting Technical Problem
- Most of what we know in astronomy comes from the observation of radiation

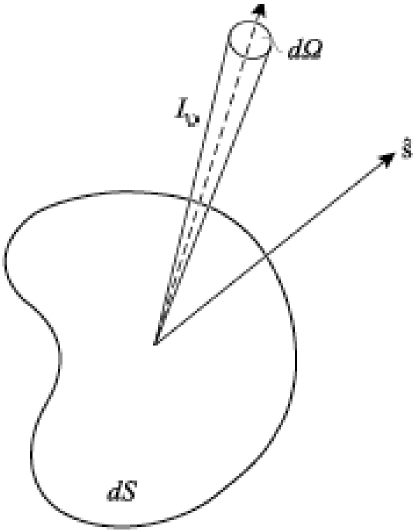
Bottom line: Seek Understanding

Definition of Specific Intensity I_ν

Radiation consists of an ensemble of photons that travel along geodesics at the speed of light in all frames. Since we aren't doing General Relativity that means that photons travel along straight lines at the speed of light *in all frames*.

The specific intensity $I_\nu(\mathbf{x}, t; \hat{\mathbf{n}})$ [ergs/cm^2/s/sr/Hz] at position $\mathbf{x}$, at time t, traveling in direction $\hat{\mathbf{n}}$ into solid angle $d\Omega$, with frequency in the range $(\nu, \nu + d\nu)$ is defined such that the energy in an area element $d\mathbf{S}$, in a time interval dt is

$$\begin{aligned} dE &\equiv I_\nu(\mathbf{x}, t; \hat{\mathbf{n}})\, \hat{\mathbf{n}} \cdot d\mathbf{S}\, d\Omega\, d\nu\, dt \ \text{[ergs]} \\ &\equiv I_\lambda(\mathbf{x}, t; \hat{\mathbf{n}})\, \hat{\mathbf{n}} \cdot d\mathbf{S}\, d\Omega\, d\lambda\, dt \ \text{[ergs]} \\ &\equiv h\nu c \psi_\nu(\mathbf{x}, t; \hat{\mathbf{n}})\, \hat{\mathbf{n}} \cdot d\mathbf{S}\, d\Omega\, d\lambda\, dt \ \text{[ergs]} \end{aligned}$$



$d\Omega$
I_ν
$\hat{s}$
dS

Photon Number Density

$$\psi_\nu = I_\nu / h\nu c \text{ photons/cm}^3\text{/Sr/Hz}$$

$$\psi(\mathbf{x}, t) = \int_0^\infty d\nu \oint d\Omega \psi_\nu \text{ photons/cm}^3$$

Relationship between I_ν and Distribution Function I

$$\mathbf{p} = h\nu/c\hat{\mathbf{n}}$$

f_α is the number of particles having spin α per unit volume of phase space. Photons are massless bosons with spin 1, and since they are massless they can only have two polarization states, aligned and anti-aligned. Thus, $\alpha = 1, 2$. For unpolarized light $f_1 = f_2 = 2f$. We will assume unpolarized light in what follows. So the occupation number of photons in a volume of phase space

$$dn_\gamma = \frac{2}{h^3} f h^3 d^3x\, d^3p$$

Note the extra h^3, this is a definition, see SA3 page 63. Let us define the total radiation distribution function as

$$f_R \equiv \sum_\alpha f_\alpha = 2f$$

Relationship between I_ν and Distribution Function II

We can write the momentum space volume as

$$d^3p = p^2 dp d\Omega = h^3\nu^2/c^3 d\Omega\, d\nu$$

$$f_R(\mathbf{x}, t; \mathbf{p}) \to h^3\nu^2/c^3 f_R(\mathbf{x}, t; \hat{\mathbf{n}}, \nu) d^3x d\Omega d\nu$$

