Solutions

Physics 4803
Homework Assignment 4
Due Sept 24 at 5:00

1) A particle of mass m is placed in a finite spherical well.

$$V(r) = \begin{cases} 0, & \text{if r} \leq \text{a;} \\ V_o, & \text{if r} > \text{a} \end{cases}$$

Find the ground state by solving the radial equation with ℓ=0. Show that there is no bound state at all if $V_o a^2 < \pi^2\hbar^2/8\text{m}$.

2) Show that the expression $< L^2 > = \hbar^2\ell(\ell+1)$ is implied directly by the two assumptions
a) The only possible values that the components of the angular momentum can have on any axis are $\hbar(-\ell, ..., +\ell)$
b) All of these components are equally probable.
(Hint $\sum_{m=1}^{\ell} m^2 = \frac{\ell(\ell+1)(2\ell+1)}{6}$)

3) Consider a system of 2 particles. If the particles are in the states $|2,1\rangle$ and $|1/2,1/2\rangle$, what is the probability that the total angular momentum is in the state $|3/2,3/2\rangle$. Do not use the Clebsch-Gordon tables, derive the expression.

4) **Derive** all of the Clebsch-Gordon coefficients when adding two spin 1 particles.

5) Using the Clebsch-Gordon coefficients. Write down all $|J,M\rangle$ states when adding a spin 3/2 particle to a spin 1/2 particle.

1. $V(r) = \begin{cases} 0 & r \le a \\ V_0 & r > a \end{cases}$

letting $u(r) = r R(r)$

Radial Equation becomes

$$\frac{-\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$$

for $l = 0$

we get

$$\frac{-\hbar^2}{2m}\frac{d^2u}{dr^2} + Vu = Eu$$

for $r \le a$ $\quad V = 0 \Rightarrow \quad u(r) = A\sin kr + \overset{0}{\cancel{B}}\cos kr$

$u(r) = A\sin kr \qquad k = \dfrac{\sqrt{2mE}}{\hbar} \qquad r \le a$

$r > a$

$$u(r) = \overset{0}{\cancel{C}}e^{\lambda r} + De^{-\lambda r}$$

$u(r) = De^{-\lambda r} \qquad \lambda = \dfrac{\sqrt{2m(V_0 - E)}}{\hbar} \qquad r > a$

Boundary conditions

at $r = a$ $\quad \left.\begin{array}{l} A\sin(ka) = De^{-\lambda a} \\ Ak\cos(ka) = -D\lambda e^{-\lambda a} \end{array}\right\}$ Divide $\Rightarrow \dfrac{1}{k}\tan ka = -1/\lambda$

$k = -\lambda \tan ka \qquad$ let $z = ka$

$\frac{z}{a} = -\lambda \tan z \Rightarrow z = -\lambda a \tan z \Rightarrow z = -\sqrt{2mV_0/\hbar^2 - 2mE_0/\hbar^2}\; a \tan z$

$$z = -\sqrt{\frac{2mV_0a^2}{\hbar^2} - \underbrace{\frac{2mE_0}{\hbar^2}}_{k^2}a^2}\,\tan z \quad \Rightarrow z = -\sqrt{\underbrace{\frac{2mV_0a^2}{\hbar^2}}_{\text{Define as } z_0^2} - \underbrace{k^2a^2}_{z^2}}\;\tan z$$

