Zeta-Functions for Non-Minimal Operators

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Abstract

We evaluate zeta-functions $\zeta(s)$ at $s = 0$ for invariant non-minimal 2nd-order vector and tensor operators defined on maximally symmetric even dimensional spaces. We decompose the operators into their irreducible parts and obtain their corresponding eigenvalues. Using these eigenvalues, we are able to explicitly calculate $\zeta(0)$ for the cases of Euclidean spaces and N-spheres. In the N-sphere case, we make use of the Euler-Maclaurin formula to develop asymptotic expansions for the required sums. The resulting $\zeta(0)$ values for dimensions 2 to 10 are given in the Appendix.

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I. INTRODUCTION

The effective potential formalism has been used by Appelquist and Chodos [1] and many others (see [2] for a complete list) to consider the problem of spontaneous compactification in Kaluza-Klein(KK) theories. The hope was to explain the smallness of the extra dimensions by using quantum gravity effects. It was soon realized that the standard effective action produced results that were dependent on the choice of the quantum gauge fixing condition [3,4] and that all conclusion about stability drawn from this standard effective action (sometimes called the naive effective action) were questionable. This problem was resolved [5,6] by the use of a new effective action, first introduced by Vilkovisky [7] and DeWitt [8]. This new effective action, now known as the Vilkovisky-DeWitt(VD) effective action, has the merit of being gauge choice independent.

However, progress in compactification has never recuperated from the setback [9, 10]. The primary reason is that even at the one-loop level the VD effective action involves determinants of operators with complicated non-local terms (in most gauges). In [11], we considered the six dimensional case for a general background spacetime using the method of Barvinsky and Vilkovisky [12] to deal with the non-local terms. Due to the complexity of this calculation, it seems quite impossible to push this method to higher dimensions.

The situation can be improved if one chooses the Landau-DeWitt gauge. Since the VD effective action is independent of gauge choice, one can of course choose whatever gauge is convenient to work with, without altering the final result. In the Landau-DeWitt gauge the non-local terms are identically zero [13]. Although the operators simplify tremendously, they remain non-minimal (see [12] and [14]), that is, they involve 2nd-order covariant derivative terms other than just the Laplacian. More explicitly, one has to deal with vector operators of the form,

$$M_V{}^{\alpha}{}_{\beta} = -\delta^{\alpha}_{\beta}\Box + a\nabla^{\alpha}\nabla_{\beta} - P^{\alpha}{}_{\beta} \ , \tag{1.1}$$

and the tensor operators of the form,

$$M_T{}^{\alpha\beta}{}_{\rho\sigma} = -\delta_\rho^{(\alpha}\delta_\sigma^{\beta)}\Box + a_1\delta_{(\rho}^{(\alpha}\nabla^{\beta)}\nabla_{\sigma)} + a_2 g_{\rho\sigma}\nabla^{(\alpha}\nabla^{\beta)} + a_3 g^{\alpha\beta}\nabla_{(\rho}\nabla_{\sigma)} - P^{\alpha\beta}{}_{\rho\sigma} \ , \tag{1.2}$$

where $A^{(\alpha}B^{\beta)} \equiv (A^\alpha B^\beta + A^\beta B^\alpha)/2$. Here we have included the most frequently encountered non-minimal second-order terms.

To date progress in evaluating determinants of such operators for dimensions beyond six has only been made for KK backgrounds of the form $R_n \times T_{d-n}$ (with R_n usually taken to be flat) and consequently progress with KK compactification beyond these simple backgrounds has halted [15]. More interesting backgrounds like S_d or $R_4 \times S_{d-4}$ have, so far, resisted all efforts [16]. In this paper we make progress in evaluating determinants of such non-minimal operators on non-flat backgrounds of the form S_d.

Using ζ-function regularization the one-loop quantum contribution to the effective action can be expressed in terms of the ζ-function for the appropriate operator as

$$\Gamma_1 = -\frac{1}{2}\zeta^{'}(0) - \zeta(0)\ \ln\mu \ , \tag{1.3}$$

where μ is the renormalization scale. As a first step towards obtaining the VD effective action, we shall in this paper concentrate on the evaluation of the zeta-function $\zeta(s)$ at $s = 0$ for the non-minimal vector and tensor operators in even dimensions. Of course, these considerations will also be useful in the usual effective action formalism when one chooses to work with gauges other than the Feynman gauge where the operators are minimal.

