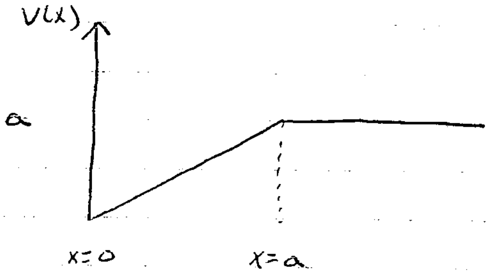
H.W. solutions
#9

1)



note since region does not extend to ∞ cannot remove Bi term

$x<0 \qquad \psi(x)=0$

$x>a \qquad \psi(x)=Ce^{-kx} \qquad k=\frac{\sqrt{2m(a-E)}}{\hbar}$

$0<x<a \qquad \psi(x)=b\,Ai\left(\frac{x-\sigma}{\rho}\right)+f\,Bi\left(\frac{x-\sigma}{\rho}\right) \qquad \rho=\left(\frac{\hbar^2}{2m}\right)^{1/3} \quad \sigma=E$

B.C. at $x=0$

$\psi(0)=0 \Rightarrow b\,Ai\left(\frac{-\sigma}{\rho}\right)+f\,Bi\left(\frac{-\sigma}{\rho}\right)=0$

① $$b=\frac{-f\,Bi\left(\frac{-\sigma}{\rho}\right)}{Ai(-\sigma/\rho)}$$

B.C at $x=a$

② $b\,Ai\left(\frac{a-\sigma}{\rho}\right)+f\,Bi\left(\frac{a-\sigma}{\rho}\right)=Ce^{-ka}$

③ $\frac{b}{\rho}Ai'\left(\frac{a-\sigma}{\rho}\right)+\frac{f}{\rho}Bi'\left(\frac{a-\sigma}{\rho}\right)=-Cke^{-ka}$

Substituting ① into ② + ③

④ $$-\frac{f\,Bi\left(\frac{-\sigma}{\rho}\right)}{Ai\left(\frac{-\sigma}{\rho}\right)}Ai\left(\frac{a-\sigma}{\rho}\right)+f\,Bi\left(\frac{a-\sigma}{\rho}\right)=Ce^{-ka}$$

⑤ $$\frac{-f\,Bi\left(\frac{-\sigma}{\rho}\right)}{\rho\,Ai(-\sigma/\rho)}Ai'\left(\frac{a-\sigma}{\rho}\right)+\frac{f}{\rho}Bi'\left(\frac{a-\sigma}{\rho}\right)=-Cke^{-ka}$$

Dividing ⑤/④

$$\Rightarrow \boxed{\frac{\frac{-Bi(-\sigma/\rho)}{\rho\,Ai(-\sigma/\rho)}Ai'\left(\frac{a-\sigma}{\rho}\right)+\frac{1}{\rho}Bi'\left(\frac{a-\sigma}{\rho}\right)}{\frac{-Bi(-\sigma/\rho)}{Ai(-\sigma/\rho)}Ai\left(\frac{a-\sigma}{\rho}\right)+Bi\left(\frac{a-\sigma}{\rho}\right)}=-k}$$

$\rho \equiv$ const so $\sec\theta = 1$ $\quad \sigma = E$ & $E = \frac{\hbar^2 k^2}{2m} - a$

$$\frac{\frac{-Bi(-E)}{Ai(-E)} Ai'(a-E) + Bi'(a-E)}{\frac{-Bi(-E)}{Ai(-E)} Ai(a-E) + Bi(a-E)} = \frac{\sqrt{2m(a-E)}}{\hbar}$$

since we set $\rho = 1 \Rightarrow \frac{\hbar^2}{2m} = 1 \Rightarrow \hbar^2 = 2m$ so right hand side $=$

$$\sqrt{(a-E)}$$

$$\boxed{\frac{\frac{-Bi(-E)}{Ai(-E)} Ai'(a-E) + Bi'(a-E)}{\frac{-Bi(-E)}{Ai(-E)} Ai(a-E) + Bi(a-E)} = \sqrt{a-E}}$$