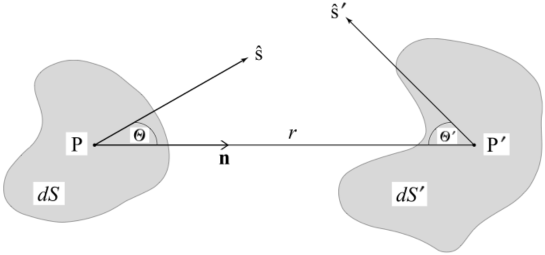
Then $(h^3\nu^2/c^3) f_R(\mathbf{x}, t; \hat{\mathbf{n}}) \hat{\mathbf{n}} \cdot d\mathbb{S}\, cdt\, d\Omega d\nu$ is the number of unpolarized photons at position $\mathbf{x}$ and time t, having frequencies in $(\nu, \nu + d\nu)$, propagating in direction $\hat{\mathbf{n}}$, thru and area $d\mathbf{S}$, into solid angle $d\Omega$ in a time interval dt. The energy they transport through $d\mathbb{S}$ is

$$\begin{aligned} dE &\equiv (h\nu)(h^3\nu^2/c^2) f_R((x), t; \hat{\mathbf{n}}, \nu) \hat{\mathbf{n}} \cdot d\mathbb{S} dt d\Omega d\nu \\ &= I_\nu(\mathbf{x}, t; \hat{\mathbf{n}}) \hat{\mathbf{n}} \cdot d\mathbb{S} dt d\Omega \end{aligned}$$

Therefore

$$f_R(\mathbf{x}, t; \hat{\mathbf{n}}, \nu) = c^2/h^4 I_\nu(\mathbf{x}, t; \hat{\mathbf{n}})/\nu^3$$

Variation with Distance I



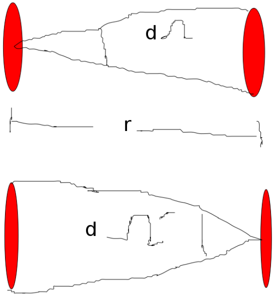
I_ν from a resolved source traveling in vacuum is an invariant. Consider a beam of photons moving through vacuum in direction $\hat{\mathbf{n}}$. The energy passing through $d\mathbf{S}$ is

$$dE = I_\nu \hat{\mathbf{n}} \cdot d\mathbf{S} d\Omega \, d\nu \, dt$$

and the energy passing through $d\mathbf{S}'$ is

$$dE' = I'_\nu \hat{\mathbf{n}} \cdot d\mathbf{S}' d\Omega' \, d\nu \, dt$$

I_ν is Invariant I



but then

$$d\Omega = dS' cos\Theta' / r^2$$

and

$$d\Omega' = dS cos\Theta / r^2$$

I_ν is Invariant II

and since energy is conserved

$$dE = dE'$$

thus

$$I_\nu = I'_\nu$$

Eddington Moments

Define $\mu \equiv cos\Theta$

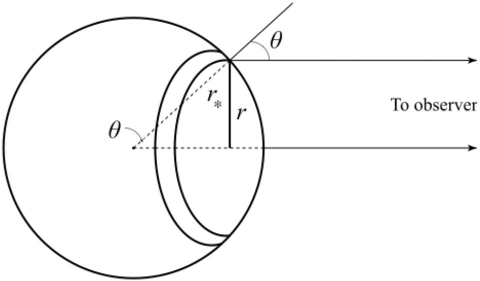
$$
\begin{aligned}
J_\nu &= \frac{1}{4\pi}\int I_\nu \, d\Omega \\
H_\nu &= \frac{1}{4\pi}\int I_\nu \, \mu d\Omega \\
K_\nu &= \frac{1}{4\pi}\int I_\nu \, \mu^2 d\Omega \\
\varepsilon_\nu &= \frac{4\pi}{c} J_\nu \\
\mathcal{F}_\nu &= 4\pi H_\nu \\
P_\nu &= \frac{4\pi}{c} K_\nu
\end{aligned}
$$