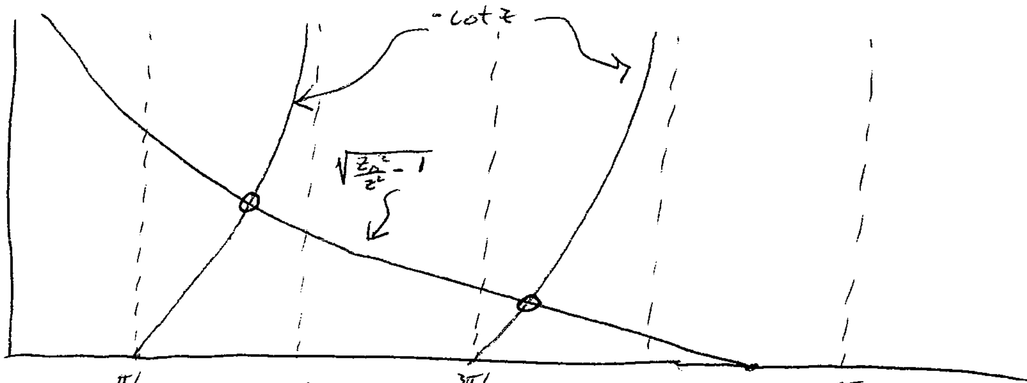
$$z = -\sqrt{z_0^2 - z^2}\ \tan z$$

or

$$-\tan z = \frac{z}{\sqrt{z_0^2 - z^2}} \qquad \text{with} \quad z \equiv ka, \quad z_0 = \frac{\sqrt{2mV_0}\,a}{\hbar}$$

Graph solution

$$-\cot z = \sqrt{\frac{z_0^2}{z^2} - 1}$$



no solution if $z_0 < \pi/2$ or $\frac{2mV_0 a^2}{\hbar^2} < \pi^2/4$

so no bound state if $\boxed{V_0 a^2 < \pi^2\hbar^2/8m}$

Ground state energy somewhere between $z = \pi/2$ & $z = \pi$

$$E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2ma^2} z^2 \text{ so } \boxed{\frac{\hbar^2\pi^2}{8ma^2} < E_0 < \frac{\hbar^2\pi^2}{2ma^2} \quad \text{actual location depends on } V_0}$$

2 show $\langle L^2 \rangle = \hbar^2 \, l(l+1)$

$\langle L^2 \rangle = \langle L_x^2 \rangle + \langle L_y^2 \rangle + \langle L_z^2 \rangle$

assumption b) => all equally probable so $\langle L^2 \rangle = 3 \langle L_z^2 \rangle$

assumption a) => only possible values are $m\hbar$ where m is from --

$\langle L^2 \rangle = 3 \langle m^2 \hbar^2 \rangle$ where m goes from $-l$ to l

expectation (average) of $\langle m^2 \hbar^2 \rangle = \dfrac{\hbar^2 \sum_{m=-l}^{m=+l} m^2}{2l+1}$

Definition of average = $\dfrac{\text{sum up values}}{\text{Total \# of values}}$

since m^2 symmetric & $m=0$ term does not contribute to sum we find

$\langle m^2 \hbar^2 \rangle = 2\hbar^2 \sum_{m=1}^{l} m^2 \;/(2l+1)$

using hint we get

$\langle m^2 \hbar^2 \rangle = \dfrac{2\hbar^2 \, l(l+1)\cancel{(2l+1)}}{6\,\cancel{(2l+1)}} = \dfrac{\hbar^2 \, l(l+1)}{3}$

since $\langle L^2 \rangle = 3\langle L_z^2 \rangle = 3\langle m^2 \hbar^2 \rangle$ we find

$$\boxed{\langle L^2 \rangle = \hbar^2 \, l(l+1)}$$

3 $j_1 = 2 \quad j_2 = 1/2$

Start with largest value of J + M

$J = j_1 + j_2 = 5/2$ There is only 1 vector with $M = 5/2$

$|J = 5/2, M = 5/2\rangle = |2,2\rangle |1/2, 1/2\rangle$

apply J_- to this state

$J_- = J_{1-} + J_{2-}$

$$J_- |5/2, 5/2\rangle = \hbar \sqrt{J(J+1) - M(M-1)}\ |5/2, 3/2\rangle$$
$$= \hbar \sqrt{5/2(5/2+1) - 5/2(5/2-1)}\ |5/2, 3/2\rangle$$
$$= \sqrt{5}\hbar\ |5/2, 3/2\rangle$$

$$\Rightarrow |5/2, 3/2\rangle = \frac{1}{\sqrt{5}\hbar} J_- |5/2, 5/2\rangle$$

$$= \frac{1}{\sqrt{5}\hbar}(J_{1-} + J_{2-})|2,2\rangle|1/2, 1/2\rangle$$

$$= \frac{1}{\sqrt{5}\hbar}\left[2\hbar\,|2,1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle + \hbar\,|2,2\rangle|\tfrac{1}{2},-1/2\rangle\right]$$

$$|5/2, 3/2\rangle = \frac{2}{\sqrt{5}}|2,1\rangle|1/2, 1/2\rangle + \frac{1}{\sqrt{5}}|2,2\rangle|1/2, -1/2\rangle$$