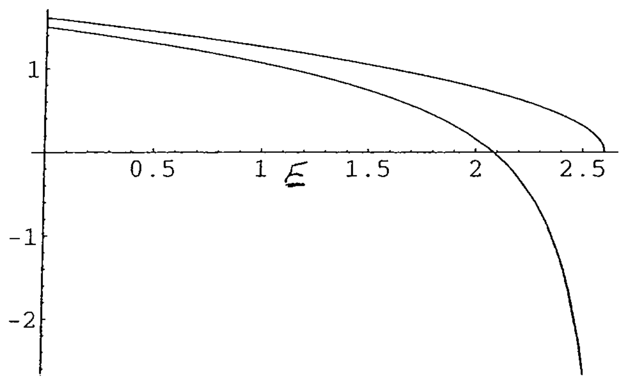
This can be solved analytically for E

using Mathematica one finds that $\boxed{a > \sim 2.7}$

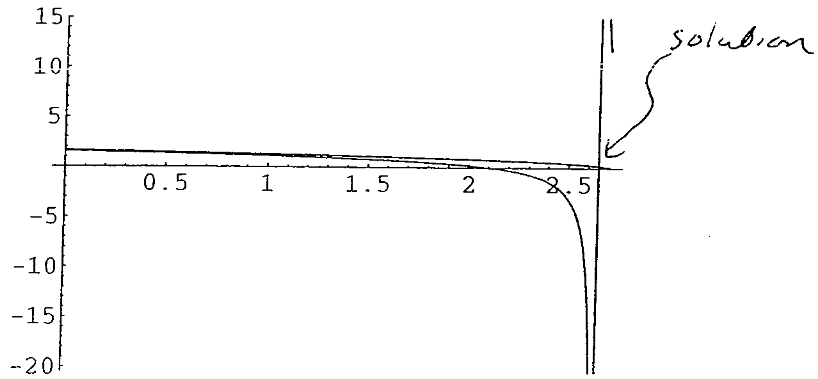
(see next page)

Plots of $$\frac{\frac{-Bi(-E)}{Ai(-E)}Ai'(a-E)+Bi'(a-E)}{\frac{-Bi(-E)}{Ai(-E)}Ai(a-E)+Bi(a-E)}$$ and $\sqrt{a-E}$

$a = 2.6$



$a = 2.7$



2) a) $\boxed{\begin{array}{ll} V(x) = mgx & x > 0 \\ V(x) = \infty & V < 0 \end{array}}$

b) $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + mgx\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = \frac{2m^2g}{\hbar^2}(x - E/mg)\psi \qquad \text{let } y = x - E/mg$$

$$\text{and } \alpha \equiv \left(\frac{2m^2g}{\hbar^2}\right)^{1/3}$$

$$\frac{d^2\psi}{dy^2} = \alpha^3 y\psi$$

if $z \equiv \alpha y \quad \frac{d^2}{dy^2} = \alpha^2 \, d^2/dz^2 \Rightarrow \alpha^2 \frac{d^2\psi}{dz^2} = \alpha^3 y \psi$

$$= \frac{d^2\psi}{dz^2} = \alpha y \psi \Rightarrow \underline{\underline{\frac{d^2\psi}{dz^2} = z\psi}}$$

so $\psi = a\,Ai(z) \Rightarrow \boxed{\psi(x) = a\,Ai(\alpha(x - E/mg)) \text{ where } \alpha = \left(\frac{2m^2g}{\hbar^2}\right)^{1/3}}$

c) $V(x) = \infty$ for $x < 0$ so $\psi(0) = 0$

$\Rightarrow Ai(\alpha(-E/mg)) = 0$

The zeroes of Airy function are:
$-2.338,\ -4.087,\ -5.521,\ -6.787$

so $\frac{-\alpha E}{mg} = y_n^+$ where y_n^+ are 0's of Airy function

$$E_n = -\frac{y_n^+ mg}{\alpha} = -y_n^+ mg\left(\frac{\hbar^2}{2m^2g}\right)^{1/3} = -y_n^+\left(\frac{1}{2}mg^2\hbar^2\right)^{1/3}$$