Azimuthal Invariance

We will assume that I_ν is independent of Φ, the azimuthal orientation of the surface, thus since $d\Omega = sin\Theta d\Theta \, d\Phi$

$$
\begin{aligned}
J_\nu &= \frac{1}{2}\int_{-1}^{1} I_\nu \, d\mu \\
H_\nu &= \frac{1}{2}\int_{-1}^{1} I_\nu \, \mu d\mu \\
K_\nu &= \frac{1}{2}\int_{-1}^{1} I_\nu \, \mu^2 d\mu
\end{aligned}
$$

Observed Fluence I



The energy per unit detector area (normal to the line of sight) received from a differential area on the stellar disk seen projected on the sky is $df_{\rm obs}(\nu) = I(r_*, \mu, \nu) d\omega$, where $d\omega$ is the solid angle subtended by the stellar surface. From the geometry $r = r_* sin\theta$ so the area of the differential annulus $dS = 2\pi rdr = 2\pi r_* sin\theta r_* cos\theta d\theta = 2\pi r_*^2 \mu d\mu$. Thus,

Observed Fluence II

$$f_{\text{obs}}(\nu) = 2\pi(r_*/D)^2 \int_0^1 I(r_*, \mu, \nu)\mu d\mu = (\frac{r_*}{D})^2 \mathcal{F}_\nu(r_*)$$

but we've pulled a fast one.

Absorption, Emission, and Scattering I

As photons travel through stellar material they are absorbed, emitted, and scattered. We describe these processes with macroscopic quantities in the transport equation. The material's total extinction coefficient χ_ν represents its ability to remove photons from the beam at frequency ν. The material's emissivity η_ν is the rate at which photons are fed into the beam at frequency ν.
The energy removed from the beam is

$$dE_\nu = -\chi_\nu(\mathbf{x}, t; \hat{\mathbf{n}}) I_\nu(\mathbf{x}, t; \hat{\mathbf{n}}) dS ds d\Omega d\nu dt$$

Where χ_ν contains terms of the form $n_i \alpha_i(\nu)$, where n_i is the number density of state i and α_i is the cross section. χ_ν has units of cm^{-1} and $1/\chi_\nu$ is the mean free path. The total extinction is usually broken up into two parts, the thermal absorption coefficient (or just the opacity) κ_ν and the scattering coefficient σ_ν

$$\chi_\nu = \kappa_\nu + \sigma_\nu$$

Absorption, Emission, and Scattering II

The total thermal emissivity consists of spontaneous and induced emission

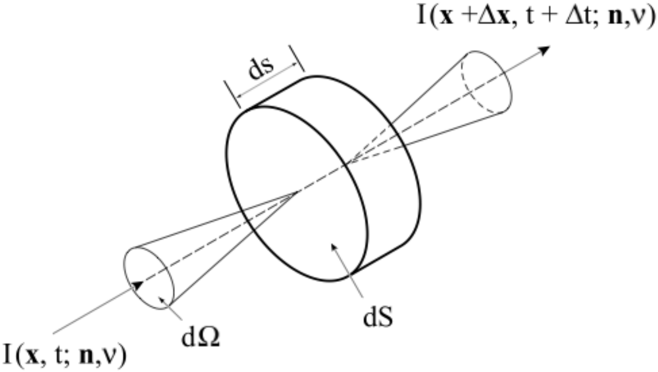
$$\eta_\nu = \eta_\nu^s + \eta_\nu^i$$

Spontaneous emission is isotropic in the co-moving frame. Induced emission is proportional to the specific intensity and thus is often treated as a correction to the opacity.

Distinction between Absorption and Scattering

In absorption a photon is destroyed and its energy is delivered to the thermal pool. In scattering a photon's direction is changed with either no change in the photon energy or only a small change. What's the difference? The rate at which energy is added or subtracted from the beam due to scattering depends on the local value of the radiation field which may have originated at a distant point and thus is only weakly coupled to the local properties of the gas. Absorption feeds energy directly into the thermal pool and thus is tightly coupled with the local gas properties. Thus, scattering processes tends to de-localize the gas-radiation equilibration process.

