Scaling Problems

Scaling problems (also called ratio or proportionality problems) are very popular and useful in physics. Learning to solve these types of problems is important. An example of a scaling problem is:

Problem: The area of a circle is given by $A = \pi r^2$. If the radius (r) of a circle is increased by 20% what is the new area of the circle?

You may think that you can't solve this problem since the original radius of the circle is unknown so and you can't determine the area of the circle. It is true that you cannot determine the original area of the circle, but you can determine how much larger the area of a circle will become after the radius is increased. This is done using a technique involving ratios. Let's solve the above problem using ratios.

Let's start by trying to write down, in equation form, what it means for the radius to increase by 20%. Often using specific examples can help in these types of problem, so let's write down some specific examples. If the original radius of a circle is 1.0 m, and the new radius is 20% larger, this means the new radius would be 1.2 m or (1.0 m + 0.2×1.0 m). If the original radius of a circle is 2.0 m, the new radius would be 2.4 m or (2.0 m + 0.2×2.0 m). We can start to see a pattern. Using the subscript "original" for any original quantity, and "new" for any quantity after making some change(s), the relationship between the original radius and the new radius is $r_{\text{new}} = r_{\text{original}} + 0.2 \times r_{\text{original}} = 1.2 \times r_{\text{original}}$. This pattern works for all changes in radius. We can look at other examples. If the radius increased by 60%, $r_{\text{new}} = r_{\text{original}} + 0.6 \times r_{\text{original}} = 1.6 \times r_{\text{original}} = 1.6 r_{\text{original}}$. If the radius increased by 100% (i.e. the radius is doubled) $r_{\text{new}} = r_{\text{original}} + 1.0 \times r_{\text{original}} = 2 r_{\text{original}}$.

Now that we know how to write in equation form the relationship between the original radius and the new radius we can go back to the original problem and solve it.

We know the relationship between the area and radius of a circle, $A = \pi r^2$, which means that $A_{\text{original}} = \pi r_{\text{original}}^2$ and $A_{\text{new}} = \pi r_{\text{new}}^2$. As mentioned above, we do not know the numerical value of the radius so we cannot actually compute the areas of the circles. However, by taking the ratio of the above equations we can solve the problem. We always take the ratios with the new quantity in the numerator. Taking the ratio of the two equations above we see

$$\frac{A_{\text{new}}}{A_{\text{original}}} = \frac{\pi r_{\text{new}}^2}{\pi r_{\text{original}}^2}$$

From above we know that $r_{\text{new}} = 1.2 r_{\text{original}}$. So we can substitute this expression for r_{new} into our ratio, noting that the whole value of $1.2 r_{\text{original}}$ must be squared,

$$\frac{A_{\text{new}}}{A_{\text{original}}} = \frac{\pi (1.2 r_{\text{original}})^2}{\pi r_{\text{original}}^2} = \frac{\pi (1.2)^2 r_{\text{original}}^2}{\pi r_{\text{original}}^2}$$

Note that we can now cancel out the π and r_{original}^2 from the numerator and denominator leaving

$$\frac{A_{\text{new}}}{A_{\text{original}}} = (1.2)^2$$

or

$$A_{\text{new}} = (1.2)^2 A_{\text{original}} = 1.44 A_{\text{original}}$$

We see that the new area is 1.44 times larger than the original area or, equivalently, 44% larger than the original area.

From this example, we notice a few important features of taking ratios. Note that the value π cancelled out in the above ratio. In general, any constant or number will cancel out in a ratio, so it does not affect the final answer. We illustrate this principle with another example.

<u>Problem:</u> If we know the relationship that K=0.5 mv^2 and v increases by 20%, how much larger is K?

We don't actually need to know what K, m, and v represent in this equation. All we need to know is the relationship and what variable is changing. Using the above technique we note $v_{\text{new}} = 1.2 v_{\text{original}}$ so

$$\frac{K_{\text{new}}}{K_{\text{original}}} = \frac{0.5m(v_{\text{new}})^2}{0.5mv_{\text{original}}^2} = \frac{0.5m(1.2v_{\text{original}})^2}{0.5mv_{\text{original}}^2} = \frac{0.5m(1.2)v_{\text{original}}^2}{0.5mv_{\text{original}}^2} = (1.2)^2 = 1.44$$

We see the same answer as above. In this example, 0.5 is a constant and m did not change and so it is also a constant. Therefore, both 0.5 and m cancel in the numerator and denominator. The answer to this question and the first question is the same because we see that A is proportional to r^2 and K is proportional to v^2 so if both v and r increase by 20% both K and A will increase by 44%.

Let's look at another example using K=0.5 mv^2.

<u>Problem:</u> How much does K change if m is reduced to 1/3 of its original value and v is doubled?

Writing down what we know:
$m_{\text{new}} = (m_{\text{original}})/3$
$v_{\text{new}} = 2v_{\text{original}}$
$K_{\text{new}} = 0.5\ m_{\text{new}}\ v_{\text{new}}^2$
$K_{\text{original}} = 0.5\ m_{\text{original}}\ v_{\text{original}}^2$

Taking the ratio gives

$$\frac{K_{\text{new}}}{K_{\text{original}}} = \frac{0.5m_{\text{new}}v_{\text{new}}^2}{0.5m_{\text{original}}v_{\text{original}}^2} = \frac{0.5(m_{\text{original}}/3)(2v_{\text{original}})^2}{0.5m_{\text{original}}v_{\text{original}}^2} = \frac{0.5(m_{\text{original}}/3)4v_{\text{original}}^2}{0.5m_{\text{original}}v_{\text{original}}^2}$$

Canceling out the 0.5, m_{original} and v_{original}^2 gives us

$$\frac{K_{\text{new}}}{K_{\text{original}}} = \frac{4}{3}$$

So the new K is 4/3 times larger than the original K.