$$\left(\frac{1}{2}mg\hbar^2\right)^{1/3} = \left[\frac{1}{2}(0.1\,kg)(9.8\,m/s^2)^2(1.055\times10^{-34}\,J\cdot s)^2\right]^{1/3} = 3.77\times10^{-23}\,J$$

so $\boxed{\begin{aligned} E_1 &= 8.81 \times 10^{-23}\,J \\ E_2 &= 1.54 \times 10^{-22}\,J \\ E_3 &= 2.08 \times 10^{-22}\,J \\ E_4 &= 2.56 \times 10^{-22}\,J \end{aligned}}$

d) $2\langle T\rangle = \langle x\, dV/dx\rangle$ $\qquad \frac{dV}{dx} = mg$ so $\langle x\, dV/dx\rangle = \langle mgx\rangle = \langle V$

so $\langle T\rangle = \frac{1}{2}\langle V\rangle$

$\langle T\rangle + \langle V\rangle = E_n \Rightarrow \ {}^3/_2\langle V\rangle = E_n \Rightarrow \langle V\rangle = \frac{2}{3}E_n$

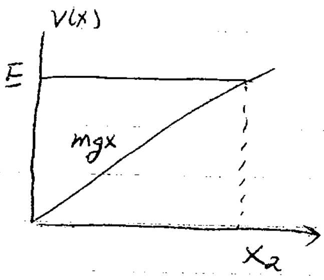
$\langle V\rangle = mg\langle x\rangle$ so $\qquad \langle x\rangle = \dfrac{2E_n}{3mg}$

for electron $E_1 = 1.84 \times 10^{-32}\,J$ or $\boxed{1.15 \times 10^{-13}\,eV}$

$$\langle x\rangle = \frac{2(1.84 \times 10^{-32}\,J)}{3(9.11 \times 10^{-31}\,kg)(9.8\,m/s^2)} = 1.37 \times 10^{-3}\,m \text{ or } \boxed{1.37\,mm}$$

3) a) Since potential is infinite at $x=0$

$$\int_0^{x_2} p(x)\,dx = (n-1/4)\pi\hbar \qquad p(x)=\sqrt{2m(E-mgx)}$$



$$E = mgx_2 \Rightarrow x_2 = E/mg$$

$$\int_0^{x_2} p(x)\,dx = \sqrt{2m}\int_0^{x_2}\sqrt{E-mgx}\,dx = \sqrt{2m}\left[-\frac{2}{3mg}(E-mgx)^{3/2}\right]\Big|_0^{x_2}$$

$$= -\frac{2}{3}\sqrt{\frac{2}{m}}\,\frac{1}{g}\left\{(E-mgx_2)^{3/2}-E^{3/2}\right\} = \frac{2}{3}\sqrt{\frac{2}{m}}\,\frac{1}{g}E^{3/2}$$

so $\frac{1}{3\sqrt{m}\,g}(2E)^{3/2} = (n-1/4)\pi\hbar$ or

$$\boxed{E_n = \left[\frac{9}{8}\pi^2 m g^2\hbar^2(n-1/4)^2\right]^{1/3}}$$

b) $\left(\frac{9}{8}\pi^2 m g^2\hbar^2\right)^{1/3} = 1.0588\times10^{-22}\text{ J} = E$

$$\boxed{\begin{aligned} E_1 &= E\,(3/4)^{2/3} = 8.74\times10^{-23}\text{ J}\\ E_2 &= E\,(7/4)^{2/3} = 1.54\times10^{-22}\text{ J}\\ E_3 &= E\,(11/4)^{2/3} = 2.08\times10^{-22}\text{ J}\\ E_4 &= E\,(15/4)^{2/3} = 2.56\times10^{-22}\text{ J}\end{aligned}}$$

c) From 2d $\langle x\rangle = \frac{2E_n}{3mg} \Rightarrow 1\text{ m} = \frac{2}{3}\,\frac{1.0588\times10^{-22}\text{ J}}{(1\text{ kg})(9.8\text{ m/s}^2)}(n-1/4)^{2/3}$