General Coordinates



The Equation of Radiative Transfer I

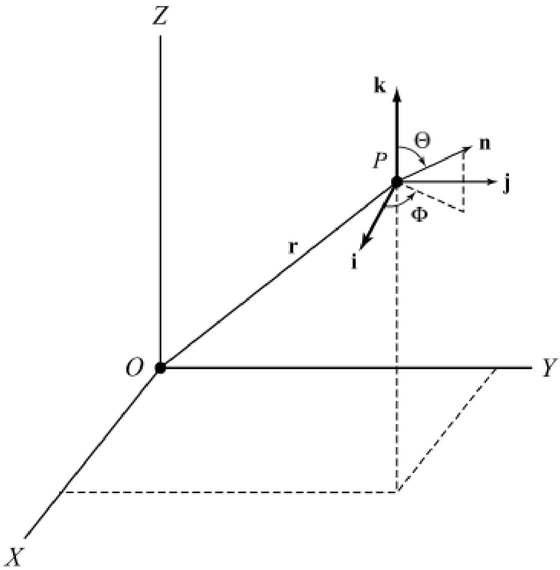
$$\delta I_\nu(\mathbf{x}, t; \hat{\mathbf{n}}) dS d\Omega d\nu dt = [\eta_\nu(\mathbf{x}, t; \hat{\mathbf{n}}) - \chi_\nu(\mathbf{x}, t; \hat{\mathbf{n}}) I_\nu(\mathbf{x}, t; \hat{\mathbf{n}})] dS d\Omega d\nu dt ds$$

$$\frac{dI_\nu(\mathbf{x}, t; \hat{\mathbf{n}})}{ds} = \eta_\nu(\mathbf{x}, t; \hat{\mathbf{n}}) - \chi_\nu(\mathbf{x}, t; \hat{\mathbf{n}}) I_\nu(\mathbf{x}, t; \hat{\mathbf{n}})$$

$\frac{dI}{ds}$ is the total derivative of I with respect to the pathlength s along a ray, which is a geodesic in spacetime. Introducing a specific set of coordinates we write

$$\begin{aligned} \frac{dI_\nu(\mathbf{x}, t; \hat{\mathbf{n}})}{ds} &= \frac{dt}{ds}\frac{\partial I_\nu}{\partial t} + \sum_{i=1}^{3} \frac{dx^i}{ds}\frac{\partial I_\nu}{dx^i} + \frac{d\Theta}{ds}\frac{\partial I_\nu}{\partial \Theta} + \frac{d\Phi}{ds}\frac{\partial I_\nu}{\partial \Phi} + \frac{d\nu}{ds}\frac{\partial I_\nu}{\partial \nu} \\ &= \eta_\nu(\mathbf{x}, t; \hat{\mathbf{n}}) - \chi_\nu(\mathbf{x}, t; \hat{\mathbf{n}}) I_\nu(\mathbf{x}, t; \hat{\mathbf{n}}) \end{aligned}$$

Cartesian Coordinates I



RTE in Cartesian Coordinates I

$$\begin{aligned}\hat{\mathbf{n}} &= n_x\hat{\mathbf{i}} + n_y\hat{\mathbf{j}} + n_z\hat{\mathbf{k}} \\ &= sin\Theta cos\Phi\hat{\mathbf{i}} + sin\Theta sin\Phi\hat{\mathbf{j}} + cos\Theta\hat{\mathbf{k}}\end{aligned}$$