Be very careful, $(2v_{\text{original}})^2 = 4v_{\text{original}}^2$. A common mistake is to forget to square the 2.

$$(2v_{\text{original}})^2 \neq 2v_{\text{original}}^2$$

One last example:

<u>Problem:</u> If $K=0.5\,mv^2$ and if K increases by a factor of 4, and m stays constant, how much does v change?

Notice that we want to know how v changes, but v is not by itself on the left hand side of the equation. So, in order to solve this, we must first solve the equation for v. Dividing both sides of the equation for K by $0.5m$ and taking the square root gives

$$v = \sqrt{\frac{K}{0.5m}}$$

Now we solve it exactly the way we solved the above problems by taking the ratio.

$$\frac{v_{\text{new}}}{v_{\text{original}}} = \frac{\sqrt{K_{\text{new}}/(0.5m_{\text{new}})}}{\sqrt{K_{\text{original}}/(0.5m_{\text{original}})}}$$

since m does not change so that $m_{\text{new}} = m_{\text{original}}$ this simplifies to

$$\frac{v_{\text{new}}}{v_{\text{original}}} = \frac{\sqrt{K_{\text{new}}}}{\sqrt{K_{\text{original}}}}$$

Since the question states K increases by a factor of 4, we know $K_{\text{new}} = 4K_{\text{original}}$ so the ratio becomes

$$\frac{v_{\text{new}}}{v_{\text{original}}} = \frac{\sqrt{4K_{\text{original}}}}{\sqrt{K_{\text{original}}}} = \sqrt{4} = 2$$

The new v is twice the original v.

SHORTHAND: Once you know how to do these ratio problems, there is a slightly shorter way to do them. You first solve for the variable that you want to know how it changes. Then you simply neglect anything that is constant (i.e. it doesn't change) and put in the numerical factor for anything that does changes. Let's redo the four examples above using this shorthand.

<u>Problem 1:</u> The area of a circle is given by $A=\pi r^2$. If the radius (r) of a circle is increased by 20% what is the new area of the circle?
<u>Solution 1:</u>

Solve for A: $A = \pi r^2$
π doesn't change so we can ignore it, and r increases by a factor of 1.2 so where we have r we simply write 1.2. Since r is squared, the 1.2 must be squared as well. The area changes by a factor of
$A_{new} / A_{original} = (1.2)^2 = 1.44$
The new A is 44% larger.

<u>Problem 2:</u> If we know the relationship that $K = 0.5\ mv^2$ and v increases by 20%, how much larger is K?

<u>Solution 2:</u>
Solve for K: $K = 0.5\ mv^2$
0.5 and m don't change so we can ignore them, and v increases by a factor of 1.2 so where we have v we write 1.2. Since v is squared, the 1.2 must be squared as well. K changes by a factor of
$K_{new} / K_{original} = (1.2)^2 = 1.44$
The new K is 44% larger.

<u>Problem 3:</u> How much does K change if m is reduced to 1/3 of its original value and v is doubled?
$K = 0.5\ mv^2$

<u>Solution 3:</u>
Solve for K: $K = 0.5\ mv^2$
0.5 doesn't change so we can ignore it, v increases by a factor of 2 so where we have v we write 2, and m decreases by 1/3 so where we have m we write 1/3. K changes by a factor of
$K_{new} / K_{original} = (1/3) \times (2)^2 = 4/3$
The new K is 4/3 larger.

<u>Problem 4:</u> If $K = 0.5\ mv^2$ and if K increases by a factor of 4, and m stays constant, how much does v change?

<u>Solution 4:</u>
Since we want to know how v changes, we first solve for v.

$$v = \sqrt{\frac{K}{0.5m}}$$

0.5 and m don't change so we can ignore them, and K increases by a factor of 4 so where we have we K we write 4. Since K is inside a square root sign, we must take the square root of 4. v changes by a factor of
$v_{new} / v_{original} = \sqrt{4} = 2$
The new v is twice as large.

One additional problem.

<u>Problem 5:</u> If $K = 0.5\ mv^2$ and if m decreases by a factor of 1/4, and K stays constant, how much does v change?

<u>Solution 5:</u>

Since we want to know how *v* changes, we first solve for *v*.

$$v = \sqrt{\frac{K}{0.5m}}$$

0.5 and *K* don't change so we can ignore them, and *m* decreases by a factor of 1/4 so where we had *m* we simply write 1/4. *v* changes by a factor of

$$v_{\text{new}} / v_{\text{original}} = \sqrt{\frac{1}{(1/4)}} = \sqrt{4} = 2$$

The new *v* is twice as large.

Practice problems

1) $A = B$
If *B* doubles, by what factor does *A* change?

2) $A = B/5$
If *B* doubles, by what factor does *A* change?

3) $A = 0.3B^2$
If *B* triples, by what factor does *A* change?

4) $A = 1.2B^3$
If *B* is halved, by what factor does *A* change?

5) $A = 1/B$
If *B* doubles, by what factor does *A* change?

6) $A = 1/B^2$
If *B* doubles, by what factor does *A* change?

7) $A = 1/B^2$
If A doubles, by what factor does *B* change?

8) $A = BC$
If *B* doubles and *C* is halved, by what factor does *A* change?

9) $A = B/C$
If *B* doubles and *C* quadruples, by what factor does *A* change?

10) $A = B/C^2$
If *C* doubles and *A* doubles, by what factor does *B* change?

Answers to Practice Problems:

1) Doubles
2) Doubles
3) 9 times larger
4) 1/8 as large
5) Half as large
6) One fourth as large
7) $B = 1/\sqrt{A}$ so $B = 1/\sqrt{2}$ times smaller
8) No change
9) Half as large
10) 8 times larger