$\Rightarrow (n-1/4)^{2/3} = 1.388\times10^{22}$ $\quad n = 1/4 + (1.388\times10^{22})^{3/2} \Rightarrow \boxed{n = 1.64\times10^{33}}$

4 $V(x) = \frac{-\hbar^2 a^2}{m} \operatorname{sech}^2(ax)$

$\int p(x)\,dx = (n-1/2)\pi\hbar$

since $V(x)$ symmetric we get

$$2\int_0^{x_2} \sqrt{2m\left(E + \frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax)\right)} = (n-1/2)\pi\hbar$$

x_2 is defined where $E = -\frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax_2)$

rewrite integral

$$2\sqrt{2}\,\hbar a \int_0^{x_2} \sqrt{\operatorname{sech}^2(ax) + \frac{mE}{\hbar^2 a^2}}\,dx$$

let $b = \frac{-mE}{\hbar^2 a^2}$ and $z = \operatorname{sech}^2(ax)$

$x = \frac{1}{a}\operatorname{sech}^{-1}\sqrt{z}$ $\qquad dx = \frac{1}{a}\left(\frac{-1}{\sqrt{z}\sqrt{1-z}}\right)\frac{1}{2\sqrt{z}}\,dz = \frac{-1}{2a}\,\frac{1}{z\sqrt{1-z}}\,dz$

$$(n-\tfrac{1}{2})\pi = 2\sqrt{2}\,a\left(\frac{-1}{2a}\right)\int_{z_1}^{z_2} \frac{\sqrt{z-b}}{z\sqrt{1-z}}\,dz$$

limits $x=0$ $\quad z = \operatorname{sech}^2(0) = 1$

at $x = x_2$ $\quad V = E \implies \operatorname{sech}^2(ax_2) = \frac{-mE}{\hbar^2 a^2} = b = z_2$

so we get

$$(n-\tfrac{1}{2})\pi = \sqrt{2}\int_b^1 \frac{1}{z}\sqrt{\frac{z-b}{1-z}}$$ look up integral

$$\left(n-\tfrac{1}{2}\right)\pi = \sqrt{2}\left\{-2\tan^{-1}\sqrt{\frac{1-z}{z-b}} - \sqrt{b}\,\sin^{-1}\left[\frac{(1+b)z-2b}{z(1-b)}\right]\right\}\Bigg|_b^1$$

$$= \sqrt{2}\left\{-2\tan^{-1}(0) + 2\tan^{-1}(\infty) - \sqrt{b}\,\sin^{-1}(1) + \sqrt{b}\,\sin^{-1}(-1)\right\}$$

$$= \sqrt{2}\left(0 + 2\,\tfrac{\pi}{2} - \sqrt{b}\,\tfrac{\pi}{2} - \sqrt{b}\,\tfrac{\pi}{2}\right) = \sqrt{2}\left(\pi(1-\sqrt{b})\right)$$

so

$$\left(n-\tfrac{1}{2}\right)\cancel{\pi} = \sqrt{2}\left(\cancel{\pi}(1-\sqrt{b})\right) \Rightarrow \frac{n-1/2}{\sqrt{2}} = 1-\sqrt{b}$$

$\sqrt{b} = 1 - \frac{1}{\sqrt{2}}\left(n-\tfrac{1}{2}\right)$ LHS positive so

$\left(n-\tfrac{1}{2}\right) < \sqrt{2}$ so $n < 1.914$ so only solution is $n=1$

so for $n=1$

$$\sqrt{b} = 1 - \frac{1}{2\sqrt{2}} \Rightarrow b = \frac{9}{8} - \frac{1}{\sqrt{2}}$$

so $\boxed{E_1 = -\frac{\hbar^2 a^2}{m}\left(\frac{9}{8} - \frac{1}{\sqrt{2}}\right)}$

