SOME RELATIVISTIC COSMOLOGICAL MODELS

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PREFACE

The purpose of this paper is to present the spatially homogeneous dust models which do not permit a simply transitive group of motions.

I wish to thank my supervising Professor R. K. Sachs for suggesting this problem and for the many suggestions he made during the calculations. I would also like to thank the other members of my committee, Professor E. L. Schucking, Professor J. Ehlers, and Professor R. P. Kerr, for their suggestions; and Dr. George Ellis for finding an error in an earlier draft of this paper.

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1. INTRODUCTION

This thesis will present some relativistic cosmological models which are spatially homogeneous and partially anisotropic. The models are similar to the Friedmann solutions, and for dust comparisons have already been made between the two.[1] The motivation for our calculations was to complete all the possible types of spatially homogeneous models. There are two types of spatially homogeneous models. The first type is that covered by the Taub-Heckmann-Schucking formalism.[2] It contains the models which permit a simply transitive group of motions that generates the spacelike homogeneous hypersurfaces.[3] The models which do not fall into the above type are presented in this thesis; they can thus be characterized as those spatially homogeneous models which do not permit a simply transitive group of motions. In addition some related models which are of the Taub-Heckmann-Schucking type are presented.

The stress-energy tensor is taken to be that of a perfect fluid and solutions are presented for four different cases. The first two are for vanishing pressure, with and without a cosmological constant, *i.e.,* dust models. The second two are for vanishing cosmological

[1] R. Kantowski and R. K. Sachs, *J. Math. Phys.* **7**, 443 (1966).
[2] L. Witten, *Gravitation* (John Wiley and Sons, New York, 1962),Ch. 11.
[3] For group terminology consult L. P. Eisenhart, *Continuous Groups of Transformations* (Dover Inc. New York, 1961).

constant with the respective equations of state $p = 1/3\rho$ (photon gas) and $p = \rho$ (stiff matter).

The difficulties of having an isotropic pressure and an anisotropic fluid motion are also considered briefly. In addition, in Appendix I, a nice proof for the non-existence of a four dimensional simple Lie algebra is given.

2. SPATIALLY HOMOGENEOUS MODELS

We will define a cosmological model to be an ordered triple (M_4, g_{ab}, u^a); a four dimensional differentiable manifold M_4, a metric on this manifold with signature $(+,-,-,-)$, g_{ab}, and a time-like congruence of curves, the tangents of which are u^a. The indices are to run $0,\ldots,3$ and the time-like congruence is to be parameterized by metric distance so that $u^a u_a = 1$.

A cosmological model is called spatially homogeneous if and only if it permits an isometry group (1) whose minimum invariant varieties are spacelike hypersurfaces and (2) which leaves the vector field u^a invariant. By isometry group we mean the complete isometry group.

Now we wish to separate spatially homogeneous models into three classes depending on a property of the isometry group.

In the manifold consider the isotropy group of a point P, *i.e.,* that subgroup of the isometry group which leaves the point P fixed. The point P lies in one of the spacelike homogeneous hypersurfaces. The tangent space to the hypersurface at P is mapped isometrically into itself by the isotropy group. In fact the isotropy group is isomorphic to an isometry group in this tangent space which leaves the origin fixed. Since the hypersurface is spacelike the tangent space is an E_3 and the full isometry group of an E_3 which leaves the origin fixed is the rotation group in three dimensions R_3. Therefore the isotropy group must be isomorphic to a subgroup of

R_3 . A subgroup of the rotation group may be zero, one, or three parametric but not two. If the isotropy group is zero parametric, *i.e.,* the identity, we call the model anisotropic; if it is one parametric we call it partially anisotropic; and if it is three parametric we call it isotropic. Since the dimension of the isometry group equals the sum of the dimension the isotropy group plus the dimension of the minimum invariant varieties[4], we are only interested in three, four, or six parametric isometry groups.

A relativistic cosmological model is a cosmological model which satisfies the Einstein field equations:[5]

$$R^{ab} - (1/2)g^{ab}R = -T^{ab} - \Lambda g^{ab}, \tag{1}$$

where

$$T^{ab} = (\rho + p)u^a u^b - pg^{ab}. \tag{2}$$

The T^{ab} above is the usual stress-energy tensor for a perfect fluid.

For a brief but clear discussion of the kinematical classification of hydrodynamic motion for fluids the reader is referred to J. Ehlers.[6]

The energy density ρ and pressure p are scalars and spatial homogeneity requires them to be constant on the hypersurfaces. Such models therefore invite the introduction of a cosmic time, *i.e.*, a coordinate that is constant on the spacelike hypersurfaces of transitivity.

[4] R. P. Kerr, *J. Math and Mech.* **12** (1963).

[5] The units are chosen such that the velocity of light, c, is 1 and the quantity $8\pi k = 1$, where k is the Newtonian gravitational constant.

[6] J. Ehlers, *Recent Developments in General Relativity* (Pergamon Press Inc. New York, 1962), pp.201-207.

The generators of the isometry group form a Lie algebra, a basis of which will be called $\xi_\alpha,\ \alpha = 1, 2, \ldots, r;\ r \geq 3$. r is the number of independent parameters in the group. The ξ_α satisfy the commutation relations and the Killing equations[7]

$$[\xi_\alpha, \xi_\beta] = C^{\nu}_{\alpha\beta}\xi_\nu, \tag{3}$$

$$\xi_{\alpha(a;b)} = 0, \tag{4}$$

where the $C^{\nu}_{\alpha\beta}$ are the structure constants of the algebra.

If x^0 is introduced as a cosmic time coordinate, then the Killing vectors ξ_α have components ξ^a_α only in the space directions, $\xi^0_\alpha = 0$.

The problem of finding spatially homogeneous relativistic models is attacked in the following manner. One assumes a particular Lie algebra (3) and solves simultaneously equations (3) and (4). These equations give some restrictions on the metric. The invariance of u^a, the field equations (1) , and an equation of state $p = p(\rho)$ are then invoked and one comes out with a solution for g^{ab}, u^a, and ρ. Now that a statement of the problem has been made, it is appropriate to classify the possible models by a property of the group involved. Those models which have a simply transitive subgroup of motions are to be put into the Taub-Heckmann-Schucking class and can be handled by their procedure. The group that does not fall into that class will be dealt with in this thesis. The rest

[7][ ] are Lie brackets, *i.e.*, $[\xi_\alpha, \xi_\beta]^a = \xi^b_\alpha \xi^a_{\beta,b} - \xi^b_\beta \xi^a_{\alpha,b}$.

of this chapter is devoted to finding the groups for all non-Taub-Heckmann-Schucking types. Anisotropic models possess three parametric symmetry groups which act in a simply transitive way and are hence of no interest. The next models to consider are the partially anisotropic ones. The important observation here will be that all four parametric Lie groups have three parametric subgroups (Appendix I.). There are two possibilities for the three parametric subgroups; (a) they act on three dimensional subspaces; (b) they act on two dimensional subspaces. (a) is not of interest because the subgroup would be simply transitive. If (b) is the case, then the subspaces are spaces of $n = 2$ dimensions with a symmetry group of $n(n+1)/2 = 3$ parameters; therefore they are spaces of constant curvature.[8]

The three two-dimensional constant curvature spaces are (locally) the closed sphere, the two flat plane, and the Lobatchewsky two plane. The curvatures are plus, zero, and minus respectively. The three group algebras are, respectively,

$$[\xi_1, \xi_2] = \xi_3, \quad [\xi_2, \xi_3] = \xi_1, \quad [\xi_3, \xi_1] = \xi_2, \tag{5}$$

$$[\xi_1, \xi_2] = 0, \quad [\xi_2, \xi_3] = \xi_1, \quad [\xi_3, \xi_1] = \xi_2, \tag{6}$$

$$[\xi_1, \xi_2] = \xi_1, \quad [\xi_2, \xi_3] = \xi_3, \quad [\xi_3, \xi_1] = 2\xi_2. \tag{7}$$

As two-spaces imbedded in flat three-spaces the two spaces can be given respectively by

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = 1, \tag{8}$$

[8]L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, N.J., 1949), p. 239.

$$
\begin{aligned}
A_\alpha x^\alpha &= \text{constant; } A_\alpha = \text{constant}, &(9)\\
(x^1)^2 &+ (x^2)^2 - (x^3)^2 = -1. &(10)
\end{aligned}
$$

The respective signatures for the imbedding spaces of (8), (9) and (10) are $(+++)$, $(+++)$, and $(++-)$.

Since we are considering three parametric subgroups of four parametric groups (5), (6), and (7) must be subalgebras of four dimensional Lie algebras. To find what algebras they can be subalgebras of, you simply introduce a fourth generator η and see what restrictions the Jacobi identity (A.3) places on the commutator of η with ξ_α ($\alpha = 1, 2, 3$). When the algebra of the plane (6) is completed to a four dimensional algebra, it will have a three dimensional subalgebra other than those of types (5), (6), or (7). In fact, it will have an abelian three dimensional subalgebra or will have the following structure (Appendix II):

$$
\begin{aligned}
&[\xi_1, \xi_2] = 0, \quad [\xi_2, \xi_3] = \xi_1, \quad [\xi_3, \xi_1] = \xi_2, \quad [\eta, \xi_1] = \xi_1,\\
&[\eta, \xi_2] = \xi_2, \quad [\eta, \xi_3] = 0.
\end{aligned} \tag{11}
$$

When the algebra of the Lobatchewsky plane (7) is completed to a four dimensional algebra it has the following structure (Appendix II):

$$
[\xi_1, \xi_2] = \xi_1, \ [\xi_2, \xi_3] = \xi_3, \ [\xi_3, \xi_1] = 2\xi_2, \ [\xi_\alpha, \eta] = 0. \tag{12}
$$

In (11) and (12) ξ_1, ξ_2, and η form a subalgebra not of type (5), (6), or (7). This means that four parametric groups with three parametric subgroups of type (6) or (7) must have three parametrlc subgroup which act transitively on three dimensional hypersurfaces and are therefore of the Taub-Heckmann-Schucking type. The only four dimensional Lie algebras of interest are therefore those which have a subalgebra of type (5) above. The final set of groups are the six parametric ones (the isotropic models). The maximum isometry group that a space of n dimensions can possess is one of the $n(n+1)/2$ parameters, and if a space possesses such a group it is a space of constant curvature.[9] In the case of $n = 3$, the maximum number is six. The models which correspond to the three possible cases of positive, negative, or zero curvature are the Robertson-Walker models.[10] The groups which go with each case all possess three parametric subgroups which act transitively on the homogeneous hypersurfaces. Locally the metrics are

$$ds^2 = dt^2 - R^2(t)\, g_{\alpha\beta}\, dx^\alpha dx^\beta \qquad (\alpha, \beta = 1, 2, 3),$$

where

$$g_{\alpha\beta} = \delta_{\alpha\beta}/[1 + (k/4)x^\sigma x^\sigma] \qquad (k = 1, 0, -1). \tag{13}$$

In the coordinate system, $y^\alpha = x^\alpha/[1 - (k/4)x^\sigma x^\sigma]$, the metrics take on the form

$$\bar{g}_{\alpha\beta} = \delta_{\alpha\beta}/[1 + k\, y^\sigma y^\sigma] - k\, y^\alpha y^\beta/[1 + k\, y^\sigma y^\sigma]^2. \tag{14}$$

The six Killing vectors take on the simple form,

$$\xi^\alpha = \epsilon^{\alpha\beta} y^\beta + \epsilon^\alpha + k\, y^\alpha \epsilon^\beta y^\beta, \tag{15}$$

$\epsilon^{\alpha\beta} = \epsilon^{[\alpha\beta]}$ and ϵ^α are arbitrary constants. To demonstrate the existence of a simply transitive three parametric subgroup we will

[9] Eisenhart, *op. cit.*
[10] H. Bondi, *Cosmology* (Cambridge University Press, London, 1960), p. 102.

just list the generators of such a subgroup. The simple transitivity can be checked by noticing that at each point y^α the three generators are linearly independent. The appropriate generators and commutation relations for the cases $k = 1, 0, -1$ are

$k = 1$

$$\xi^\alpha_\rho = 1/2\left[\epsilon^{\rho\alpha\beta} y^\beta + \delta^\alpha_\rho + y^\rho y^\alpha\right],$$
$$[\xi_\rho, \xi_\sigma] = \epsilon_{\rho\sigma\omega}\, \xi_\omega, \tag{16}$$

$k = 0$

$$\xi^\alpha_\rho = \delta^\alpha_\rho,$$
$$[\xi_\rho, \xi_\sigma] = 0, \tag{17}$$

$k = -1$

$$\xi^\alpha_\rho = \epsilon^{\rho\alpha\beta} y^\beta + \epsilon^{1\rho\beta}\left(\delta^\alpha_\beta - y^\beta y^\alpha\right),$$
$$[\xi_1, \xi_2] = \xi_3,\ [\xi_1, \xi_3] = -\xi_2,\ [\xi_2, \xi_3] = 0. \tag{18}$$

The only model to which the Taub-Heckmann-Schucking formalism will not apply is a partially anisotropic model whose isometry group contains a three parametric subgroup whose algebra has been labeled (5) on page six.

As pointed out in Appendix II, (5) and (7) are equivalent if the algebras are defined over the field of complex numbers rather than the real numbers. A result of this is that the field equations for space-times with the symmetries of (5) and (7) are similar. The solutions for both cases will be presented in this paper.

Incidentally, it is not necessary that the groups corresponding to the algebras (5) and (7) generate two-spaces; they may just as well generate three dimensional spaces and hence be simply transitive subgroups. Much work has been done on the models which correspond to (5) acting as a simply transitive group.[11,12] The rest of this thesis will be devoted to giving the cosmological solutions corresponding to (5) and (7) acting on two dimensional subspaces.

[11] C. Behr, *Analytische und Numerische Untersuchung der Singularitaten Homogener Rotierender Weltmodelle*, (Dissertation Hamburg, 1965)
[12] L. C. Shepley, *SO(3,R)-Homogeneous Cosmologies*, (Dissertation, Princeton University, 1965).