In the next section, we obtain eigenvalues for the vector operator M_V and the tensor operator M_T in maximally symmetric spaces, by decomposing the eigenfunctions for M_V into transverse and longitudinal parts, and for M_T into transverse-traceless (TT), longitudinal-transverse-traceless (LTT), longitudinal-longitudinal-traceless (LLT) and trace (TR) parts [17]. In Section III, we explicitly evaluate $\zeta(0)$ in Euclidean space. Then in Section IV, we extend the results to N-spheres using the Euler-Maclaurin formula to develop asymptotic expansions for the relevant summations [18]. Finally, the conclusions are given in Section V. In the Appendix we summarize $\zeta(0)$ values for various cases.

II. EIGENVALUES IN MAXIMALLY SYMMETRIC SPACES

The ζ-function for an operator M is defined as

$$\zeta_M(s) \equiv \sum_{\lambda} \lambda^{-s} \ , \tag{2.1}$$

where λ's are the eigenvalues of the operator M. Therefore, to calculate $\zeta(s)$ for M_V and M_T, we must first obtain eigenvalues for these operators. Here we assume that our background spacetime is a maximally symmetric space in which the Riemann tensor, the Ricci tensor, and the scalar curvature are given by,

$$R_{\mu\nu\alpha\beta} = \kappa(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}) \ , \tag{2.2}$$

$$R_{\mu\nu} = \kappa(N-1)g_{\mu\nu} \ , \tag{2.3}$$

$$R = \kappa N(N-1) \ , \tag{2.4}$$

where N is the dimension of the space and κ is a constant.

A. Vector Case

We first consider the vector operator M_V of (1.1). For M_V to be invariant in maximally symmetric spaces, the function $P^{\alpha}{}_{\beta}$ must be proportional to δ^{α}_{β}, i.e.,

$$M_V{}^{\alpha}{}_{\beta} = \delta^{\alpha}_{\beta}(-\Box - P) + a\nabla^{\alpha}\nabla_{\beta} \ , \tag{2.5}$$

with P and a constants. The eigenfunctions V^{α} of M_V can be decomposed [17] into a transverse part T^{α},

$$\nabla_{\alpha}T^{\alpha} = 0 \ , \tag{2.6}$$

and a longitudinal part L^{α}, which is the gradient of a scalar function S,

$$L^{\alpha} = \nabla^{\alpha}S \ , \tag{2.7}$$

with

$$V^\alpha = T^\alpha + L^\alpha \ . \tag{2.8}$$

Acting with the operator M_V on T^α gives

$$\begin{aligned} {M_V}^\alpha{}_\beta T^\beta &= \left[\delta^\alpha_\beta(-\Box - P) + a\nabla^\alpha\nabla_\beta\right] T^\beta \ , \\ &= (-\Box - P)T^\alpha \ , \end{aligned} \tag{2.9}$$

because of the transverse property of T^α. For L^α the result is

$$\begin{aligned} {M_V}^\alpha{}_\beta L^\beta &= \left[\delta^\alpha_\beta(-\Box - P) + a\nabla^\alpha\nabla_\beta\right] \nabla^\beta S \ , \\ &= \left[-(1-a)\Box - P\right] \nabla^\alpha S + a[\nabla^\alpha, \Box]S \ . \end{aligned} \tag{2.10}$$

To evaluate the commutator, we use the defining identity for the Riemann tensor,

$$[\nabla_\alpha, \nabla_\beta]T^{\rho\sigma\cdots} = -R^{\xi\rho}{}_{\alpha\beta}T_\xi{}^{\sigma\cdots} - R^{\xi\sigma}{}_{\alpha\beta}T^\rho{}_\xi{}^{\cdots} + \cdots \ , \tag{2.11}$$

which for maximally symmetric spaces simplifies to

$$[\nabla_\alpha, \nabla_\beta]T^{\rho\sigma\cdots} = \kappa(\delta^\rho_\alpha T_\beta{}^{\sigma\cdots} - \delta^\rho_\beta T_\alpha{}^{\sigma\cdots}) + \kappa(\delta^\sigma_\alpha T^\rho{}_\beta{}^{\cdots} - \delta^\sigma_\beta T^\rho{}_\alpha{}^{\cdots}) + \cdots \ . \tag{2.12}$$

The commutator in Eq. (2.10) becomes,

$$\begin{aligned} [\nabla^\alpha, \Box]S &= g^{\rho\sigma}[\nabla^\alpha, \nabla_\rho\nabla_\sigma]S \ , \\ &= \kappa(1-N)\nabla^\alpha S \ , \end{aligned} \tag{2.13}$$

and thus,

$${M_V}^\alpha{}_\beta L^\beta = \left[-(1-a)\Box - P - a\kappa(N-1)\right] L^\alpha \ . \tag{2.14}$$

Using Eqs. (2.9) and (2.14) one can obtain the eigenvalues of the vector operator M_V when the corresponding eigenvalues for the Laplacian are known.