5) $V_{eff} = V + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}$ $\quad V = \frac{-e^2}{4\pi\epsilon_0 r}$

$$V_{eff} = \frac{-e^2}{4\pi\epsilon_0 r} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}$$

$$(n-1/2)\pi\hbar = \int_{r_1}^{r_2} p(x)\,dx$$

$$(n-1/2)\pi\hbar = \int_{r_1}^{r_2} \sqrt{2m\left(E + \frac{e^2}{4\pi\epsilon_0 r} - \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right)}\,dr$$

$$= \sqrt{-2mE}\int_{r_1}^{r_2}\sqrt{-1 + A/r - B/r^2}\,dr$$

where $A = \frac{-e^2}{4\pi\epsilon_0}\frac{1}{E}$ $\quad B = \frac{-\hbar^2}{2m}\frac{l(l+1)}{E}$

note $A \,\&\, B > 0$ since $E < 0$

$$\Rightarrow (n-1/2)\pi\hbar = \sqrt{-2mE}\int_{r_1}^{r_2}\frac{\sqrt{-r^2+Ar-B}}{r}\,dr$$

$r_1 \,\&\, r_2$ are points at which $E = V$ ~~since~~ or $E - V = 0$

$\Rightarrow$ these are the roots of numerator

$E - V = \sqrt{-2mE}\,\frac{\sqrt{-r^2+Ar-B}}{r}$ this only occurs when $-r^2+Ar-B = 0$

so $(-r^2+Ar-B) = (r-r_1)(r_2-r)$

$$(n-1/2)\pi\hbar = \sqrt{-2mE}\int_{r_1}^{r_2}\frac{\sqrt{(r-r_1)(r_2-r)}}{r}\,dr = \sqrt{-2mE}\,\frac{\pi}{2}\left(\sqrt{r_2}-\sqrt{r_1}\right)^2$$

$$2(n-1/2)\hbar = \sqrt{-2mE}\,(r_2 + r_1 - 2\sqrt{r_1 r_2})$$

rewrite in terms of $A \,\&\, B \Rightarrow -r^2 + Ar - B = (r-r_1)(r_2-r) = -r^2 + (r_1+r_2)r - r_1r_2$

$\Rightarrow A = r_1 + r_2 \quad B = r_1 r_2$

so $2(n-1/2)\hbar = \sqrt{-2mE}\,(A - 2\sqrt{B})$

$$2(n-\tfrac{1}{2})\hbar = \sqrt{-2mE}\left(\frac{-e^2}{4\pi\varepsilon_0}\frac{1}{E} - 2\sqrt{\frac{-\hbar^2}{2m}\frac{\ell(\ell+1)}{E}}\right)$$

$$= \frac{e^2}{4\pi\varepsilon_0}\sqrt{\frac{-2m}{E}} - 2\hbar\sqrt{\ell(\ell+1)}$$

so

$$\frac{e^2}{4\pi\varepsilon_0}\sqrt{\frac{-2m}{E}} = 2\hbar\left[n-\tfrac{1}{2}+\sqrt{\ell(\ell+1)}\right]$$

$$\frac{-E}{2m} = \frac{\left(e^2/4\pi\varepsilon_0\right)^2}{4\hbar^2\left[n-\tfrac{1}{2}+\sqrt{\ell(\ell+1)}\right]^2}$$

$$E = \frac{-\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2}{\left[n-\tfrac{1}{2}+\sqrt{\ell(\ell+1)}\right]^2} = \boxed{\frac{-13.6\,eV}{\left[n-\tfrac{1}{2}+\sqrt{\ell(\ell+1)}\right]^2} = E}$$

for $n \gg \ell$ + $n \gg \tfrac{1}{2}$

$$\boxed{E = \frac{-13.6\,eV}{n^2}}$$