3. EQUATIONS AND GENERAL SOLUTIONS

In this chapter we will present the general solutions for the spatially homogeneous perfect fluid models with symmetries corresponding to (19) and (20) below, and equations of state indicated in the introduction. Just as the Friedmann solutions with cosmological constant are given in terms of a first order differential equation in $R(t)$, [$R(t)$ here is defined by equation (13)], the general solutions presented here are given in terms of one or more first order differential equations. For particular values of the arbitrary constants that appear in the general solutions, the first order differential equations are integrable. We present some of these explicit solutions in the next chapter.

In the last chapter we proved that any spatially homogeneous space-time which does not have a simply transitive subgroup must be four parametric and have a subgroup whose Lie algebra is (5). However, because of the similarity of (5) and (7) we will solve both cases simultaneously. The two subgroups are classified by Bianchi as types IX and VIII respectively.[13] The algebras of the four parametric groups are found by adding a fourth generator η to (5) and (7) and requiring the Jacobi identity be satisfied. When this is done you find that η must essentially commute with the generators of the subgroup (5) and (7) (Appendix II).

[13] L. Bianchi, *Lezioni sulla teoria dei quippi continue finite di trasformazioni* (Spoerri, Pisa, 1918), p. 550.

The four dimensional Lie algebras of interest are therefore

$$[\xi_1,\xi_2]=\xi_3,\ [\xi_2,\xi_3]=\xi_1,\quad [\xi_3,\xi_1]\ =\xi_2,\ [\eta,\xi_\alpha]=0, \tag{19}$$

$$(\alpha=\ 1,2,3)$$

$$[\xi_1,\xi_2]=\xi_1,\ [\xi_2,\xi_3]=\xi_3,\quad [\xi_3,\xi_1]\ =2\xi_2,\ [\eta,\xi_\alpha]=0. \tag{20}$$

As indicated in the last chapter the procedure used here will be to take (19) or (20) and simultaneously solve Killing's equations (4). Coordinates x^a are chosen such that x^0 is constant on the homogeneous hypersurfaces and the tangents to the lines $x^\alpha=$ constant are orthogonal to the hypersurfaces. The existence of such a local coordinate system is seen by the following argument. First label the homogeneous hypersurfaces by a parameter x^0 and the points in one of these hypersurfaces by x^α. At each point x^α construct the geodesics orthogonal to the hypersurface. The geodesics are to be parameterized by proper time s and labeled by x^α, The tangents to these geodesics are denoted by $v^a=v^a(s,x^\alpha)$. It is now easy to show that these geodesics are orthogonal to all the hypersurfaces. The generators of the isometry group ξ_α span the tangent space of the hypersurfaces at each point in space-time. On the original hypersurface $v^a\xi_{\alpha a}=0$. We now compute how $v^a\xi_{\alpha a}$ changes as we proceed along the geodesics .

$$\frac{dv^a\xi_{\alpha a}}{ds}=(v^a\xi_{\alpha a})_{;b}v^b=v^a_{;b}v^b\xi_{\alpha a}+v^a\xi_{\alpha a;b}v^b=0.$$

The first term vanishes because v^a is a geodesic and the second because the ξ_α are Killing vectors. v^a then remains orthogonal to the vectors ξ_α and hence to all the hypersurfaces. The desired coordinate system is then given by labeling a point P by x^0, the label for the hypersurface that P lies in, and x^α, the label for the geodesic which goes through P. In this coordinate system $g_{0\alpha} = 0$ and also the Killing vectors have vanishing '0' component. Because the symmetry group is transitive on the hypersurfaces, $g_{00} = g_{00}(x^0)$, and you can introduce a coordinate $t = t(x_0)$ such that the metric has the following form:

$$ds^2 = dt^2 - g_{\alpha\beta}(x^\nu, t)\, dx^\alpha dx^\beta \qquad (\alpha, \beta, \nu = 1, 2, 3). \tag{21}$$

One consequence of Killing's equations is now that

$$\frac{\partial \xi_\alpha}{\partial t} = \frac{\partial \eta}{\partial t} = 0.$$

We have assumed that the subgroups (5) and (7) act transitively on two dimensional subspaces. The metric $g_{\alpha\beta}(x^\nu, t)$ can be put into a simple form by choosing coordinates x^ν (in one of the hypersurfaces t = constant) such that $x^1 = r$ labels the two-spaces generated by (5) or (7) and such that $\eta^a = \delta^a_1$. We can find such coordinates by the following construction. We let $\overline{x}^1$ label the two-spaces generated by (5) or (7) and let $\overline{x}^A$ $(A = 2, 3)$ label the points in one of these two-spaces. Now look at the trajectories of the one parametric subgroup whose generator is η and which go through the points $\overline{x}^A$. We label these trajectories by $\overline{x}^A$. Now we label the point P in the hypersurface by $\overline{x}^1$, the label for the two-space in which it lies, and $\overline{x}^A$, the label for the trajectory which goes through P. In this hypersurface and in this coordinate system $\xi^1_\alpha = 0$ and $\eta^\alpha = \eta\delta^\alpha_1$. Since the group structure requires $[\eta, \xi_\alpha] = 0$ we have $\eta^\alpha = \eta(\overline{x}^1)\delta^\alpha_1$. Now we introduce the coordinates $r = \int d\overline{x}^1/\eta(\overline{x}^1)$, $x^A = \overline{x}^A$. η^a now has the desired form $\eta^a = \delta^a_1$. Because the group generators are independent of t it follows that $\eta^a = \delta^a_1$ everywhere and r labels the two-spaces generated by (5) or (7) in every hypersurface. The commutation relations and Killing equations then require the metric to have the following form

$$ds^2 = dt^2 - X(t)^2 dr^2 - Y(t)^2 g_{AB}\, dx^A dx^B \qquad (A, B = 2, 3), \tag{22}$$

where

$$g_{AB} = g_{AB}(x^2, x^3).$$

The two dimensional subspaces $r =$ constant, $t =$ constant, are spaces of constant curvature, positive for (19) and negative for (20). By using the available coordinate transformations g_{AB} can be put in the following form[14] for (19)

$$g_{AB}\, dx^A dx^B = \left[\sin^2\theta (d\phi)^2 + (d\theta)^2\right], \tag{23}$$

for (20)

$$g_{AB}\, dx^A dx^B = \left[\sinh^2\theta (d\phi)^2 + (d\theta)^2\right]. \tag{24}$$

The spatially homogeneous metrics which have the symmetries of (18) and (19) are

Case I.

$$ds^2 = dt^2 - X(t)^2 dr^2 - Y(t)^2 \left[\sin^2\theta (d\phi)^2 + (d\theta)^2\right], \tag{25}$$

Case II.

$$ds^2 = dt^2 - X(t)^2 dr^2 - Y(t)^2 \left[\sinh^2\theta (d\phi)^2 + (d\theta)^2\right]. \tag{26}$$

[14]These computations can be checked with A. S. Petrov, *EinsteinRaume* (Akademie-Verlag, Berlin, 1964), p. 60 and pp. 225-6.

The Killing vectors are

Case I.

$$\eta = \frac{\partial}{\partial r}, \quad \xi_1 = \frac{\partial}{\partial \phi}, \quad \xi_2 = \sin\phi\,\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\,\frac{\partial}{\partial\phi},$$

$$\xi_3 = \cos\phi\,\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\,\frac{\partial}{\partial\phi}, \tag{27}$$

Case II.

$$\eta = \frac{\partial}{\partial r}, \quad \xi_1 = -\cos\phi\,\frac{\partial}{\partial\theta} + (\coth\theta\sin\phi - 1)\,\frac{\partial}{\partial\phi},$$

$$\xi_2 = \sin\phi\,\frac{\partial}{\partial\theta} + \coth\theta\cos\phi\,\frac{\partial}{\partial\phi}, \quad \xi_3 = \cos\phi\,\frac{\partial}{\partial\theta} - (\coth\theta\sin\phi + 1)\,\frac{\partial}{\partial\phi}. \tag{28}$$

The field equations are[15]

Case I.

$$\begin{aligned}
\Lambda + T_0^0 &= 2\frac{\dot{X}\dot{Y}}{XY} + \frac{1+\dot{Y}^2}{Y^2}, && (29)\\
\Lambda + T_1^1 &= 2\frac{\ddot{Y}}{Y} + \frac{1+\dot{Y}^2}{Y^2}, && (30)\\
\Lambda + T_2^2 &= \Lambda + T_3^3 = \frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + \frac{\dot{X}\dot{Y}}{XY}, && (31)\\
T_j^i &= 0 \qquad (i \neq j). && (32)
\end{aligned}$$

[15]These field equations can be checked with H. Bondi, *M.N.* **107** (1947). However, the integrations performed there do not apply because Y is not a function of r.

Case II.

$$
\begin{aligned}
\Lambda + T^0_0 &= 2\,\frac{\dot X \dot Y}{XY} - \frac{1-\dot Y^2}{Y^2}, & (33)\\
\Lambda + T^1_1 &= 2\,\frac{\ddot Y}{Y} - \frac{1-\dot Y^2}{Y^2}, & (34)\\
\Lambda + T^2_2 &= \Lambda + T^3_3 = \frac{\ddot X}{X} + \frac{\ddot Y}{Y} + \frac{\dot X \dot Y}{XY}, & (35)\\
T^i_j &= 0 \qquad (i \neq j). & (36)
\end{aligned}
$$

For a perfect fluid energy-momentum tensor,

$$T^{ab} = (\rho + p)\,u^a u^b - p\,g^{ab} \qquad u^a u_a = 1, \tag{37}$$

equations (32) or (36) imply $u^a = \delta^a_0$, *i.e.,* u^a is hypersurface orthogonal. It is geodesic because

$$u^a{}_{;b} u^b = \left\{ \begin{array}{c} a \\ 0\ 0 \end{array} \right\} = 0.$$

The fluid flow lines therefore form a geodesic normal congruence and are hence irrotational. The models are not, however, expansion or shear free.

The results presented in this chapter will be the general solutions for equations of state of the form $p = A\,\rho$, where $A = 0$, $1/3$, and 1. Special explicit solutions will be presented in the next chapter. The contracted Bianchi identities give

$$\frac{\dot\rho}{\rho + p} + \frac{[X\dot Y^2]}{[XY^2]} = 0. \tag{38}$$

For $p = A\,p$, (38) integrates to give a constant of the motion;

$$\rho = \frac{\rho_0}{[XY^2]^{A+1}}, \qquad \rho_0 = \text{constant}. \tag{39}$$

In some instances (39) will aid in finding solutions of the field equations. In general the method of finding solutions is to find linear combinations of (29) through (31) and (33) through (35) which are homogeneous and easily integrable. We will not present the details but simply list the solutions. They can be checked by plugging them into the appropriate field equations.

Notice that metric (25) goes over into (26) under a transformation $\theta \rightarrow i\theta$, $Y \rightarrow iY$; and also that the field equations for the two cases become identical. In some cases it should be possible to obtain solutions to (26) provided the solutions to (25) are known. The obvious sufficient condition is that the constants in Y be alterable (real - complex) in such a way that $Y \rightarrow iY$. This technique was used in the following calculations.

Case I.

(i) Dust
$\Lambda = 0$, $p = 0$

A function $\eta(t)$ is introduced to simplify the form of the metric.

$$\begin{aligned} X &= \epsilon + (\epsilon\,\eta + b)\tan\eta, \\ Y &= a\cos^2\eta, \\ t - t_0 &= a\,(\eta + \frac{1}{2}\sin 2\eta), \\ \rho &= \frac{\epsilon\sec^4\eta}{a^2\,[1 + (\eta + b)\,\tan\eta]}. \end{aligned} \tag{40}$$

ϵ, a and b are constants satisfying[16]

$$\epsilon = 0,\, 1; \quad -\infty < a < \infty,\, a \neq 0; \quad -\pi/2 < b \leq \pi/2.$$

(ii) Dust
$\Lambda \neq 0,\ p = 0$

For the general solutions, t is expressed as an elliptic integral of Y, and X as a function of Y and $\dot{Y}$.

$$\begin{aligned} t - t_0 &= \int \left[\frac{\Lambda}{3}\, Y^2 + \frac{a}{Y} - 1\right]^{-1/2} dY, \\ X &= \dot{Y}\left[b + \int \frac{dY}{\dot{Y}^3 Y}\right], \\ \rho &= \frac{2}{XY^2}. \end{aligned} \tag{41}$$

a and b are constants and $\dot{Y} = \left[\frac{\Lambda}{3} Y^2 + \frac{a}{Y} - 1\right]^{1/2}$. The first equation is just the Friedmann equation. The special Case of $a = 0$ is explicitly integrable and is considered in the next chapter.

(iii) Photon gas
$\Lambda = 0,\ p = \frac{1}{3}\rho$

These solutions are most easily expressed in terms of functions $\beta(t)$ and $\nu(t)$,

$$\begin{aligned} X &= \frac{\beta^2}{\nu}, \\ Y &= \frac{\nu^2}{\beta}, \\ \rho &= 3\left(\frac{a}{3}\right)^2 \frac{1}{\nu^4}, \end{aligned}$$

[16]This solution has been compared with the Friedmann solutions, see footnote 1. It has been independently found by K. Thorne in a form which contains a magnetic field; *Geometrodynamics of Cylindrical Systems*, (Dissertation, Princeton, 1965).

where

$$\nu = \left[c\,\beta + \beta^2 - \frac{\beta^4}{a^2} \right]^{1/2},$$

and β is related to t by

$$\int \nu\, d\beta = \frac{a}{3}(t - t_0). \tag{42}$$

a and c are constants and $a \neq 0$. The solution which corresponds to $a = 0$ is presented in the next chapter along with the explicit solution for $c = 0$.

