1) Using the recursion relation:

\[ a_{n+2} = \frac{(m_\ell + n)(m_\ell + n + 1)}{(n + 1)(n + 2)} - k a_n \]

where \( k = \ell(\ell + 1) \), and the definition of the Associated Legendre Polynomials:

\[ P^{m_\ell}_\ell(x) = (1 - x^2)^{m_\ell/2} \sum_{n=0}^{\infty} a_n x^n \]

and the definition of the Spherical Harmonics:

\[ Y^{m_\ell}_\ell = (-1)^{m_\ell} \sqrt{\frac{(2\ell + 1)(\ell - |m_\ell|)!}{4\pi (\ell + |m_\ell|)!}} e^{im_\ell \phi} P^{m_\ell}_\ell(\cos \theta) \]

and using the normalization condition

\[ \int |Y^{m_\ell}_\ell|^2 d\Omega = 1 \]

calculate all of the Spherical Harmonics starting with \( \ell = 0 \) up to \( \ell = 2 \).
Note \( \ell \geq m_\ell \) and only consider \( m_\ell \geq 0 \).

2) Consider a 2-Dimensional potential well centered on the origin with \( E > V_0 \) where the potential is:

\[ V(r) = \begin{cases} 
0, & \text{if } r \leq a; \\
V_0, & \text{if } r > a
\end{cases} \]

Solve Schrodinger’s equation for this potential assuming \( \Psi \) is separable. Using the change of variables \( \rho = kr \) (with \( k = \sqrt{2m(E - V_0)/\hbar} \)) use a series solution in the form:

\[ R = \rho^\lambda \sum_{j=0}^{\infty} a_j \rho^j \]

a) If \( a_0 \) is not zero, what is \( \lambda \)?

b) Show that \( a_1 \) must be 0.

c) Determine the recursion relationship between the coefficients.

3) A particle of mass \( m \) is placed in a finite spherical well.

\[ V(r) = \begin{cases} 
0, & \text{if } r \leq a; \\
V_0, & \text{if } r > a
\end{cases} \]

Find the ground state by solving the radial equation with \( \ell = 0 \). Show that there is no bound state at all if \( V_0a^2 < \pi^2\hbar^2/8m \).
4) Show that the expression \( \langle L^2 \rangle = \hbar^2 \ell (\ell + 1) \) is implied directly by the two assumptions
a) The only possible values that the components of the angular momentum can have on any axis are \( \hbar (-\ell, \ldots, +\ell) \)
b) All of these components are equally probable.
(Hint: \( \sum_{m=1}^{\ell} m^2 = \frac{\ell(\ell+1)(2\ell+1)}{6} \))