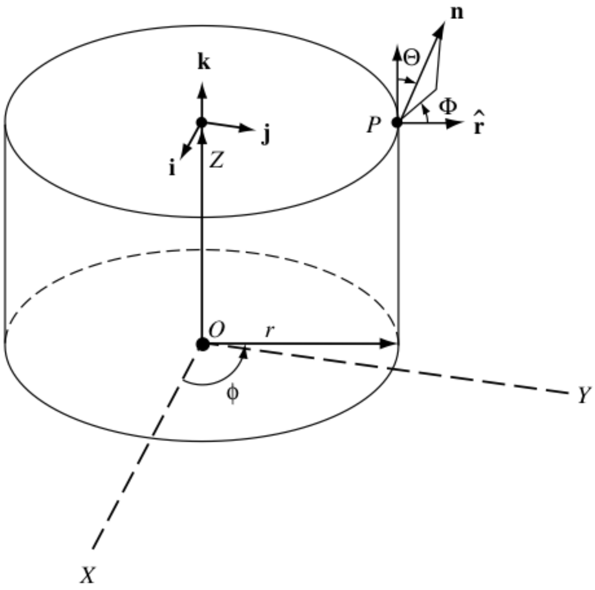
$$\begin{aligned}\frac{dt}{ds} &= \frac{1}{c} \\ \frac{dx}{ds} &= n_x \quad , \frac{dy}{ds} = n_y, \ \frac{dz}{ds} = n_z \\ \frac{d\Theta}{ds} &= 0 \quad , \frac{d\Phi}{ds} = 0, \ \frac{d\nu}{ds} = 0\end{aligned}$$

RTE in Cartesian Coordinates II

$$
\begin{aligned}
\frac{dI_\nu}{ds} &= \frac{1}{c}\frac{\partial I_\nu}{\partial t} + n_x \frac{\partial I_\nu}{\partial x} + n_y \frac{\partial I_\nu}{\partial y} + n_z \frac{\partial I_\nu}{\partial z} \\
&= \frac{1}{c}\frac{\partial I_\nu}{\partial t} + \hat{\mathbf{n}} \cdot \nabla I_\nu \\
&= \eta_\nu(\mathbf{x}, t; \hat{\mathbf{n}}) - \chi_\nu(\mathbf{x}, t; \hat{\mathbf{n}}) I_\nu(\mathbf{x}, t; \hat{\mathbf{n}})
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{c}\frac{\partial I_\nu}{\partial t} &+ (1-\mu^2)^{1/2} cos\Phi \frac{\partial I_\nu}{\partial x} + (1-\mu^2)^{1/2} sin\Phi \frac{\partial I_\nu}{\partial y} + \mu \frac{\partial I_\nu}{\partial z} \\
&= \eta_\nu(\mathbf{x}, t; \hat{\mathbf{n}}) - \chi_\nu(\mathbf{x}, t; \hat{\mathbf{n}}) I_\nu(\mathbf{x}, t; \hat{\mathbf{n}})
\end{aligned}
$$

Cylindrical Coordinates I



RTE in Cylindrical Coordinates I

$$\hat{\mathbf{r}} = cos\phi\hat{\mathbf{i}} + sin\phi\hat{\mathbf{j}},\ \hat{\mathbf{z}} = \hat{\mathbf{k}},\ \hat{\phi} = -sin\theta\hat{\mathbf{i}} + cos\theta\hat{\mathbf{j}}$$

$$\hat{\mathbf{n}} = \sin\Theta cos\Phi\hat{\mathbf{r}} + \cos\Theta\hat{\mathbf{k}} + \sin\Theta\sin\Phi\hat{\phi}$$

$$\hat{\mathbf{n}} = (\cos\Phi\cos\phi - \sin\Phi\sin\phi)\sin\Theta\hat{\mathbf{i}} + (\cos\Theta\sin\phi + \sin\Phi\cos\phi)\sin\Theta\hat{\mathbf{j}} + \cos\Theta\hat{\mathbf{k}}$$

$$d\mathbf{s} = (\sin\Theta cos\Phi\hat{\mathbf{r}} + cos\Theta\hat{\mathbf{k}} + \sin\Theta sin\Phi\hat{\phi})ds$$

$$dr = \hat{\mathbf{r}} \cdot d\mathbf{s} = \sin\Theta\cos\Phi ds$$

$$dz = \hat{\mathbf{k}} \cdot d\mathbf{s} = \cos\Theta ds$$

$$rd\phi = \hat{\phi} \cdot d\mathbf{s} = \sin\Theta\sin\Phi ds$$

From above, due to ds the basis vectors $\hat{\mathbf{r}}$ and $\hat{\mathbf{phi}}$ at P' will have rotated by an angle $d\phi$ around the $z-$axis with respect to those at