(iv) Stiff matter
$\Lambda = 0,\ p = \rho$

For the general solution, t is given as an integral of X. $\dot{X}$ is assumed different from zero.[17]

$$\begin{aligned}
(t - t_0) &= a^2 c \int \frac{dX}{X^2 \cosh^2(a \ln X)}, \\
Y &= \frac{c\, a}{X \cosh(a \ln X)}, \\
\rho &= \frac{X^2}{a^2 c^2} \cosh^4(a \ln X) \left(1 - \frac{1}{a^2} \right),
\end{aligned} \tag{43}$$

a and c are constants, $a \geq 1,\ c \neq 0$.

Case II.

The solutions for Case II are similar to the solutions for Case I.

(i) Dust
$\Lambda = 0,\ p = 0$

Two different solutions arise for this case; we call them (a) and (b).[18] In both cases a function $\eta(t)$ is introduced to simplify

[17]For $\dot{X} = 0$ the equations simplify and the solutions have already been given by Thorne, footnote 16.
[18]See footnote 1.

the form of the solutions.

(a)

$$
\begin{aligned}
X &= \epsilon - (\epsilon\,\eta + b)\tanh\eta, \\
Y &= a\,\cosh^2\eta, \\
t - t_0 &= a\,(\eta + \frac{1}{2}\sinh 2\eta), \\
\rho &= -\epsilon\,\frac{\mathrm{sech}^4\eta}{a^2\,[1 - (\eta + b)\tanh\eta]},
\end{aligned}
\tag{44}
$$

(b)

$$
\begin{aligned}
X &= \epsilon - (\epsilon\,\eta + b)\coth\eta, \\
Y &= a\sinh^2\eta, \\
t - t_0 &= a\,(\eta - \frac{1}{2}\sinh 2\eta), \\
\rho &= -\epsilon\,\frac{\mathrm{csch}^4\eta}{a^2\,[1 - (\eta + b)\coth\eta]}.
\end{aligned}
\tag{45}
$$

In both (a) and (b); ϵ, a, and b are constants subject to the restrictions:

$$\epsilon = 0, 1; \quad -\infty < a < \infty,\ a \neq 0; \quad 0 \leq b < \infty.$$

(ii) Dust

$\Lambda \neq 0,\ p = 0$

For the general solution, t and X are expressed as elliptic integrals of Y,

$$
\begin{aligned}
t - t_0 &= \int \left[\frac{\Lambda}{3}Y^2 - \frac{a}{Y} + 1\right]^{-1/2} dY, \\
X &= \dot{Y}\left[b + \int \frac{dY}{\dot{Y}^3 Y}\right], \\
\rho &= \frac{2}{XY^2}.
\end{aligned}
\tag{46}
$$

a and b are constants and $\dot{Y} = \left[\frac{\Lambda}{3}Y^2 - \frac{a}{Y} + 1\right]^{1/2}$.

The special case of $a = 0$ is presented in the next chapter.

(iii) Photon gas

$\Lambda = 0, \ p = 1/3\rho$

These solutions are most easily expressed in terms of functions $\beta(t)$ and $\nu(t)$.

$$
\begin{aligned}
X &= \frac{\beta^2}{\nu}, \\
Y &= \frac{\nu^2}{\beta}, \\
\rho &= 3\left(\frac{a}{3}\right)^2 \frac{1}{\nu^4},
\end{aligned}
$$

where

$$\nu = \left[c\,\beta + \beta^2 + \frac{\beta^4}{a^2}\right]^{1/2},$$

and β satisfies

$$t - t_0 = \frac{3}{a}\int \nu\, d\beta. \tag{47}$$

In the above a and c are constants and $a \neq 0$. The special case which corresponds to $a = 0$ gives a negative energy density ρ and is therefore of no interest. The special case of $c = 0$ is presented in the next chapter.

(iv) Stiff matter

$\Lambda = 0, \ p = \rho$

For the general solution, t is given as an integral of X. $\dot{X}$ is assumed different from zero. The special case of $\dot{X} = 0$ is given in the next chapter.

$$
\begin{aligned}
t - t_0 &= a^2 c \int \frac{dX}{X^2 \sinh^2(a \ln X)}\,, \\
Y &= \frac{c\,a}{X \sinh(a \ln X)}\,, \\
\rho &= \frac{X^2}{a^2 c^2} \sinh^4(a \ln X) \left(1 - \frac{1}{a^2}\right),
\end{aligned}
\tag{48}
$$

a and c are constants, $a \geq 1,\ c \neq 0$.

4. PARTICULAR SOLUTIONS AND GENERAL PROPERTIES

In this chapter we present some explicit solutions to the equations presented in the last chapter. These solutions correspond to particular values of the constants in the general solutions for which the equations are completely integrable. The expansion Θ and shear σ are also computed and plotted along with the energy density ρ to show the overall behavior of the various solutions.

The expansion and shear are defined as follows

$$\Theta = u^a{}_{;a}\,, \tag{49}$$

$$\sigma_{ab} = h^c{}_{(b}u_{a);c} - \frac{1}{3}\,\Theta\, h_{ab}\,, \tag{50}$$

$$\sigma = \frac{1}{\sqrt{2}}\left(\sigma_{ab}\,\sigma^{ab}\right)^{1/2}, \tag{51}$$

where

$$h_{ab} = g_{ab} - u_a u_b\,.$$

For metrics (25) and (26), which I repeat here,

$$ds^2 = dt^2 - X(t)^2 dr^2 - Y(t)^2\left[\sin^2\theta\,(d\phi)^2 + (d\theta)^2\right], \tag{25}$$

$$ds^2 = dt^2 - X(t)^2 dr^2 - Y(t)^2\left[\sinh^2\theta\,(d\phi)^2 + (d\theta)^2\right], \tag{26}$$

and a perfect fluid energy-momentum tensor, the fluid flow lines reduced to $u^a = \delta^a_0$. The expansion and shear reduce to:

$$\Theta = \frac{\dot{X}}{X} + 2\,\frac{\dot{Y}}{Y}\,, \tag{52}$$

$$\sigma = \frac{1}{\sqrt{3}}\left\|\frac{\dot{X}}{X} - \frac{\dot{Y}}{Y}\right\|\,. \tag{53}$$

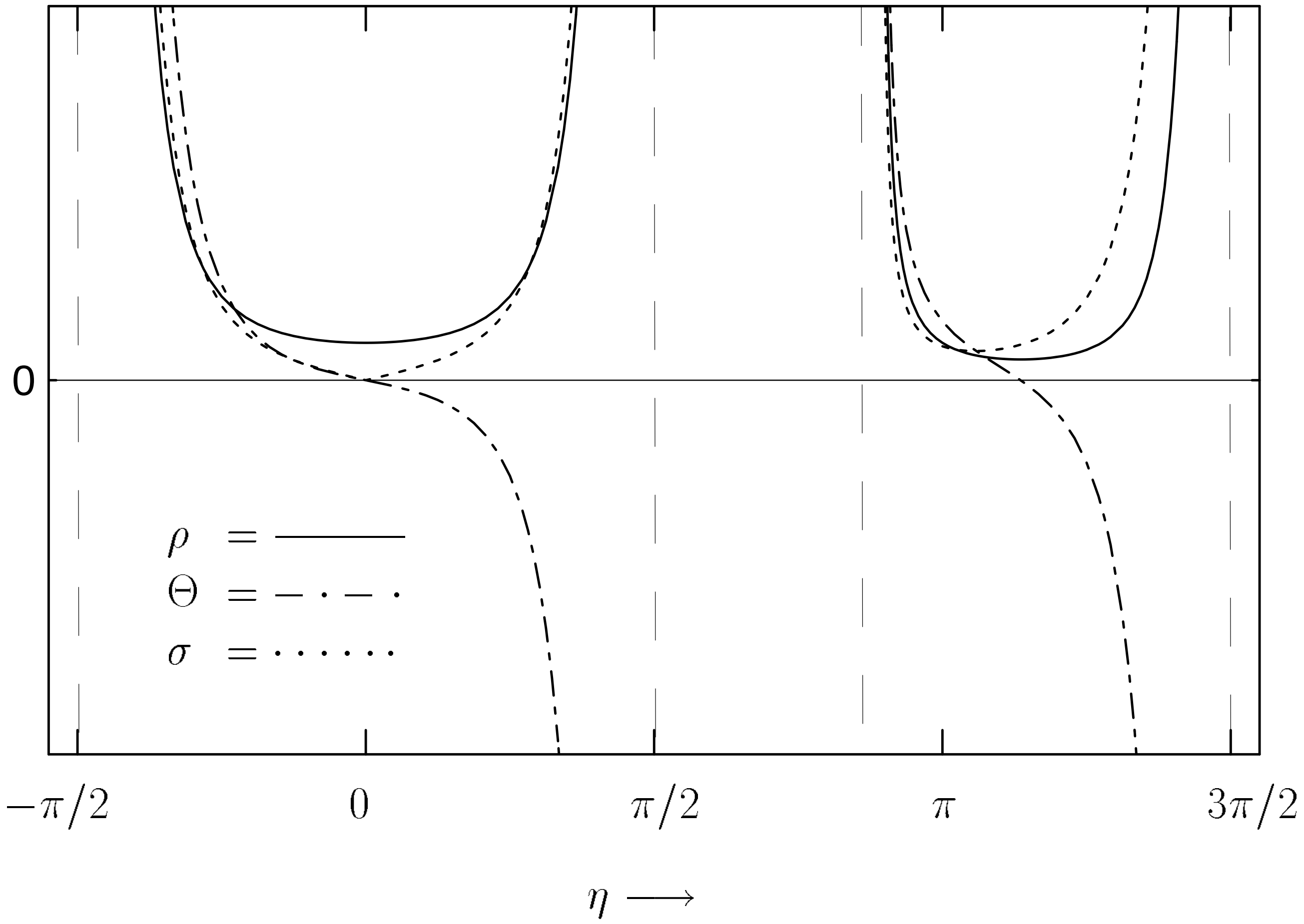

Fig.1. Graphs of ρ, Θ, and σ for General Case of $\Lambda = 0$, $p = 0$; $b = 0$, $a = 1$.

Case I.
$\Lambda = 0$
$p = 0$

In this case the general solution is completely integrated and has been given in (40). It is repeated here for completeness.

$$
\begin{aligned}
X &= 1 + (\eta + b)\tan\eta, \\
Y &= a\,\cos^2\eta, \\
t - t_0 &= a(\eta + \frac{1}{2}\sin 2\eta), \\
\rho &= \frac{\sec^4\eta}{a^2\,[1 + (\eta + b)\tan\eta]}, \\
\Theta &= \frac{\sec^2\eta}{2a}\left[\frac{(\eta + b) - 3\tan\eta - 3\,(\eta + b)\tan^2\eta}{1 + (\eta + b)\tan\eta}\right], \\
\sigma &= \frac{\sec^2\eta}{2|a|\sqrt{3}}\left\|\frac{(\eta + b) + 3\tan\eta + 3\,(\eta + b)\tan^2\eta}{1 + (\eta + b)\tan\eta}\right\|.
\end{aligned}
$$

a and b are constants satisfying

$$-\infty < a < \infty, \quad a \neq 0, \quad -\pi/2 < b \leq \pi/2.$$

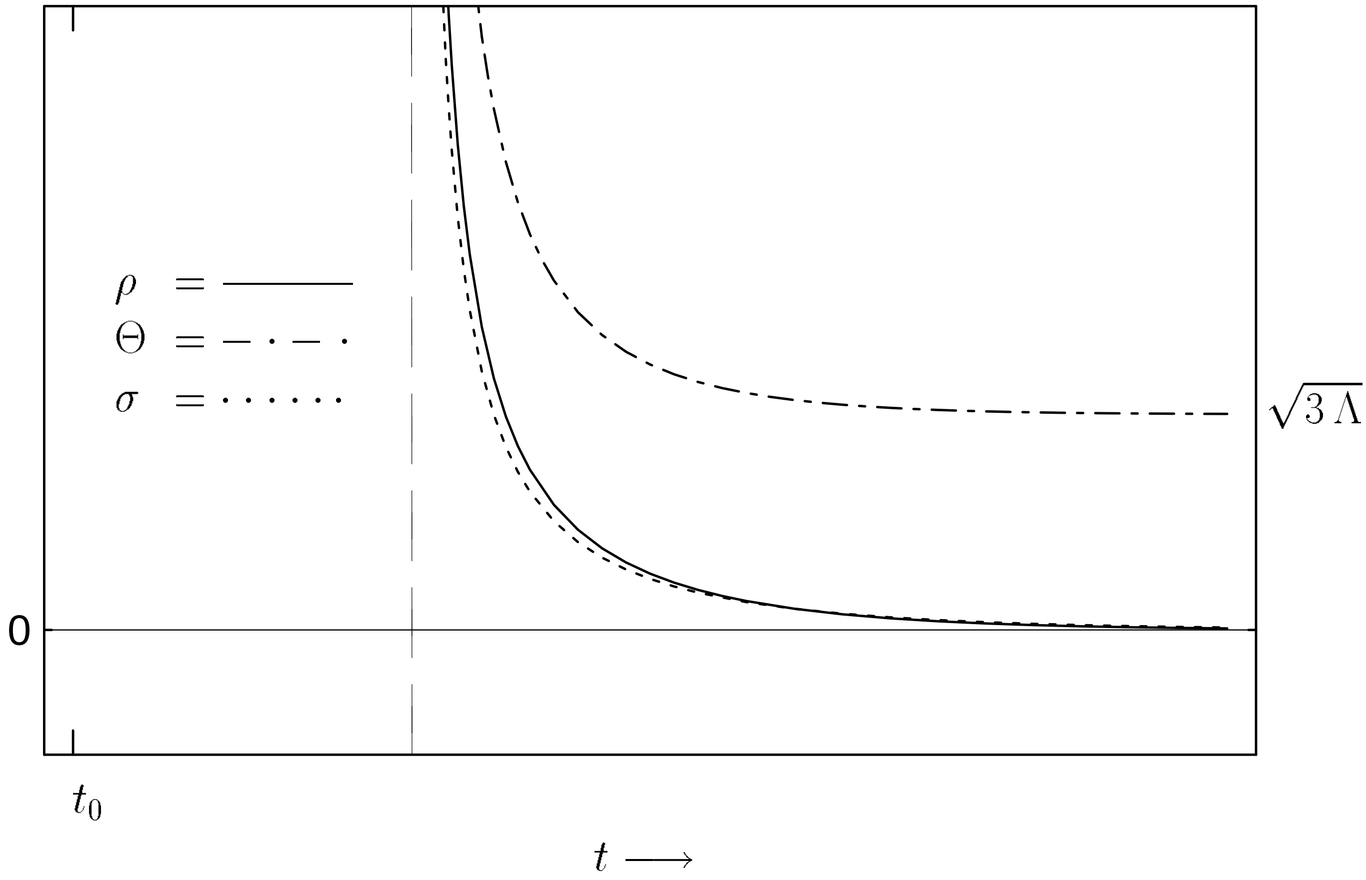

Fig.2. Graphs of ρ, Θ, and σ for Special Case of $\Lambda \neq 0$, $p = 0$; $a = 0$, $b < -\pi/2$.