we also want to find $|3/2, 3/2\rangle$ state. This will be a linear combination of $|2,1\rangle|1/2, 1/2\rangle$ + $|2,2\rangle|1/2, -1/2\rangle$

$$|3/2, 3/2\rangle = \alpha\,|2,1\rangle|1/2, 1/2\rangle + \beta\,|2,2\rangle|1/2, -1/2\rangle$$

this must be orthogonal to $|5/2, 3/2\rangle \Rightarrow \frac{2\alpha}{\sqrt{5}} + \beta/\sqrt{5} = 0 \Rightarrow \alpha = -\beta/2$

it must also be normalized to 1 $\langle 3/2, 3/2 | 3/2, 3/2\rangle = 1$

$$\Rightarrow \alpha^2 + \beta^2 = 1 \Rightarrow \beta^2 + 4\beta^2 = 1 \Rightarrow \beta = -\sqrt{4/5}$$
$$\alpha = +\sqrt{1/5}$$

so $\quad |5/2, 3/2\rangle = \sqrt{4/5}\, |2,1\rangle |1/2, 1/2\rangle + \sqrt{1/5}\, |2,2\rangle |1/2, -1/2\rangle$

$\quad |3/2, 3/2\rangle = \sqrt{1/5}\, |2,1\rangle |1/2, 1/2\rangle - \sqrt{4/5}\, |2,2\rangle |1/2, -1/2\rangle$

solving for $|2,1\rangle |1/2, 1/2\rangle$

$|2,1\rangle |1/2, 1/2\rangle = \sqrt{1/5}\, |3/2, 3/2\rangle + \sqrt{4/5}\, |5/2, 3/2\rangle$

so probability to be in $|3/2, 3/2\rangle$ is $|\sqrt{1/5}|^2 = \boxed{1/5}$

4) Derive all of the Clebsch-Gordon coefficients when adding 2 spin 1 particles

$j_1 = 1 \quad j_2 = 1$ There will be $(2j_1+1)(2j_2+1) = \underline{\underline{9}}$ possible states

Start with largest value of $J = 2$, there is only 1 value of M for this J $M=2$

$$\boxed{① \quad |J=2, M=2\rangle = |1,1\rangle|1_2,1\rangle}$$

Apply J_- to ①

$$J_-|2,2\rangle = \hbar\sqrt{2(2+1) - 2(2-1)}\,|2,1\rangle = 2\hbar\,|2,1\rangle$$

so

$$|2,1\rangle = \frac{1}{2\hbar} J_-|2,2\rangle = \frac{1}{2\hbar}(J_{1-}+J_{2-})|1,1\rangle|1,1\rangle = \frac{1}{2\hbar}\{\sqrt{2}\hbar|1,0\rangle|1,1\rangle + \sqrt{2}\hbar|1,1\rangle|1,0\rangle\}$$

$$\boxed{② \quad |J=2, M=1\rangle = \frac{1}{\sqrt{2}}|1,0\rangle|1,1\rangle + \frac{1}{\sqrt{2}}|1,1\rangle|1,0\rangle}$$

Apply J_- to ②

$$J_-|2,1\rangle = \hbar\sqrt{6}\,|2,0\rangle$$

$$|2,0\rangle = \frac{1}{\sqrt{6}\hbar}J_-|2,1\rangle = \frac{1}{\sqrt{6}\hbar}(J_{1-}+J_{2-})\{\tfrac{1}{\sqrt{2}}|1,0\rangle|1,1\rangle + 1/\sqrt{2}\,|1,1\rangle|1,0\rangle\}$$

$$= \frac{1}{\sqrt{2}\sqrt{6}}\frac{\hbar}{\hbar}\{\sqrt{2}|1,-1\rangle|1,1\rangle + \sqrt{2}|1,0\rangle|1,0\rangle + \sqrt{2}|1,0\rangle|1,0\rangle + \sqrt{2}|1,1\rangle|1,-1\rangle\}$$

$$\boxed{③ \quad |J=2, M=0\rangle = \frac{1}{\sqrt{6}}|1,-1\rangle|1,1\rangle + \frac{2}{\sqrt{6}}|1,0\rangle|1,0\rangle + \frac{1}{\sqrt{6}}|1,1\rangle|1,-1\rangle}$$