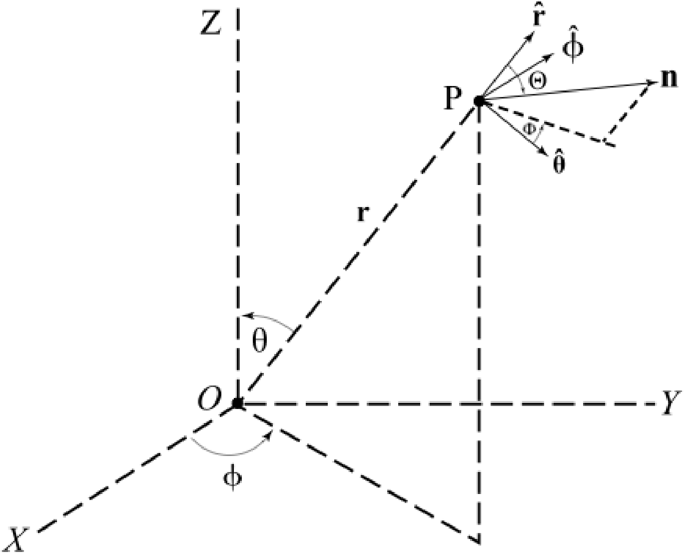
RTE in Cylindrical Coordinates II

point P. This implies that, in principle, Φ may change to $\Phi + d\Phi$ and Θ may change to $\Theta + d\Theta$. But $\hat{\mathbf{n}}$ is fixed in space, so (n_x, n_y, n_z) are constant WRT the fixed basis, $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. Thus, from the above expression for $\hat{\mathbf{n}}$ in $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ you can derive (dn_x, dn_y, dn_z) due to variations in Φ and Θ and demand that they are zero. This gives:

$$
\begin{aligned}
\frac{dr}{ds} &= sin\Theta cos\Phi, \quad \frac{dz}{ds} = cos\Phi, \quad \frac{d\phi}{ds} = sin\Theta sin\Phi/r \\
\frac{d\Theta}{ds} &= 0, \qquad \frac{d\Phi}{ds} = -sin\Theta cos\Phi/r, \quad \frac{d\nu}{ds} = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{c}\frac{\partial I_\nu}{\partial t} &+ sin\Theta cos\Phi \frac{\partial I_\nu}{\partial r} + cos\Theta \frac{\partial I_\nu}{\partial z} + \frac{sin\Theta sin\Phi}{r}\frac{\partial I_\nu}{\partial \phi} - \frac{sin\Theta sin\Phi}{r}\frac{\partial I_\nu}{\partial \Phi} \\
&= \eta_\nu(\mathbf{x}, t; \hat{\mathbf{n}}) - \chi_\nu(\mathbf{x}, t; \hat{\mathbf{n}}) I_\nu(\mathbf{x}, t; \hat{\mathbf{n}})
\end{aligned}
$$

Spherical Coordinates I



RTE in Spherical Coordinates I

$$\hat{\mathbf{r}} = sin\theta cos\phi\hat{\mathbf{i}} + sin\theta sin\phi\hat{\mathbf{j}} + cos\theta\hat{\mathbf{k}}$$

$$\hat{\theta} = cos\theta cos\phi\hat{\mathbf{i}} + cos\theta sin\phi\hat{\mathbf{j}} - sin\theta\hat{\mathbf{k}}$$

$$\hat{\phi} = -sin\theta\hat{\mathbf{i}} + cos\theta\hat{\mathbf{j}}$$

$$\hat{\mathbf{n}} = \cos\Theta\hat{\mathbf{r}} + sin\Theta cos\Phi\hat{\theta} + \sin\Theta sin\Phi\hat{\phi}$$