Case I.
$\Lambda \neq 0$
$p = 0$

This special case corresponds to $a = 0$ in equation (41). Λ must be greater than zero; the solution is:

$$
\begin{aligned}
Y &= \sqrt{\frac{3}{\Lambda}} \cosh\left\{\sqrt{\frac{\Lambda}{3}}(t - t_0)\right\}, \\
X &= b \sinh\omega + 1 + \sinh\omega \tan^{-1}(\sinh\omega), \\
\rho &= -\frac{2}{3}\frac{\Lambda}{\cosh^2\omega \left[b \sinh\omega + 1 + \sinh\omega \tan^{-1}(\sinh\omega)\right]}, \\
\Theta &= \sqrt{\frac{\Lambda}{3}} \frac{\left[(3\cosh\omega - 2\,\mathrm{sech}\,\omega)\left(b + \tan^{-1}(\sinh\omega)\right) + 3\tanh\omega\right]}{\left[b \sinh\omega + 1 + \sinh\omega \tan^{-1}(\sinh\omega)\right]}, \\
\sigma &= \frac{\sqrt{\Lambda}}{3} \frac{\mathrm{sech}\,\omega \left[b + \tan^{-1}(\sinh\omega)\right]}{\left[b \sinh\omega + 1 + \sinh\omega \tan^{-1}(\sinh\omega)\right]},
\end{aligned}
$$

where

$$\omega = \sqrt{\frac{\Lambda}{3}}(t - t_0).$$

b is the same constant that appears in (41), $b < -\pi/2$. The models start with $\rho = \infty$ and expand to $\rho = 0$ in an infinite proper time.

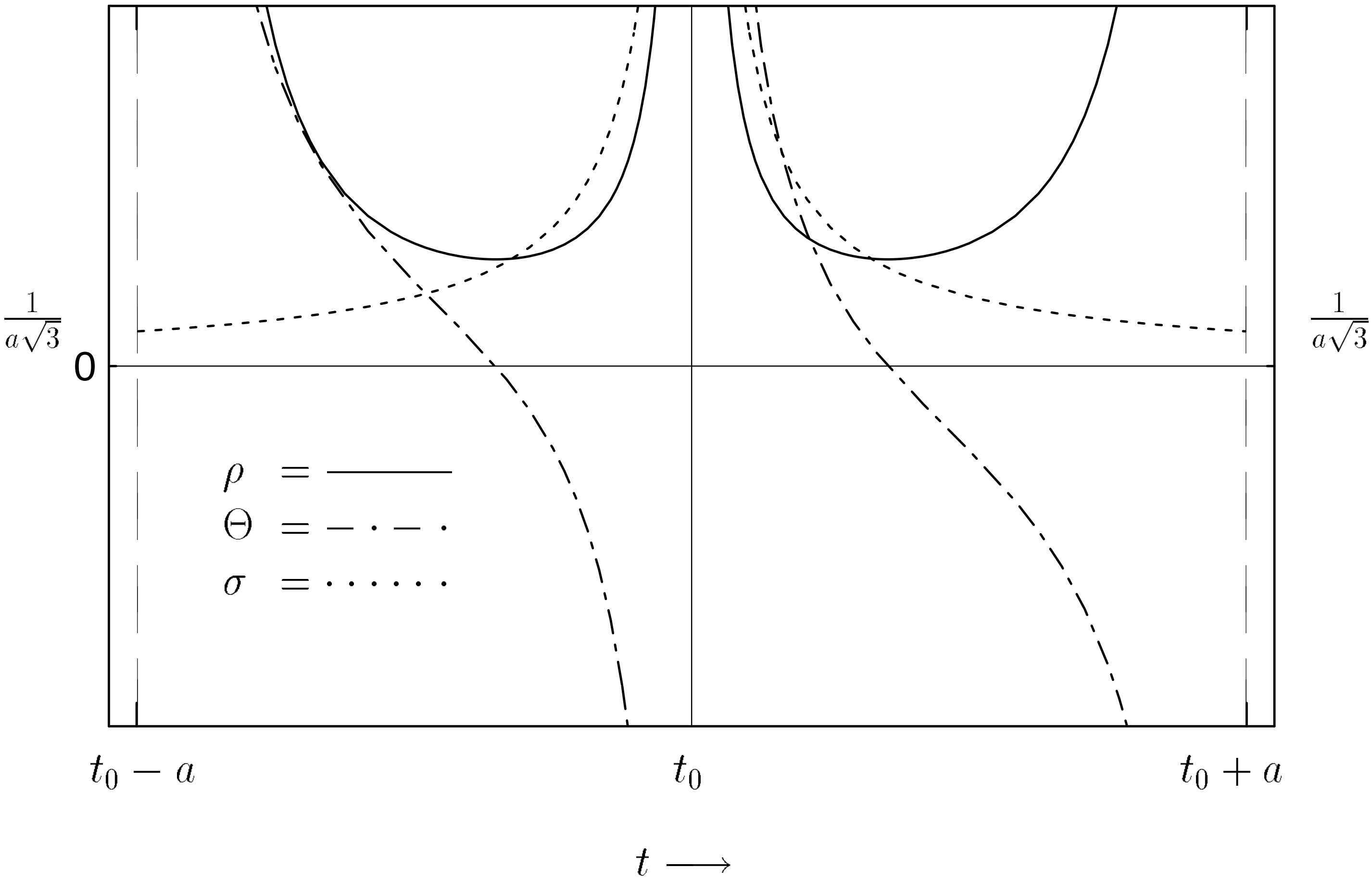

Fig.3. Graphs of ρ, Θ, and σ for Special Case of $\Lambda = 0$, $p = 1/3\,\rho$; $c = 0$.

Case I.
$\Lambda = 0$
$p = \frac{1}{3}\rho$

The first explicit solution corresponds to $c = 0$ in equation (42).

$$
\begin{aligned}
Y &= a\left[1-\left(\frac{t-t_0}{a}\right)^{2/3}\right]^{1/2}\left(\frac{t-t_0}{a}\right)^{2/3}, \\
X &= a\left[\frac{1}{\left(\frac{t-t_0}{a}\right)^{2/3}}-1\right]^{1/2}, \\
\rho &= \frac{1}{3a^2\left[\left(1-\left(\frac{t-t_0}{a}\right)^{2/3}\right)\left(\frac{t-t_0}{a}\right)^{2/3}\right]^2}, \\
\Theta &= \frac{1}{t-t_0}\frac{\left[1-2\left(\frac{t-t_0}{a}\right)^{2/3}\right]}{\left[1-\left(\frac{t-t_0}{a}\right)^{2/3}\right]}, \\
\sigma &= \frac{1}{\sqrt{3}}\left\|\frac{1}{t-t_0}\right\|.
\end{aligned}
$$

$a, a \neq 0$, is the same constant as in (42). These models all start in a singular state ($\rho = \infty$) and end in a singular state. The universe exists for a proper time equal to a.

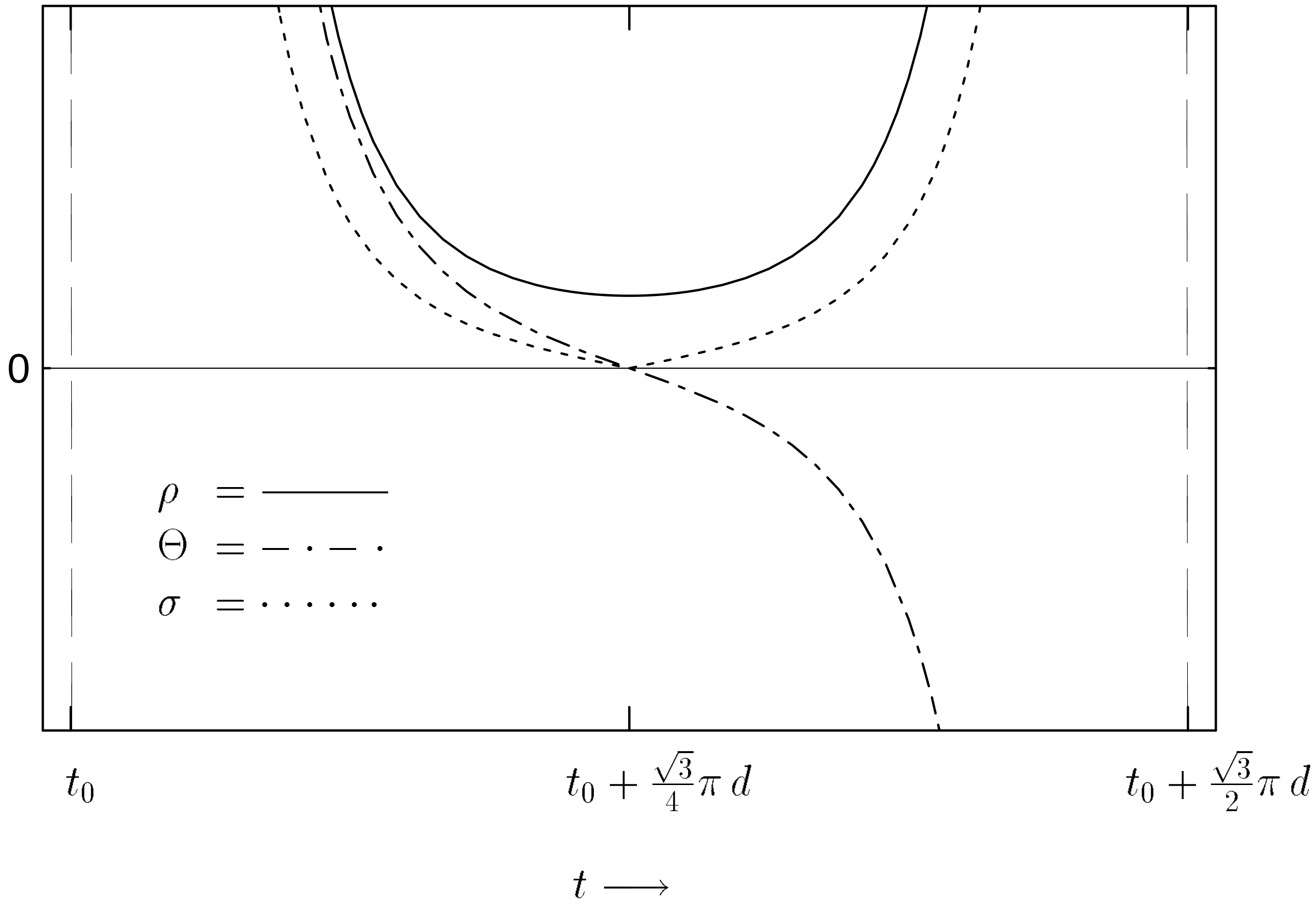

Fig.4. Graphs of ρ, Θ, and σ for Special Case of $\Lambda = 0$, $p = 1/3\,\rho$; $a = 0$.

Case I.
$\Lambda = 0$
$p = \frac{1}{3}\rho$

This again is an explicit solution corresponding to (42) but for the constant $a = 0$. The equations to solve are slightly different and the following explicit solutions cannot be gotten from the general solution (42) by any obvious limiting procedure. They are best expressed in terms of a function of $\chi(t)$.

$$
\begin{aligned}
Y &= d \sin^2 \chi, \\
X &= \frac{1}{\sin \chi}, \\
\rho &= \frac{1}{d^2 \sin^4 \chi}, \\
\Theta &= \frac{\sqrt{3}}{d} \frac{\cos \chi}{\sin^3 \chi}, \\
\sigma &= \left\| \frac{1}{d} \frac{\cos \chi}{\sin^3 \chi} \right\|,
\end{aligned}
$$

where χ is related to t by

$$t - t_0 = \frac{\sqrt{3}}{2} d \left[\chi - \frac{\sin 2\chi}{2} \right].$$

d is a constant, $d \neq 0$. These models start with $\rho = \infty$ and end with $\rho = \infty$. The proper time for this evolution is $\frac{\sqrt{3}}{2} d\ \pi$.