Apply J_- to ③

$$J_-|2,0\rangle = \hbar\sqrt{6}\,|2,-1\rangle$$

$$|2,-1\rangle = \frac{1}{\sqrt{6}\hbar}J_-|2,0\rangle$$

$|2,-1\rangle = \frac{1}{\sqrt{6}\hbar}(J_{1-}+J_{2-})\frac{1}{\sqrt{6}}\{|1,-1\rangle|1,1\rangle + 2|1,0\rangle|1,0\rangle + |1,1\rangle|1,-1\rangle\}$

$= \frac{\hbar}{6\hbar}\{0 + 2\sqrt{2}|1,-1\rangle|1,0\rangle + \sqrt{2}|1,0\rangle|1,-1\rangle + \sqrt{2}|1,-1\rangle|1,0\rangle + 2\sqrt{2}|1,0\rangle|1,-1\rangle +$

$= \frac{1}{6}\{3\sqrt{2}|1,-1\rangle|1,0\rangle + 3\sqrt{2}|1,0\rangle|1,-1\rangle\}$

$\boxed{④\ |J=2, M=-1\rangle = \frac{1}{\sqrt{2}}|1,-1\rangle|1,0\rangle + 1/\sqrt{2}|1,0\rangle|1,-1\rangle}$

apply J_- to ④ however $|J=2, M=-2\rangle$ is lowest possible M state can only be formed by adding 2 lowest states

$\boxed{⑤\ |J=2, M=-2\rangle = |1,-1\rangle|1,-1\rangle}$

we have found all $J=2$ states, now take next lowest state $J=1$, largest value of $M=1$ for this state.
We have already found 1 $M=1$ state. The state $|J=1, M=1\rangle$ must be $\perp$ to this state.

$|J=1, M=1\rangle = \alpha|1,0\rangle|1,1\rangle + \beta|1,1\rangle|1,0\rangle$

This state is $\perp$ to ② $\Rightarrow \frac{\alpha}{\sqrt{2}} + \beta/\sqrt{2} = 0 \Rightarrow \alpha = -\beta$
Norm $=1 \Rightarrow \alpha^2+\beta^2=1 \Rightarrow 2\alpha^2=1 \quad \alpha = 1/\sqrt{2}, \ \beta = -1/\sqrt{2}$

$\boxed{⑥\ |J=1, M=1\rangle = \frac{1}{\sqrt{2}}|1,0\rangle|1,1\rangle - 1/\sqrt{2}|1,1\rangle|1,0\rangle}$

Apply J_- to ⑥

$J_-|1,1\rangle = \sqrt{2}\hbar|1,0\rangle \Rightarrow |1,0\rangle = \frac{1}{\sqrt{2}\hbar}J_-|1,1\rangle$

$|1,0\rangle = \frac{1}{\sqrt{2}\hbar}(J_{1-}+J_{2-})\{\frac{1}{\sqrt{2}}|1,0\rangle|1,1\rangle - 1/\sqrt{2}|1,1\rangle|1,0\rangle\}$

$= \frac{\hbar}{2\hbar}\{\sqrt{2}|1,-1\rangle|1,1\rangle - \sqrt{2}|1,0\rangle|1,0\rangle + \sqrt{2}|1,0\rangle|1,0\rangle - \sqrt{2}|1,1\rangle|1,-1\rangle$

$\boxed{⑦\ \Rightarrow |J=1, M=0\rangle = \frac{1}{\sqrt{2}}|1,-1\rangle|1,1\rangle - \frac{1}{\sqrt{2}}|1,1\rangle|1,-1\rangle}$

Apply J_- to (7)

$J_-|1,0\rangle = \sqrt{2}\hbar\,|1,-1\rangle \Rightarrow |1,-1\rangle = \frac{1}{\sqrt{2}\hbar} J_-\,|1,0\rangle$

$|1,-1\rangle = \frac{1}{\sqrt{2}\hbar}\left(J_{1-} + J_{2-}\right)\left(\frac{1}{\sqrt{2}}|1,-1\rangle|1,1\rangle - \frac{1}{\sqrt{2}}|1,1\rangle|1,-1\rangle\right)$