B. Tensor Case

Similar consideration can be applied to the invariant tensor operator M_T. In maximally symmetric spaces, the functions $P^{\alpha\beta}{}_{\rho\sigma}$ can involve only two different invariant tensors, $\delta^{(\alpha}_\rho\delta^{\beta)}_\sigma$ and $g^{\alpha\beta}g_{\rho\sigma}$. One can thus write

$$P^{\alpha\beta}{}_{\rho\sigma} = P\delta_\rho^{(\alpha}\delta_\sigma^{\beta)} + Qg^{\alpha\beta}g_{\rho\sigma} \ , \tag{2.15}$$

with P and Q being constants. The tensor operator thus becomes,

$$M_T{}^{\alpha\beta}{}_{\rho\sigma} = (-\Box - P)\delta_\rho^{(\alpha}\delta_\sigma^{\beta)} - Qg^{\alpha\beta}g_{\rho\sigma} + a_1\delta_{(\rho}^{(\alpha}\nabla^{\beta)}\nabla_{\sigma)} + a_2 g_{\rho\sigma}\nabla^{(\alpha}\nabla^{\beta)} + a_3 g^{\alpha\beta}\nabla_{(\rho}\nabla_{\sigma)} \ . \tag{2.16}$$

The eigenfunctions $H^{\alpha\beta}$ of M_T can be decomposed [17] into:

the TT part, $T^{\alpha\beta}$, where

$$\nabla_\alpha T^{\alpha\beta} = 0 \ , \tag{2.17}$$

$$T_\alpha{}^\alpha = 0 \ ; \tag{2.18}$$

the LTT part, $L^{T\alpha\beta}$, where

$$L^{T\alpha\beta} = \nabla^\alpha T^\beta + \nabla^\beta T^\alpha \ , \tag{2.19}$$

for some transverse vector T^α with

$$\nabla_\alpha T^\alpha = 0 \ ; \tag{2.20}$$

the LLT part, $L^{L\alpha\beta}$, where

$$L^{L\alpha\beta} = \nabla^\alpha\nabla^\beta L + \nabla^\beta\nabla^\alpha L - \frac{2}{N}g^{\alpha\beta}\Box L \ , \tag{2.21}$$

for some scalar function L; and the TR part $g^{\alpha\beta}H^\mu{}_\mu/N$. Therefore,

$$H^{\alpha\beta} = T^{\alpha\beta} + L^{T\alpha\beta} + L^{L\alpha\beta} + \frac{1}{N}g^{\alpha\beta}H^\mu{}_\mu \ . \tag{2.22}$$

Acting with M_T on $T^{\alpha\beta}$ and $L^{T\alpha\beta}$, and with the help of the identity in Eq. (2.12), we have

$$M_T{}^{\alpha\beta}{}_{\rho\sigma}T^{\rho\sigma} = (-\Box - P)T^{\alpha\beta} \ , \tag{2.23}$$

$$M_T{}^{\alpha\beta}{}_{\rho\sigma}L^{T\rho\sigma} = \left[-(1 - \frac{1}{2}a_1)\Box - P - a_1\kappa\right] L^{T\alpha\beta} \ . \tag{2.24}$$

However, when acting on the LLT part and the TR part,

$$
\begin{aligned}
M_T{}^{\alpha\beta}{}_{\rho\sigma} L^{L\rho\sigma} &= \left[-\left(1 - \left(1 - \frac{1}{N}\right) a_1\right) \Box - P - \kappa a_1 (N-1) \right] L^{L\alpha\beta} \\
&\quad + \left[2\left(1 - \frac{1}{N}\right)(a_1 + a_3 N)(\Box + \kappa N) \right] \left(\frac{1}{N} g^{\alpha\beta} \Box L \right) ,
\end{aligned}
\tag{2.25}
$$

$$
\begin{aligned}
M_T{}^{\alpha\beta}{}_{\rho\sigma} \left(\frac{1}{N} g^{\rho\sigma} H^\mu_\mu \right) &= \left[-\left(1 - \left(\frac{a_1}{N} + a_2 + a_3\right)\right) \Box - P - NQ \right] \left(\frac{1}{N} g^{\alpha\beta} H^\mu_\mu \right) \\
&\quad + \left[\frac{1}{2} \left(\frac{a_1}{N} + a_2 \right) (\Box - 2\kappa N) \right] \times \\
&\quad \left(\nabla^\alpha \nabla^\beta + \nabla^\beta \nabla^\alpha - \frac{2}{N} g^{\alpha\beta} \Box \right) \left(\frac{1}{\Box} H^\mu{}_\mu \right) .
\end{aligned}
\tag{2.26}
$$