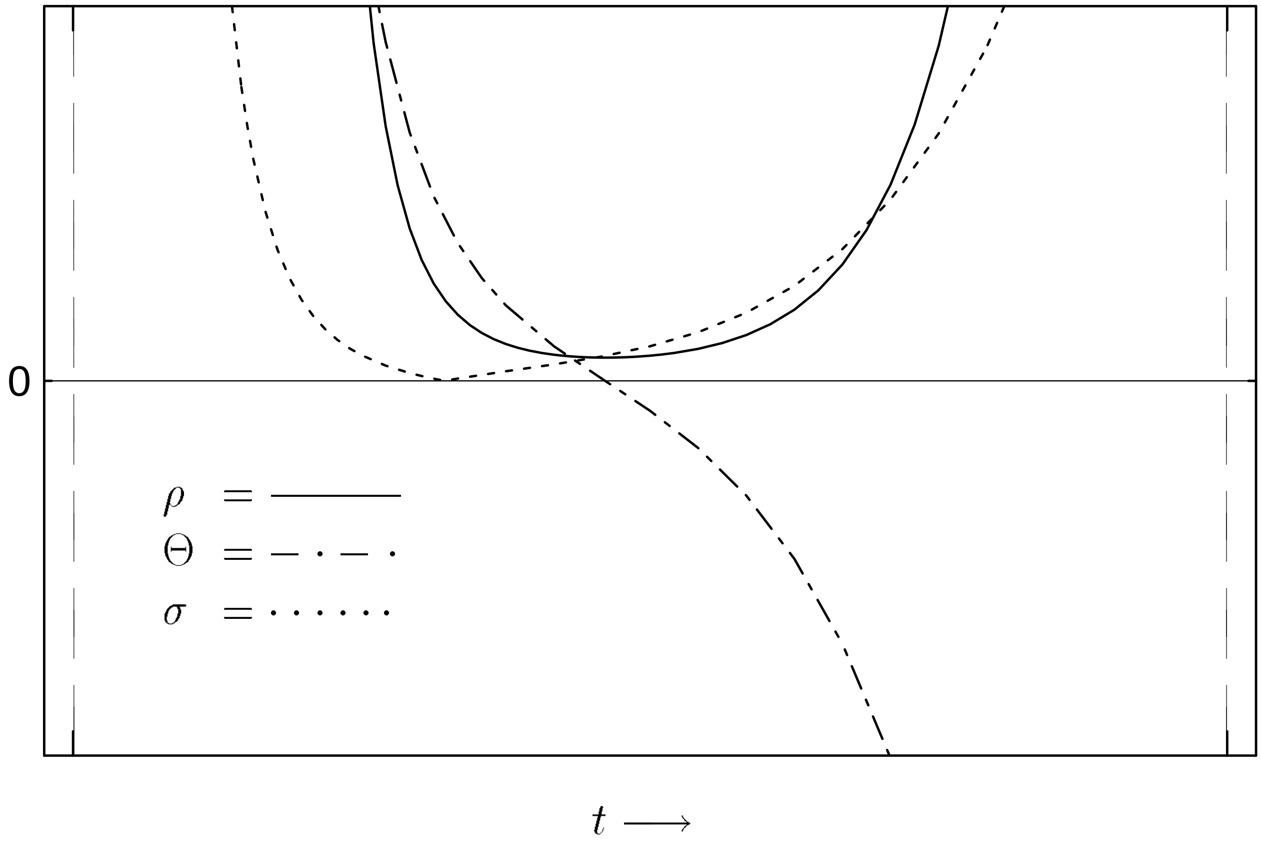

Fig.5. Graphs of ρ, Θ, and σ for General Case of $\Lambda = 0$, $p = \rho$; $c = 1$, $a > 2$.

Case I.
Stiff Matter
$\Lambda = 0$
$p = \rho$

There are no obvious special cases (see footnote 17) of the general solutions (43). I repeat the general solution here and indicate its properties by the corresponding graphs.

$$
\begin{aligned}
t - t_0 &= a^2 c \int \frac{dX}{X^2 \cosh^2(a \ln X)}, \\
Y &= \frac{c\, a}{X \cosh(a \ln X)}, \\
\rho &= \frac{X^2}{a^2 c^2} \cosh^4(a \ln X) \left(1 - \frac{1}{a^2}\right), \\
\Theta &= \frac{-X}{a^2 c} [1 + 2a \tanh(a \ln X)] \cosh^2(a \ln X), \\
\sigma &= \frac{1}{\sqrt{3}} \left\| \frac{X}{a^2 c} [2 + a \tanh(a \ln X)] \cosh^2(a \ln X) \right\|,
\end{aligned}
$$

a and c are constants, $a \geq 1,\ c \neq 0$. The general solution starts in a singular state and ends in a singular state. The proper time for this evolution is finite.

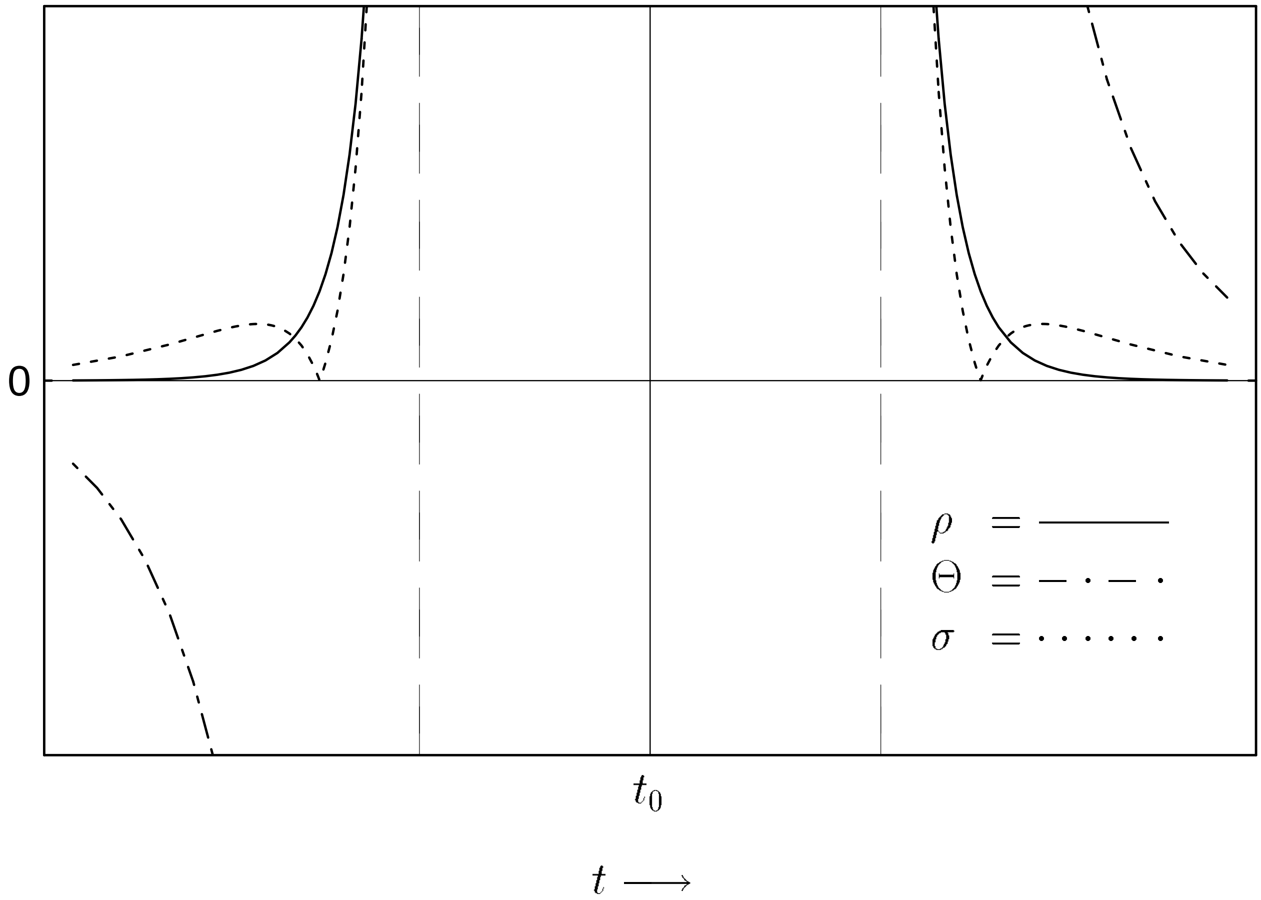

Fig.6. Graphs of ρ, Θ, and σ for General Case of $\Lambda = 0$, $p = 0$; $b = 0$.

Case II.
$\Lambda = 0$
$p = 0$

The general solutions are completely integrated in (44) and (45). I repeat the general solution (44) to indicate their general properties .

$$
\begin{aligned}
X &= 1-(\eta+b)\tanh\eta, \\
Y &= a\,\cosh^2\eta, \\
t-t_0 &= a(\eta+\frac{1}{2}\sinh 2\eta), \\
\rho &= -\frac{\mathrm{sech}^4\eta}{a^2\,[1-(\eta+b)\tanh\eta]}, \\
\Theta &= -\frac{\mathrm{sech}^2\eta}{2a}\left[\frac{(\eta+b)-3\tanh\eta+3\,(\eta+b)\tanh^2\eta}{1-(\eta+b)\tanh\eta}\right], \\
\sigma &= \frac{\mathrm{sech}^2\eta}{2|a|\sqrt{3}}\left\|\frac{(\eta+b)+3\tanh\eta-3\,(\eta+b)\tanh^2\eta}{1-(\eta+b)\tanh\eta}\right\|.
\end{aligned}
$$

a and b are constants subject to the restrictions:

$$-\infty < a < \infty,\ a \neq 0\,;\quad 0 \leq b < \infty.$$

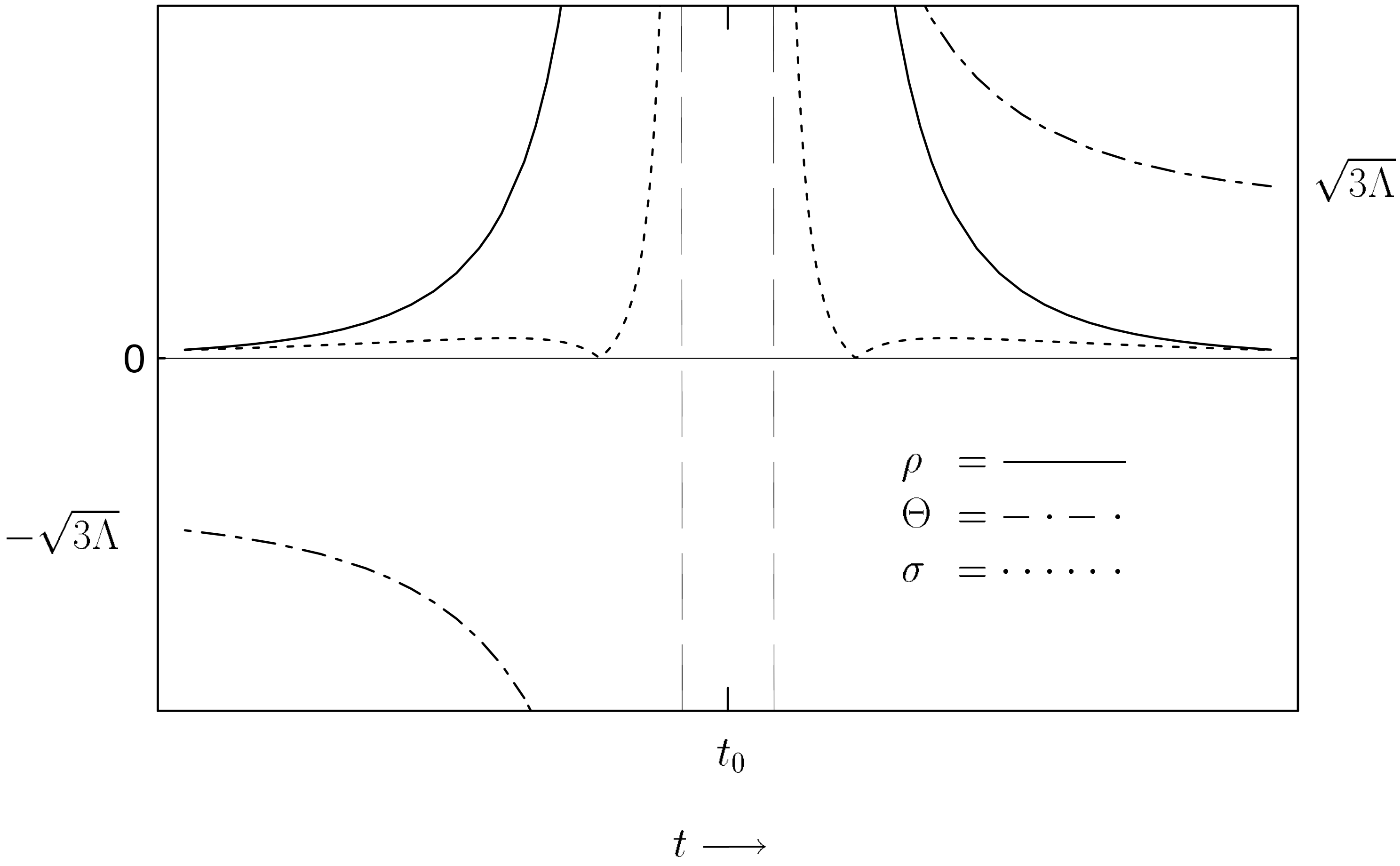

Fig.7a. Graphs of ρ, Θ, and σ for Special Case of $\Lambda > 0$, $p = 0$ $a = 0$, $b > 0$.