$= \frac{\hbar}{2\hbar}\left\{ 0 - \sqrt{2}\,|1,0\rangle|1,-1\rangle + \sqrt{2}\,|1,-1\rangle|1,0\rangle - 0 \right\}$

$= \frac{1}{\sqrt{2}}|1,-1\rangle|1,0\rangle - \frac{1}{\sqrt{2}}|1,0\rangle|1,-1\rangle$ (negative sign arbitrary

(8) $\boxed{|J=1, M=-1\rangle = \frac{1}{\sqrt{2}}|1,0\rangle|1,-1\rangle - 1/\sqrt{2}\,|1,-1\rangle|1,0\rangle}$

There is one last state $J=0, M=0$

This must be $\perp$ to other $M=0$ states & a linear combination of states

$|0,0\rangle = \alpha|1,-1\rangle|1,1\rangle + \beta|1,0\rangle|1,0\rangle + \gamma|1,1\rangle|1,-1\rangle$

orthogonal to (3) $\Rightarrow \frac{\alpha}{\sqrt{6}} + \frac{2\beta}{\sqrt{6}} + \frac{\gamma}{\sqrt{6}} = 0 \Rightarrow \alpha + 2\beta + \gamma = 0$

orthogonal to (7) $\Rightarrow \frac{\alpha}{\sqrt{2}} - \frac{\gamma}{\sqrt{2}} = 0 \Rightarrow \alpha = \gamma$

normalized $\Rightarrow \alpha^2 + \beta^2 + \gamma^2 = 1$

since $\alpha + 2\beta + \gamma = 0$ & $\alpha = \gamma \Rightarrow 2\alpha + 2\beta = 0 \Rightarrow \alpha = -\beta$

$\alpha^2 + (-\alpha)^2 + \alpha^2 = 1 \Rightarrow \alpha = 1/\sqrt{3},\ \beta = -1/\sqrt{3},\ \gamma = 1/\sqrt{3}$

so final state is

(9) $\boxed{|J=0, M=0\rangle = \frac{1}{\sqrt{3}}|1,-1\rangle|1,1\rangle - 1/\sqrt{3}\,|1,0\rangle|1,0\rangle + 1/\sqrt{3}\,|1,1\rangle|1,-1\rangle}$

5) spin $3/2$ + spin $1/2$

$|2,2\rangle = |3/2, 3/2\rangle|\tfrac{1}{2}, \tfrac{1}{2}\rangle$

$|2,1\rangle = \sqrt{1/4}\ |3/2, 3/2\rangle|1/2, -1/2\rangle + \sqrt{3/4}\ |3/2, 1/2\rangle|\tfrac{1}{2}, \tfrac{1}{2}\rangle$

$|1,1\rangle = \sqrt{3/4}\ |3/2, 3/2\rangle|\tfrac{1}{2}, -1/2\rangle - \sqrt{\tfrac{1}{4}}\ |3/2, \tfrac{1}{2}\rangle|\tfrac{1}{2}, \tfrac{1}{2}\rangle$

$|2,0\rangle = \sqrt{1/2}\ |3/2, \tfrac{1}{2}\rangle|\tfrac{1}{2}, -1/2\rangle + \sqrt{\tfrac{1}{2}}\ |3/2, -1/2\rangle|\tfrac{1}{2}, \tfrac{1}{2}\rangle$

$|1,0\rangle = \sqrt{\tfrac{1}{2}}\ |3/2, \tfrac{1}{2}\rangle|\tfrac{1}{2}, -1/2\rangle - \sqrt{\tfrac{1}{2}}\ |\tfrac{3}{2}, -1/2\rangle|\tfrac{1}{2}, \tfrac{1}{2}\rangle$

$|2,-1\rangle = \sqrt{3/4}\ |3/2, -1/2\rangle|\tfrac{1}{2}, -1/2\rangle + \sqrt{\tfrac{1}{4}}\ |3/2, -3/2\rangle|\tfrac{1}{2}, 1/2\rangle$

$|1,-1\rangle = \sqrt{1/4}\ |3/2, -1/2\rangle|\tfrac{1}{2}, -1/2\rangle - \sqrt{3/4}\ |3/2, -3/2\rangle|\tfrac{1}{2}, \tfrac{1}{2}\rangle$

$|2,-2\rangle = |3/2, -3/2\rangle|\tfrac{1}{2}, -1/2\rangle$