We see that the second term in Eq. (2.25) involves $g^{\alpha\beta}\Box L/N$, which is the trace part of the function $\nabla^{(\alpha}\nabla^{\beta)}L$, while the second term in Eq. (2.26) involves $(\nabla^\alpha\nabla^\beta + \nabla^\beta\nabla^\alpha - \frac{2}{N}g^{\alpha\beta}\Box)\left(\frac{1}{\Box}H^\mu{}_\mu\right)$, which belongs to the LLT part. Hence, the functions in the LLT part and the TR part are coupled together as long as the operator is non-minimal (unless $a_2 = a_3 = -a_1/N$). To find the eigenfunctions and the corresponding eigenvalues one must take the appropriate linear combinations of the functions in these two parts. In the following sections we shall demonstrate explicitly how this can be done for Euclidean spaces and N-spheres.

III. ζ-FUNCTIONS ON EUCLIDEAN SPACES

In this section, we calculate the ζ-functions for the vector and tensor operators in N-dimensional Euclidean spaces. In this simple case where $\kappa = 0$, the eigenvalues for the Laplacian are just $-k^2$, with the eigenfunctions Fourier transformed to momentum space.

A. Vector Case

For the vector operator, from Eqs. (2.9) and (2.14), there are $(N-1)$ eigenfunctions in the transverse part with eigenvalues,

$$
\lambda_T = k^2 - P , \tag{3.1}
$$

and one eigenfunction in the longitudinal part with eigenvalue,

$$\lambda_L = (1-a)k^2 - P \ . \tag{3.2}$$

Thus the ζ-function is,

$$\zeta_N^V(s) = (N-1)\sum_k (k^2-P)^{-s} + \sum_k [(1-a)k^2 - P]^{-s} \ . \tag{3.3}$$

The sum over k is an integral because k is a continuous variable,

$$\begin{aligned}
\sum_k (k^2-P)^{-s} &= V_N \int \frac{d^N k}{(2\pi)^N}(k^2-P)^{-s} \ , \\
&= \frac{V_N}{\Gamma(s)}\int_0^\infty d\tau \tau^{s-1}\int \frac{d^N k}{(2\pi)^N} e^{-\tau(k^2-P)} \ , \\
&= \frac{(-1)^{\frac{N}{2}-s} V_N}{(4\pi)^{N/2}} \frac{\Gamma\left(s-\frac{N}{2}\right)}{\Gamma(s)} P^{\frac{N}{2}-s} \ ,
\end{aligned} \tag{3.4}$$

where V_N is the volume of the N-dimensional space (N = even). Therefore,

$$\zeta_N^V(s) = \frac{(-1)^{\frac{N}{2}-s} V_N}{(4\pi)^{N/2}} \frac{\Gamma\left(s-\frac{N}{2}\right)}{\Gamma(s)} \left[(N-1) + (1-a)^{-N/2}\right] P^{\frac{N}{2}-s} \ , \tag{3.5}$$

and

$$\zeta_N^V(0) = \frac{V_N}{(4\pi)^{N/2}(N/2)!}\left[(1-a)^{-N/2} + (N-1)\right] P^{N/2} \ . \tag{3.6}$$

B. Tensor Case

For the tensor operator, there are $\frac{1}{2}(N-2)(N+1)$ eigenfunctions in the TT part with the eigenvalues,

$$\lambda_{TT} = k^2 - P \ , \tag{3.7}$$

and $(N-1)$ eigenfunctions in the LTT part with eigenvalues,

$$\lambda_{LTT} = \left(1 - \frac{1}{2}a_1\right) k^2 - P \ . \tag{3.8}$$

The ζ-functions corresponding to these two parts are,

$$\zeta_N^{TT}(0) = \frac{V_N}{(4\pi)^{N/2}(N/2)!} P^{N/2} \left[\frac{1}{2}(N-2)(N+1)\right] \ , \tag{3.9}$$

$$\zeta_N^{LTT}(0) = \frac{V_N}{(4\pi)^{N/2}(N/2)!} P^{N/2} \left[(N-1)\left(\frac{2}{2-a_1}\right)^{N/2}\right] \ . \tag{3.10}$$