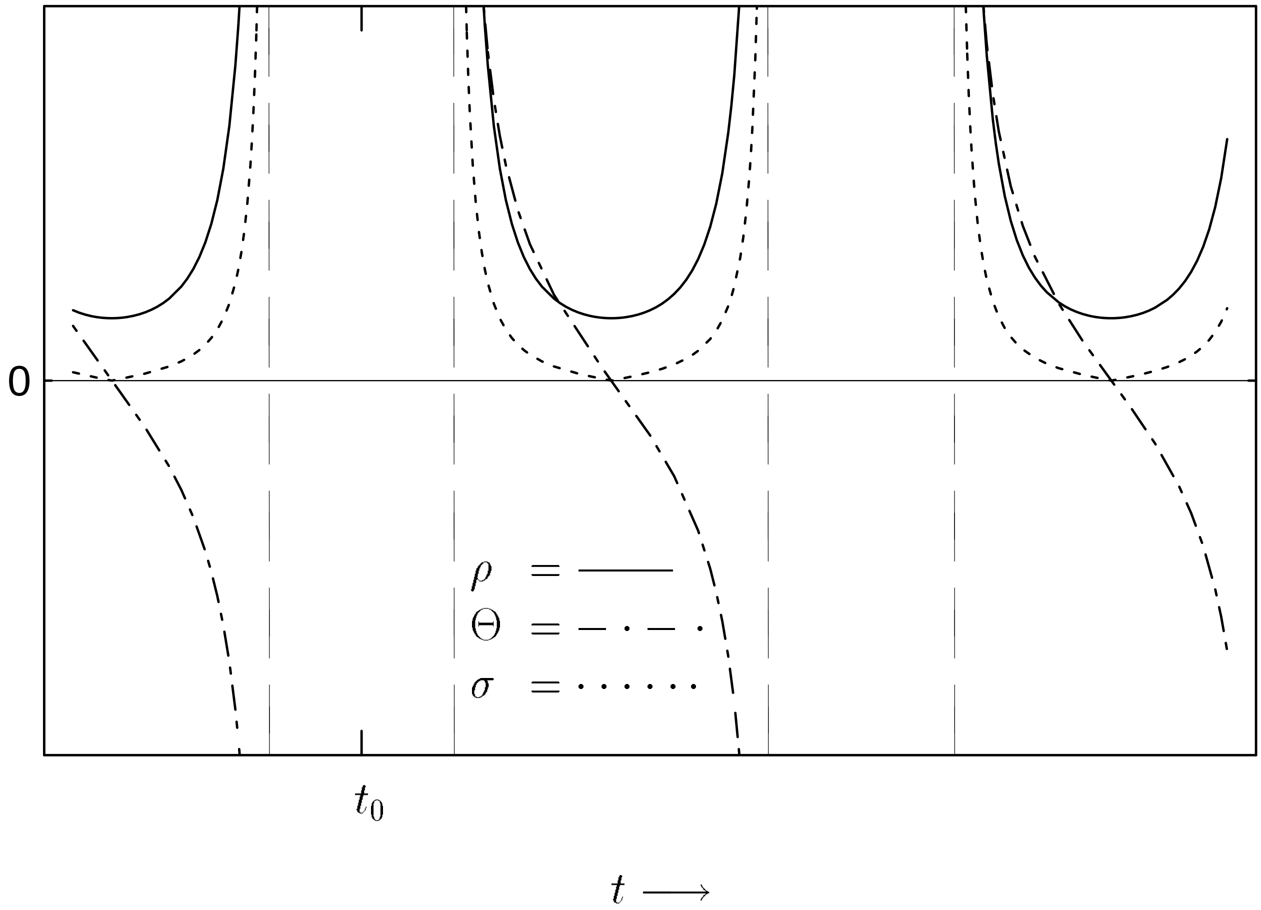

Fig.7b. Graphs of ρ, Θ, and σ for Special Case of $\Lambda < 0$, $p = 0$; $a = 0$, $b = 0$.

Case II.
$\Lambda \neq 0$
$p = 0$

Two explicit solutions arise for the case $a = 0$ in equation (46). The two solutions correspond to the two possible signs of Λ.

$$
\begin{aligned}
\Lambda &> 0, \\
Y &= \sqrt{\frac{3}{\Lambda}} \sinh\omega, \\
X &= b\cosh\omega + 1 + \cosh\omega \, \log \left\| \tanh\frac{\omega}{2} \right\|, \\
\rho &= \frac{2}{3} \frac{\Lambda \, \mathrm{csch}^{\,2}\omega}{\left[b\cosh\omega + 1 + \cosh\omega \, \log \left\| \tanh\frac{\omega}{2} \right\|\right]}, \\
\Theta &= \sqrt{\frac{\Lambda}{3}} \frac{\left[(3\sinh\omega + 2\,\mathrm{csch}\,\omega)\left(b + \log \left\| \tanh\frac{\omega}{2} \right\|\right) + 3\coth\omega\right]}{\left[b\cosh\omega + 1 + \cosh\omega \log \left\| \tanh\frac{\omega}{2} \right\|\right]}, \\
\sigma &= \frac{\sqrt{\Lambda}}{3} \left\| \frac{\left[b \,\mathrm{csch}\,\omega + \mathrm{csch}\,\omega \log \left\| \tanh\frac{\omega}{2} \right\|\right]}{\left[b\cosh\omega + 1 + \cosh\omega \log \left\| \tanh\frac{\omega}{2} \right\|\right]} \right\|.
\end{aligned}
$$

$$
\begin{aligned}
\Lambda &< 0, \\
Y &= \sqrt{\frac{3}{\|\Lambda\|}} \sin\omega, \\
X &= b\cos\omega + 1 + \cos\omega \log \left\| \tan\frac{\omega}{2} \right\|, \\
\rho &= \frac{2}{3} \frac{\|\Lambda\|}{\sin^{2}\omega \left[b \, \cos\omega + 1 + \cos\omega \log \left\| \tan\frac{\omega}{2} \right\|\right]},
\end{aligned}
$$

$$
\begin{aligned}
\Theta &= \sqrt{\frac{\|\Lambda\|}{3}}\ \frac{\left[(-3\sin\omega + 2\csc\omega)\left(b + \log\left\|\tan\frac{\omega}{2}\right\|\right) + 3\cot\omega\right]}{\left[b\,\cos\omega + 1 + \cos\omega\log\left\|\tan\frac{\omega}{2}\right\|\right]}, \\
\sigma &= \frac{\sqrt{\|\Lambda\|}}{3}\ \left\|\frac{b\csc\omega + \csc\omega\log\left\|\tan\frac{\omega}{2}\right\|}{\left[b\cos\omega + 1 + \cos\omega\log\left\|\tan\frac{\omega}{2}\right\|\right]}\right\|.
\end{aligned}
$$

In both cases $\omega = \sqrt{\frac{\|\Lambda\|}{3}}\,(t - t_0)$, and b is the same constant that appears in (46).

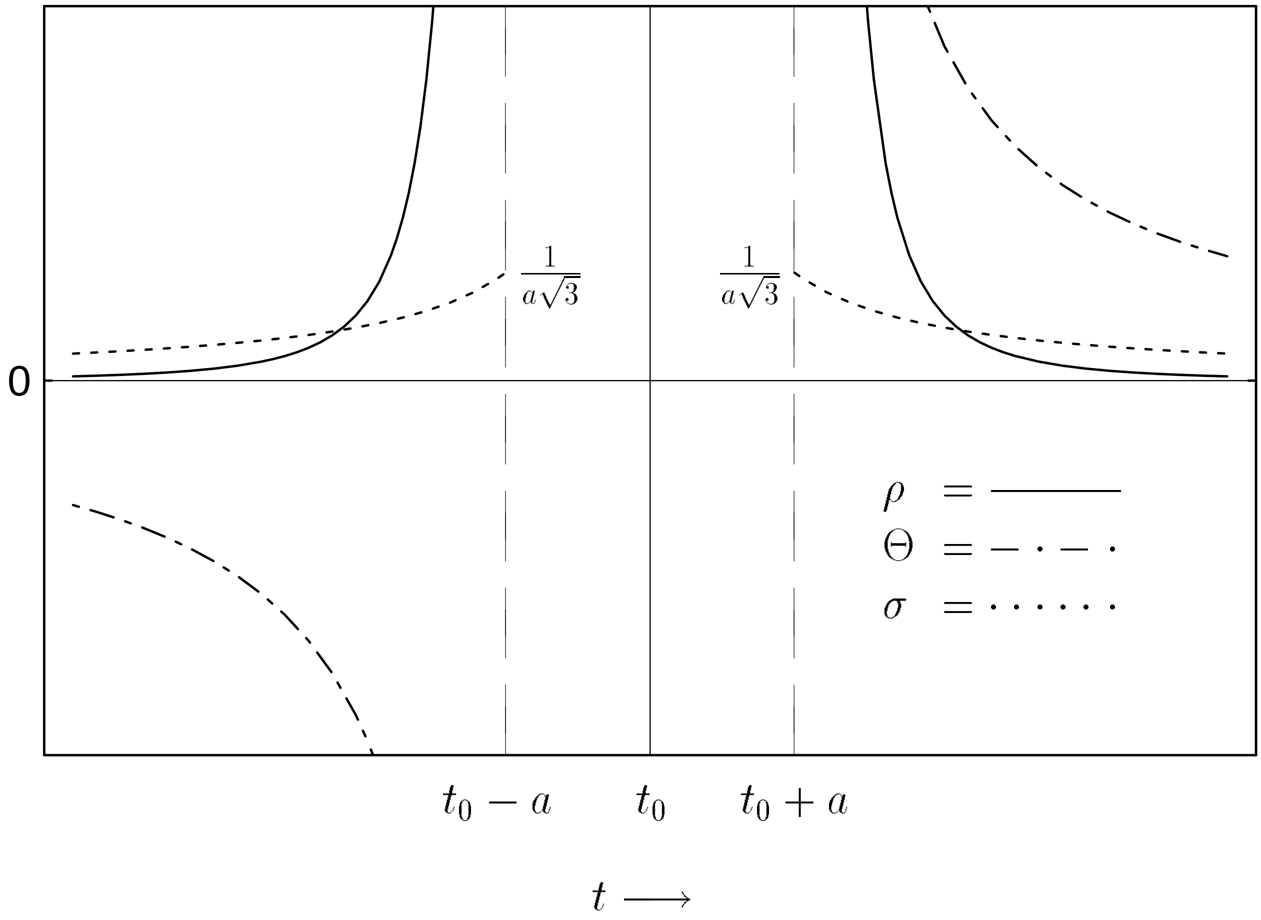

Fig.8. Graphs of ρ, Θ, and σ for Special Case of $\Lambda = 0$, $p = 1/3\,\rho$; $c = 0$.

Case II.
Photon Gas
$\Lambda = 0$
$p = \frac{1}{3}\rho$

This special solution corresponds to $c = 0$ in equation (47).

$$
\begin{aligned}
Y &= a\left[\left(\frac{t-t_0}{a}\right)^{2/3} - 1\right]^{1/2}\left(\frac{t-t_0}{a}\right)^{2/3}, \\
X &= a\left[1 - \frac{1}{\left(\frac{t-t_0}{a}\right)^{2/3}}\right]^{1/2}, \\
\rho &= \frac{1}{3a^2\left[\left(\left(\frac{t-t_0}{a}\right)^{2/3} - 1\right)\left(\frac{t-t_0}{a}\right)^{2/3}\right]^2}, \\
\Theta &= \frac{1}{t-t_0}\frac{\left[2\left(\frac{t-t_0}{a}\right)^{2/3} - 1\right]}{\left[\left(\frac{t-t_0}{a}\right)^{2/3} - 1\right]}, \\
\sigma &= \frac{1}{\sqrt{3}}\left\|\frac{1}{t-t_0}\right\|.
\end{aligned}
$$

a is a constant satisfying $a \neq 0$. These models start (end) with $\rho = \infty$ and expand to (from) the state $\rho = 0$. The evolution takes place in an infinite proper time.

The models which correspond to $a = 0$ in equation (47) have a negative energy density and are hence excluded.

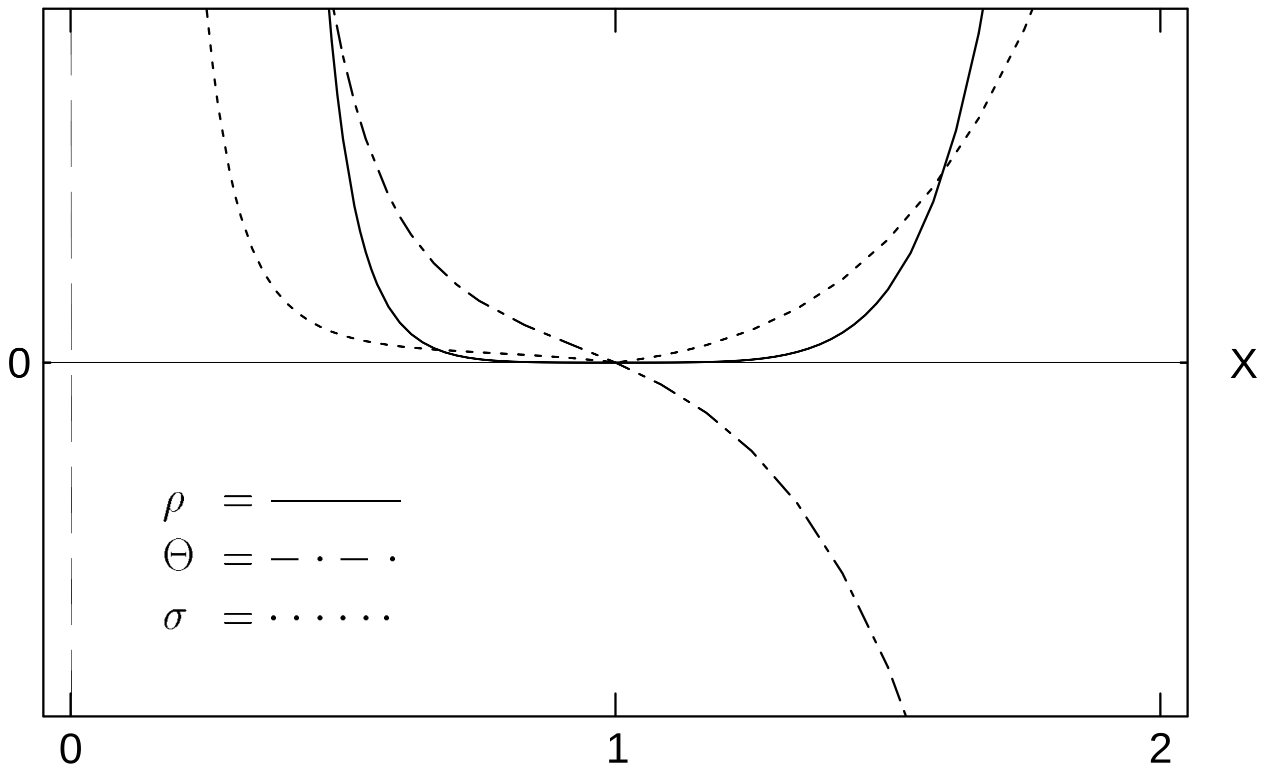

Fig.9. Graphs of ρ, Θ, and σ for General Case of $\Lambda = 0$, $p = \rho$; $a > 2$.