$$\hat{\mathbf{n}} = [\sin\theta\cos\phi\cos\Theta + (\cos\theta\cos\phi\cos\Phi - \sin\phi\sin\Phi)\sin\Theta]\hat{\mathbf{i}} + [\cos\theta\sin\phi\cos\Theta + (\cos\theta\sin\phi\cos\Phi + \cos\phi\sin\Phi)\sin\Theta]\hat{\mathbf{j}} + (\cos\theta\cos\Theta - \sin\theta\cos\Phi\sin\Theta)\hat{\mathbf{k}}$$

$$d\mathbf{s} = (\sin\Theta cos\Phi\hat{\mathbf{r}} + cos\Theta\hat{\mathbf{k}} + \sin\Theta sin\Phi\hat{\phi})ds$$

$$dr = \hat{\mathbf{r}} \cdot d\mathbf{s} = \cos\Theta ds$$

$$rd\theta = \hat{\theta} \cdot d\mathbf{s} = \sin\Theta\cos\Phi ds$$

RTE in Spherical Coordinates II

$$r \sin\theta d\phi = \hat{\phi} \cdot d\mathbf{s} = \sin\Theta \sin\Phi ds$$

$$d\mathbf{s} = (cos\Theta\hat{\mathbf{r}} + sin\Theta cos\Phi\hat{\theta} + \sin\Theta sin\Phi\hat{\phi})ds$$

$$\frac{dr}{ds} = \quad cos\Theta \quad \frac{d\theta}{ds} = \frac{sin\Theta cos\Phi}{r}, \; \frac{d\phi}{ds} = \frac{sin\Theta sin\Phi}{rsin\theta}$$

At P', both Θ and Φ have changed because in addition to a change in (r,θ,ϕ), the local basis set $(\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi})$ has rotated. But $\hat{\mathbf{n}}$ is fixed in space, so $(n_x, \; n_y, \; n_z)$ are constant WRT the fixed basis, $\hat{\mathbf{i}}, \; \hat{\mathbf{j}}, \; \hat{\mathbf{k}}$. Thus, from the above expression for $\hat{\mathbf{n}}$ in $\hat{\mathbf{i}}, \; \hat{\mathbf{j}}, \; \hat{\mathbf{k}}$ you can derive $(dn_x, \; dn_y, \; dn_z)$ due to variations in Φ and Θ and demand that they are zero. This gives:

$$\frac{d\Theta}{ds} = \quad -\frac{sin\Theta}{r} \quad \frac{d\Phi}{ds} = -\frac{sin\Theta sin\Phi cot\theta}{r}, \; \frac{d\nu}{ds} = 0$$

RTE in Spherical Coordinates III

$$\begin{aligned}
\frac{1}{c}\frac{\partial I_\nu}{\partial t} &+ cos\Theta\frac{\partial I_\nu}{\partial r} + \frac{sin\Theta cos\Phi}{r}\frac{\partial I_\nu}{\partial \theta} \\
&+ \frac{sin\Theta sin\Phi}{r sin\theta}\frac{\partial I_\nu}{\partial \phi} - \frac{sin\Theta}{r}\frac{\partial I_\nu}{\partial \Theta} - \frac{sin\Theta sin\Phi cot\theta}{r}\frac{\partial I_\nu}{\partial \Phi} \\
&= \eta_\nu(\mathbf{x}, t; \hat{\mathbf{n}}) - \chi_\nu(\mathbf{x}, t; \hat{\mathbf{n}}) I_\nu(\mathbf{x}, t; \hat{\mathbf{n}})
\end{aligned}$$

In a spherically symmetric atmosphere I depends only on the coordinate r and by symmetry is independent of Φ. Defining $\mu = cos\Theta$, the equation of radative transfer reduces to

$$\begin{aligned}
\frac{1}{c}\frac{\partial I_\nu}{\partial t} &+ \mu\frac{\partial I_\nu}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I_\nu}{\partial \mu} \\
&= \eta_\nu(r, t; \mu) - \chi_\nu(r, t; \mu) I_\nu(r, t; \mu)
\end{aligned}$$