Case II.
Stiff Matter
$\Lambda = 0$
$p = \rho$

The general solution (48) is repeated here to indicate its overall behavior. An explicit example corresponding to $\dot{X} = 0$ is given next.

$$
\begin{aligned}
t - t_0 &= a^2 c \int \frac{dX}{X^2 \sinh^2(a \ln X)}, \\
Y &= \frac{c\, a}{X \sinh(a \ln X)}, \\
\rho &= \frac{X^2}{a^2 c^2} \sinh^4(a \ln X) \left(1 - \frac{1}{a^2}\right), \\
\Theta &= \frac{-X}{a^2 c}[1 + 2a \coth(a \ln X)] \sinh^2(a \ln X), \\
\sigma &= \frac{1}{\sqrt{3}} \left\| \frac{X}{a^2 c}[2 + a \coth(a \ln X)] \sinh^2(a \ln X) \right\|,
\end{aligned}
$$

a and c are constants, $a \geq 1$, $c \neq 0$. The general solution starts (finishes) in a singular state and expands to (from) a $\rho = 0$ state. The proper time for the evolution is infinite.

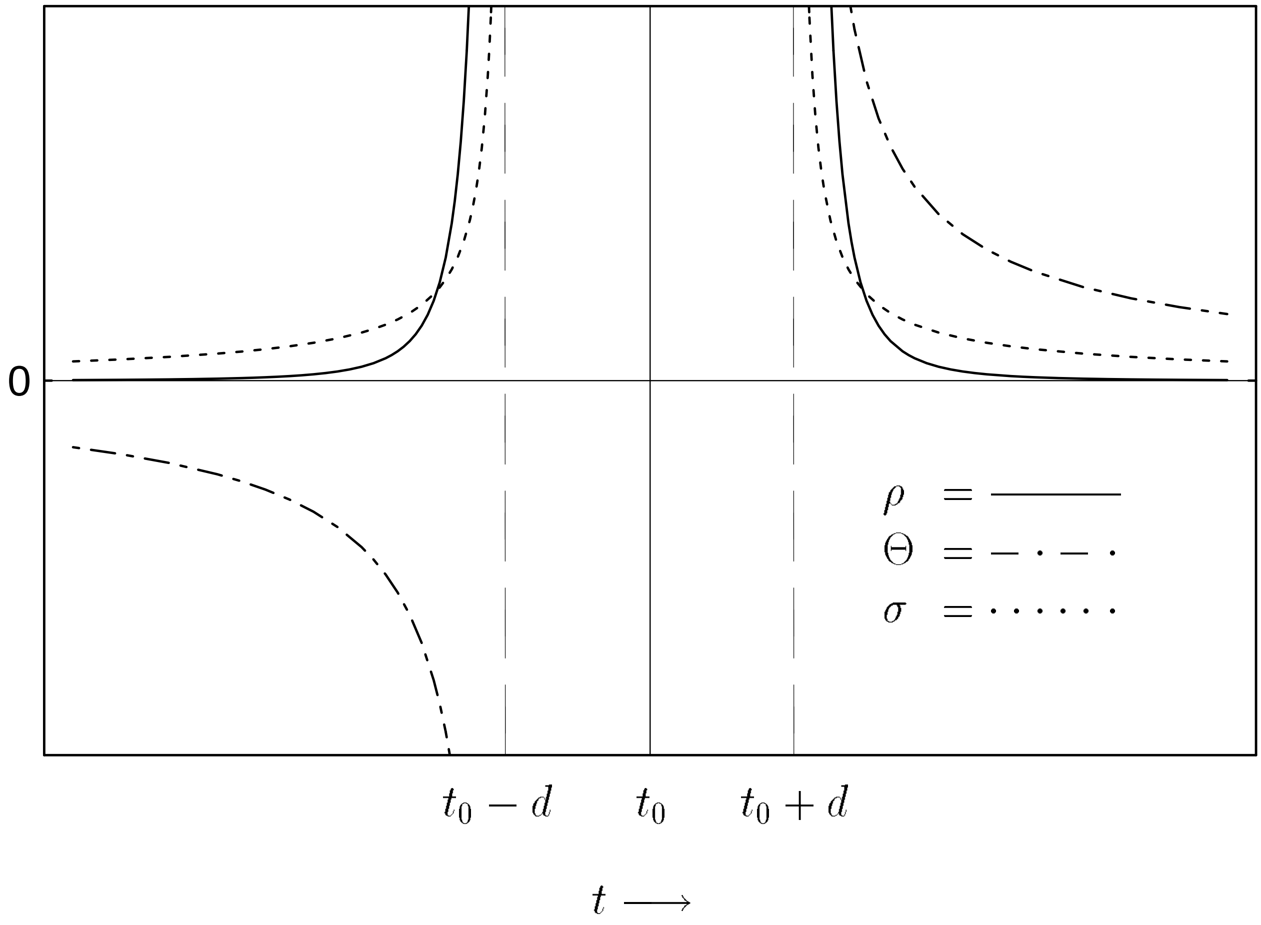

Fig.10. Graphs of ρ, Θ, and σ for Special Case of $\Lambda = 0$, $p = \rho$; $\dot{X} = 0$.

Case II.
Stiff Matter
$\Lambda = 0$
$p = \rho$

This explicit solution corresponds to the case $\dot{X} = 0$. There is no obvious way to obtain this solution as a limiting case of (48).

$$
\begin{aligned}
Y &= \left[(t-t_0)^2 - d^2\right]^{1/2}, \\
X &= 1, \\
\rho &= \frac{d^2}{\left[(t-t_0)^2 - d^2\right]^2}, \\
\Theta &= \frac{2(t-t_0)}{\left[(t-t_0)^2 - d^2\right]}, \\
\sigma &= \frac{1}{\sqrt{3}} \left\| \frac{(t-t_0)}{\left[(t-t_0)^2 - d^2\right]} \right\| .
\end{aligned}
$$

d and t_0 are constants, $d \neq 0$. These solutions start (finish) in a $\rho = \infty$ state and end (begin) in a state $\rho = 0$.

5. CONCLUDING REMARKS

1. Because the explicit models presented here are shearing they may be used to indicate the possible influence of shear on the early stages of development of the universe. Present observations indicate and opinions are that the present shearing motion is small compared to the expansion; however, this is no proof that in earlier stages the situation was the same. Because shear affects the rate of development of ρ, p, etc., these models can predict different observable quantities than, for example, the Friedmann models predict.
2. These models can be pictured in the following way. Consider one of the homogeneous hypersurfaces, *i.e.,* the surface $t = t_0$. This hypersurface can then be pictured as the topological product of a real line with a sphere for Case 1 (a Lobatchewsky plane for Case II). The measures defined on the line and on the sphere depend only on t, *i.e.,* $X = X(t)$, $Y = Y(t)$. Singularities appear in the space-time when the measures on the line or sphere vanish. G. Ellis has recently made the observation that these models are a subclass of those for which the measure on the line is also a function of position along the line, *i.e.,* $X = X(r,t)$ and that these models are easily extended to the more general solutions.[19]

[19] G . Ellis. To be published in *J. Math. Phys.* The more general solutions are obtained by replacing the constant b in equations (40), (44), and (45) by an arbitrary function of r, *i.e.,* $b = b(r)$.

3. Note that the Case I models ($\Lambda = 0$) all exist for only a finite proper time and the Case II models exist for an infinite proper time regardless of the equation of state ($A = 0, 1/3, 1$). The only deviations from this behavior occur when $\Lambda > 0$ for Case I and $\Lambda < 0$ for Case II.

4. Some difficulties arise when these models are interpreted from the point of view of kinetic theory. The kinetic theory view is that the universe consists of a gas which is described by a distribution function $F(x^a, p^b)$. This distribution function is a solution of the Boltzmann equation $p^a \frac{DF}{dx^a} = \frac{dF}{d\tau}|_{coll.}$ and is related to the geometry through the energy momentum tensor T^{ab},[20]

$$T^{ab} = \int F\, p^a p^b dp\,. \tag{54}$$

If, in order to have a perfect fluid T^{ab}, you assume F to be isotropic in p^a about the time-like congruence whose tangent is u^a the observation is made that the left-hand side of the Boltzmann equation is not necessarily isotropic in p^a. Under very reasonable assumptions about the scattering cross section, the collision term in the Boltzmann equation is isotropic in p^a about u^a if F is. For the spatially homogeneous models presented in this paper the Boltzmann equation reduces to

[20] J. Ehlers, *Akad. Wiss. Lit. Mainz* **11**, 824 (1961).

$$p^0\dot{F} + \frac{\Theta}{3}\left(m^2 - {p^0}^2\right) F' + \sigma_{ab}\; p^a p^b F' = \frac{dF}{d\tau}|_{\rm coll.}\ ,$$

where

$$F = F(p^0, t),\ \ \dot{F} = \frac{\partial F}{\partial t},\ \ F' = \frac{\partial F}{\partial p^0}\,. \qquad (55)$$

The conclusion is that if F was isotropic at time $t = t_0$, it would not remain isotropic due to the shearing term $\sigma_{ab}\, p^a p^b F'$. The point is that these solutions must be considered as 'zeroth' order approximations in the kinetic theory picture. The approximation being reasonable only if the collisions are frequent enough to keep 'large' anisotropies in the distribution function from developing.

APPENDIX I.

This appendix will show that all four dimensional Lie algebras have three dimensional subalgebras. The reader may, if he chooses check this assertion by looking at tables of all four dimensional Lie algebras.[21]

Let L be the Lie algebra in question and $L' = [L, L]$ be the derived algebra.[22] We distinguish five cases according to the dimension of L'; $(0, 1, 2, 3, 4)$. The first four cases are disposed of in the following manner. Let ξ_i $(i = 1, ..., k)$ be a basis for L'. Complete this to a basis for L by adding ξ_j $(j = k + 1, ..., 4)$. Now ξ_α $(\alpha = 1, 2, 3)$ spans a three dimensional subalgebra of L. The only remaining case is when the dimension of $L' = 4$, *i.e.*, $L' = L$. It will turn out that there are no algebras in this class. Note that if L is simple then $L' = L$ and hence the remainder of this proof will also prove that there are no simple four dimensional Lie algebras. Let ξ_a $(a = 1, ..., 4)$ be a basis for L and C^a_{bc} be the structure constants, *i.e.*,

$$[\xi_b, \xi_c] = C^a_{bc}\xi_a. \tag{A.1}$$

The definition of a Lie algebra L implies

$$C^a_{bc} = C^a_{[bc]}, \tag{A.2}$$

[21] A. S. Petrov, *Einstein-Raume* (Akademie-Verlag, Berlin, 1964), p. 60.
[22] N. Jacobson, *Lie Algebras* (Interscience, New York, 1962), Ch.l.

$$C^e_{[bc}C^a_{d]e} = 0. \tag{A.3}$$

The latter equation is called the Jacobi identity.

The assumption $L' = L$ implies the six vectors $C^a_{bc}\xi_a$ in (A.1) span L. A contradiction will be shown by exhibiting four independent linear combinations of these six vectors which vanish, *i.e.*, $C^a_{bc}\xi_a$ can at most span a two dimensional subspace of L. If the ξ_a are chosen as a basis of L, $C^a_{bc}\xi_a$ has components C^a_{bc} where a labels the component and 'bc' the six different vectors. The six sets of components then span a real four dimension vector space R_4. It then follows that

$$C^a_{bc}A_a = 0 \longrightarrow A_a = 0. \tag{A.4}$$

Now by contracting on a and b in (A.3) the contracted Jacobi identity is obtained:

$$C^e_{bc}C^a_{ae} = 0. \tag{A.5}$$

If $L' = L$, equation (A.4) implies $C^a_{ab} = 0$. Now C^a_{bc} can be interpreted as four bivectors (one for each value of a) because of the antisymmetry on the lower indices. In the usual way the duals, $C^{*a,bc}$, can be defined.[23]

$$C^{*a,bc} = \frac{1}{2}\,\epsilon^{bcde}\,C^a_{de}, \tag{A.6}$$

which implies

$$C^a_{bc} = \frac{1}{2}\,\epsilon_{bcde}\,C^{*a,de}. \tag{A.7}$$

[23] J. L. Synge and A. Schild, *Tensor Calculus* (University of Toronto Press, Toronto, 1961), p. 246.

From the condition $C'^{a}_{ab} = 0$ it follows that $C'^{*[a,bc]} = 0$, or because of the anti-symmetry on the bivector indices,

$$C'^{*a,bc} + C'^{*b,ca} + C'^{*c,ab} = 0. \tag{A.8}$$

This reduces the Jacobi identity (A.3) to

$$C'^{*a,bc} C'^{d}_{dc} = 0. \tag{A.9}$$

(A.9) represents four linear combinations of the six vectors $C'^{d}_{bc}\xi_d$ which vanish. These four linear combinations are independent if and only if

$$C'^{*a,bc} A_a = 0 \longrightarrow A_a = 0. \tag{A.10}$$

But this assertion follows from (A.7) and (A.4) and hence the dimension of $L' \leq 2$. The assumption that $L' = L$ is therefore inconsistent with the Jacobi identity for four dimensional algebras and the proof is complete.

APPENDIX II.

This appendix will give the structure of all four dimensional Lie algebras over the field of real numbers which have (5), (6), or (7) as subalgebras. I repeat (5), (6), and (7) here for convenience.

$$[\xi_1, \xi_2] = \xi_3, \ [\xi_2, \xi_3] = \xi_1, \ [\xi_3, \xi_1] = \xi_2, \tag{5}$$
$$[\xi_1, \xi_2] = 0, \ [\xi_2, \xi_3] = \xi_1, \ [\xi_3, \xi_1] = \xi_2, \tag{6}$$
$$[\xi_1, \xi_2] = \xi_1, \ [\xi_2, \xi_3] = \xi_3, \ [\xi_3, \xi_1] = 2\xi_2. \tag{7}$$

The technique will be to introduce a fourth basis vector and see what restrictions the Jacobi identity places on the structure constants of the enlarged algebra. Once this is done we introduce a simple form for the structure constants by choosing an appropriate basis. We deal with (5) and (7) first because they are simple algebras and are complex equivalent (see below).

Definition 1: H is an ideal of a Lie algebra L if and only if H is a subalgebra of L and $[L, H] \leq H$.

Definition 2: L is a simple Lie algebra if and only if L has no ideals other than 0 or L itself and the dimension of L is greater than 1.

Up to now we have been talking about algebras defined over the field of real numbers. If we now consider algebras over the field of complex numbers (5) and (7) become equivalent. The basis transformation which transforms (7) into (5) is

$$
\begin{aligned}
\underline{\xi}_1 &= i\,\frac{(\xi_1+\xi_2)}{2}, \\
\underline{\xi}_2 &= \frac{(\xi_1-\xi_3)}{2}, \\
\underline{\xi}_3 &= i\,\xi_2.
\end{aligned} \tag{A.11}
$$

In this basis the commutation relations are

$$
\left[\underline{\xi}_\alpha, \underline{\xi}_\beta\right] = \epsilon_{\alpha\beta\gamma}\,\underline{\xi}_\gamma, \tag{A.12}
$$

the same as (5). We now ask the question, what four dimensional Lie algebra over the field of complex numbers have subalgebras of the type (A.12).

If we denote the four dimensional Lie algebra by $\underline{L}$ and the subalgebra corresponding to (A.12) by $\underline{H}$, the simplicity of $\underline{H}$ implies $\underline{H} \le \underline{L}' \le \underline{L}$. The dimension of $\underline{H}$ is three and the dimension of $\underline{L}'$ (from Appendix I) is less than or equal to three and hence $\underline{H} = \underline{L}'$. We complete $\left\{\underline{\xi}_\alpha\right\}$ to a basis of L by adding $\overline{\eta}$. Since $\underline{L}' = \underline{H}$, $\underline{L}$ must have the following structure:

$$
\begin{aligned}
\left[\underline{\xi}_\alpha, \underline{\xi}_\beta\right] &= \epsilon_{\alpha\beta\gamma}\,\underline{\xi}_\gamma, \\
[\overline{\eta}\,, \xi_\alpha] &= A_{\alpha\beta}\,\xi_\beta,
\end{aligned} \tag{A.13}
$$

where $A_{\alpha\beta}$ are constants.

The Jacobi identity of course restricts the possible values of $A_{\alpha\beta}$ *i.e.*,

$$
[\overline{\eta}, [\xi_\alpha, \xi_\beta]] + [\xi_\alpha, [\xi_\beta, \overline{\eta}]] + [\xi_\beta, [\overline{\eta}, \xi_\alpha]] = 0. \tag{A.14}
$$

(A.14) reduces to the following when the structure constants (A.13) are put in:

$$\epsilon_{\alpha\beta\gamma}A_{\gamma\sigma} - \epsilon_{\alpha\gamma\sigma}A_{\beta\gamma} + \epsilon_{\beta\gamma\sigma}A_{\alpha\gamma} = 0\,. \tag{A.15}$$

We now multiply this equation by $\epsilon_{\alpha\beta\gamma}$, sum on α and β, and use the following identity

$$\epsilon_{\alpha\beta\lambda}\epsilon_{\alpha\gamma\sigma} = \delta_{\beta\gamma}\delta_{\lambda\sigma} - \delta_{\beta\sigma}\delta_{\lambda\gamma}, \tag{A.16}$$

to obtain:

$$A_{\lambda\sigma} + A_{\sigma\lambda} - \delta_{\lambda\sigma}A_{\alpha\alpha}. \tag{A.17}$$

If we sum on λ and σ in (A.17) we find $A_{\alpha\alpha} = 0$. Putting this back into (A.17) we have

$$A_{(\lambda\sigma)} = 0. \tag{A.18}$$

We can now find a simple basis for $\underline{L}$ by introducing

$$\underline{\eta} = \overline{\eta} - \frac{1}{2}\epsilon_{\alpha\beta\gamma}A_{\alpha\beta}\underline{\xi}_{\gamma}. \tag{A.19}$$

We now compute $\left[\underline{\eta}, \underline{\xi}_{\alpha}\right]$,

$$\left[\underline{\eta}, \underline{\xi}_{\alpha}\right] = A_{(\alpha\sigma)}\underline{\xi}_{\sigma} = 0. \tag{A.20}$$

The conclusion is that if a four dimensional Lie algebra $\underline{L}$ has a subalgebra whose structure is given by (A.12) then $\underline{L}$ has a structure given by

$$\begin{aligned}\left[\underline{\xi}_{\alpha}, \underline{\xi}_{\beta}\right] &= \epsilon_{\alpha\beta\gamma}\,\underline{\xi}_{\gamma},\\ \left[\underline{\eta}, \underline{\xi}_{\alpha}\right] &= 0.\end{aligned} \tag{A.21}$$

Now to apply this to (5) and (7). Since we can get from the real subalgebra (5) or (7) to the complex subalgebra (A.21) by a complex basis transformation, we must be able to generate a complex Lie algebra $\underline{L}$ of the type (A.21) from the real four dimensional L (which contains (5) or (7)) by a complex basis transformation. If we complete ξ_{α} of (5) or (7) to a basis of L by adding ω, $\underline{\eta}$ of

(A.21) must be expressible as a non-vanishing linear (complex) combination of ξ^α and ω.

$$\underline{\eta} = f^\alpha \xi_\alpha + g\ \omega, \tag{A.22}$$

where f^α and g are constants.

Since $\underline{\eta}$ commutes with $\underline{L}$ it also commutes with ω.

$$[f^\alpha \xi_\alpha + g\ \omega, L] = 0. \tag{A.23}$$

Since the structure constants of L are real, both the real and imaginary parts of $f^\alpha \xi_\alpha + g\ \omega$ must commute with L. At least one of these two parts must be different from zero; call it η and replace ω by η as a fourth basis vector of L. The structure of the real four dimensional Lie algebras which contain (5) and (7) are respectively

$$[\xi_1, \xi_2] = \ \xi_3,\ [\xi_2, \xi_3] = \ \xi_1,\ [\xi_3, \xi_1] = \ \ \xi_2,\ [\eta, \xi_\alpha] = \ \ 0, \tag{A.24}$$
$$[\xi_1, \xi_2] = \ \xi_1,\ [\xi_2, \xi_3] = \ \xi_3,\ [\xi_3, \xi_1] = \ \ 2\xi_2,\ [\eta, \xi_\alpha] = \ \ 0. \tag{A.25}$$

The solutions to the problem of finding what four dimensional Lie algebras have (6) as a subalgebra is straight forward but not as nice as the above calculation. Let L be such a four dimensional algebra. From (6) we see that the dimension of L' is greater than or equal to two and from Appendix I we see that it is equal to or less than three. If L' is three dimensional and different from (6) we introduce a vector ω which completes ξ_1 and ξ_2 to be a basis of L' and then ξ_α and ω is a basis of L. If L' is just (6) or a subalgebra of (6), ξ_α spans L' and we introduce ω to complete ξ_α to be a basis of L. Both cases can be summed up by saying that

$$\{\xi_1, \xi_2, (1-\delta)\xi_3 + \delta\omega\} \qquad (\delta = 0, 1),$$

spans L' and $\{\xi_\alpha,\ \omega\}$ spans L.

The commutators of ω with ξ_α then take on the form

$$\begin{aligned}
[\,\omega,\xi_1\,] &= A_1\,\xi_1 + B_1\,\xi_2 + C_1\,[(1-\delta)\xi_3 + \delta\omega]\,, \\
[\,\omega,\xi_2\,] &= A_2\,\xi_1 + B_2\,\xi_2 + C_2\,[(1-\delta)\xi_3 + \delta\omega]\,, \\
[\,\omega,\xi_3\,] &= A_3\,\xi_1 + B_3\,\xi_2 + C_3\,[(1-\delta)\xi_3 + \delta\omega]\,. \quad \text{(A.26)}
\end{aligned}$$

Now if we replace ω by λ as a basis vector for L where λ is defined by

$$\begin{aligned}
\lambda &= \omega + \alpha\,\xi_1 + \beta\,\xi_2, \\
\alpha &= -\frac{[A_2 + B_3C_3\delta]}{[1+(C_3\delta)^2]}, \\
\beta &= \frac{[B_3 - A_2C_3\delta]}{[1+(C_3\delta)^2]}, \quad \text{(A.27)}
\end{aligned}$$

then the new commutation relations are:

$$\begin{aligned}
[\,\lambda,\xi_l\,] &= A_1\xi_1 + B_1\xi_2 + C_1\,[(1-\delta)\xi_3 + \delta\lambda]\,, \\
[\,\lambda,\xi_2\,] &= A_2\xi_1 + B_2\xi_2 + C_2\,[(1-\delta)\xi_3 + \delta\lambda]\,, \\
[\lambda,\xi_3\,] &= C_3\,[\,(1-\delta)\xi_3 + \delta\eta]\,. \quad \text{(A.28)}
\end{aligned}$$

In (A.28) the A_1, A_2, B_1, and B_2 are not the same constants that appear in (A.27). We now apply the Jacobi identities to see what restrictions the subalgebra (6) places on the constants in (A.28). The Jacobi identity

$$[\lambda,[\xi_1,\xi_2]] + [\xi_1,[\xi_2,\lambda]] + [\xi_2,[\lambda,\xi_1]] = 0, \quad \text{(A.29)}$$

gives the following restrictions:

$$\begin{aligned}
\delta\left(C_2\overline{A}_1 - C_1\overline{A}_2\right) + (1-\delta)C_1 &= 0, \\
\delta\left(C_2\overline{B}_1 - C_1\overline{B}_2\right) + (1-\delta)C_2 &= 0. \quad \text{(A.30)}
\end{aligned}$$

The Jacobi identity

$$[\lambda,[\xi_2,\xi_3]]+[\xi_2,[\xi_3,\lambda]]+[\xi_3,[\lambda,\xi_2]]=0, \tag{A.31}$$

gives the restrictions

$$\begin{aligned} A_1-B_2+C_3\left[\delta A_2-(1-\delta)\right] &= 0,\\ A_2+B_1+C_3\,\delta B_2 &= 0,\\ C_1 &= 0. \end{aligned} \tag{A.32}$$

The Jacobi identity

$$[\lambda,[\xi_3,\xi_1]]+[\xi_3,[\xi_1,\lambda]]+[\xi_1,[\lambda,\xi_3]]=0, \tag{A.33}$$

gives the restrictions

$$\begin{aligned} A_2+B_1-\delta C_3A_1 &= 0,\\ A_1-B_2+C_3\left[\delta B_1+(1-\delta)\right] &= 0,\\ C_2 &= 0. \end{aligned} \tag{A.34}$$

The complete set of restrictions on the constants of (A.28) is therefore;

$$\begin{aligned} C_1=C_2 &= 0,\\ A_1-B_2+C_3\left[\delta A_2-(1-\delta)\right] &= 0,\\ A_1-B_2+C_3\left[\delta B_1+(1-\delta)\right] &= 0,\\ A_2+B_1+C_3\delta B_2 &= 0,\\ A_2+B_1-C_3\delta A_1 &= 0. \end{aligned} \tag{A.35}$$

The solution to these equations divide into two cases, $C_3=0$ and $C_3\neq 0$.

$$\text{If } C_3 = 0 \longrightarrow C_1=C_2=0,\ A_1=B_2,\ A_2=-B_1. \tag{A.36}$$

$$\text{If } C_3 \neq 0 \longrightarrow C_1=C_2=A_1=A_2=B_1=B_2=0 \tag{A.37}$$

The case $C_3 = 0$ is

$$\begin{aligned}
[\lambda, \xi_1] &= A_1\xi_1 + B_1\xi_2, \\
[\lambda, \xi_2] &= -B_1\xi_1 + A_1\xi_2, \\
[\lambda, \xi_3] &= 0.
\end{aligned} \tag{A.38}$$

The commutation relations can be simplified by replacing λ by $\eta = A_1^{-1}(\lambda - B_1\xi_3)$ if $A_1 \neq 0$ or with $\eta = (\lambda - B_1\xi_3)$ if $A_1 = 0$.

The result is

$$\begin{aligned}
[\eta, \xi_1] &= \delta\xi_1, \\
[\eta, \xi_2] &= \delta\xi_2, \\
[\eta, \xi_3] &= 0,
\end{aligned} \tag{A.39}$$

where $\delta = 0, 1$.

The case $C_3 \neq 0$ is

$$[\lambda, \xi_1] = 0, \ [\lambda, \xi_2] = 0, \ [\lambda, \xi_3] = C_3\lambda. \tag{A.40}$$

The conclusion is that if a four dimensional Lie algebra L has (6) as a subalgebra then L has one of the following forms

$$\begin{array}{rlrlrl}
[\xi_1, \xi_2] = & 0, & [\xi_2, \xi_3] = & \xi_1, & [\xi_3, \xi_1] = & \xi_2, \\
[\eta, \xi_1] = & \xi_1, & [\eta, \xi_2] = & \xi_2, & [\eta, \xi_3] = & 0,
\end{array} \tag{A.41}$$

or

$$\begin{array}{rlrlrl}
[\xi_1, \xi_2] = & 0, & [\xi_2, \xi_3] = & \xi_1, & [\xi_3, \xi_1] = & \xi_2, \\
[\eta, \xi_1] = & 0, & [\eta, \xi_2] = & 0, & [\eta, \xi_3] = & C\eta,
\end{array} \tag{A.42}$$

where C is a constant.

VITA

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