The functions in the LLT and the TR parts are coupled together as shown in Section II. By diagonalizing the matrix of which the elements are given by the coefficients in Eqs. (2.25) and (2.26), we can see that the two eigenvalues λ_1 and λ_2 corresponding to these two parts satisfy the equations,

$$\lambda_1 + \lambda_2 = \alpha_e k^2 + \gamma_e \ , \tag{3.11}$$

$$\lambda_1 \lambda_2 = A_e k^4 + C_e k^2 + E_e \ , \tag{3.12}$$

where

$$\alpha_e = 2 - (a_1 + a_2 + a_3) \ , \tag{3.13}$$

$$\gamma_e = -2P - NQ \ , \tag{3.14}$$

$$A_e = 1 - (a_1 + a_2 + a_3) - a_2 a_3 (N-1) \ , \tag{3.15}$$

$$C_e = -[2 - (a_1 + a_2 + a_3)]P - [N - (N-1)a_1]Q \ , \tag{3.16}$$

$$E_e = P(P + NQ) \ . \tag{3.17}$$

Since λ_1 and λ_2 are not polynomials in k^2, it is very difficult to do the k-integral to obtain the corresponding ζ-functions. In fact, we have

$$\lambda_1 = \frac{1}{2}\left[(\alpha_e k^2 + \gamma_e) + \sqrt{(\alpha_e k^2 + \gamma_e)^2 - 4(A_e k^4 + C_e k^2 + E_e)}\right] \ , \tag{3.18}$$

$$\lambda_2 = \frac{1}{2}\left[(\alpha_e k^2 + \gamma_e) - \sqrt{(\alpha_e k^2 + \gamma_e)^2 - 4(A_e k^4 + C_e k^2 + E_e)}\right] \ . \tag{3.19}$$

However, we are interested in the ζ-function at $s = 0$, and this depends only on the small τ behavior in the integrand of the τ-integral like the one in Eq. (3.4). To extract the small τ behavior from the integral over k, we need only to concentrate on the part of large k. Hence, we can expand λ_1 and λ_2 as power series in $1/k^2$, and we shall see in the following that only the first few terms will contribute to $\zeta_N^{LLT+TR}(0)$. Expanding,

$$
-\frac{1}{2}k^2\left[\sqrt{\left(\alpha_e+\frac{\gamma_e}{k^2}\right)^2-4\left(A_e+\frac{C_e}{k^2}+\frac{E_e}{k^4}\right)}-\left(\alpha_e+\frac{\gamma_e}{k^2}\right)\sqrt{1-\frac{4A_e}{\alpha_e^2}}\right]
$$
$$
\equiv \eta_0+\frac{\eta_1}{k^2}+\frac{\eta_2}{k^4}+\cdots\ , \tag{3.20}
$$

we can evaluate the ζ-function for the LLT and the TR parts,

$$
\begin{aligned}
\zeta_N^{LLT+TR}(0) &= \lim_{s\to 0} V_N \int \frac{d^Nk}{(2\pi)^N}\left[\lambda_1^{-s}+\lambda_2^{-s}\right]\ ,\\
&= \lim_{s\to 0}\frac{V_N}{\Gamma(s)}\int_0^\infty d\tau\tau^{s-1}\int\frac{d^Nk}{(2\pi)^N}\left[e^{-\tau\lambda_1}+e^{-\tau\lambda_2}\right]\ ,\\
&= \lim_{s\to 0}\frac{V_N}{\Gamma(s)}\int_0^\infty d\tau\tau^{s-1}e^{-\frac{\tau}{2}\gamma_e}\int\frac{d^Nk}{(2\pi)^N}e^{-\frac{\tau}{2}\alpha_e k^2}\times\\
&\quad\left[2-\frac{\tau\alpha_e}{2A_e}\sqrt{\alpha_e^2-4A_e}\left(\eta_0+\frac{\eta_1}{k^2}+\frac{\eta_2}{k^4}+\cdots\right)\right.\\
&\quad\left.+\frac{\tau^2}{2}\left(\frac{\alpha_e^2(\alpha_e^2-2A_e)}{4A_e^2}\right)\left(\eta_0^2+\frac{2\eta_0\eta_1}{k^2}+\cdots\right)+\cdots\right]\ .
\end{aligned}\tag{3.21}
$$

Note that the last step involves a rescaling

$$
\tau\left[1\pm\sqrt{1-4A_e/\alpha_e^2}\right]\to\tau\ . \tag{3.22}
$$

In this power series form, the integrations over k and τ can be performed. To be explicit, we consider the $N=2$ case. The k-integral in Eq. (3.21) becomes,

$$
\begin{aligned}
&\int\frac{d^Nk}{(2\pi)^N}e^{-\frac{\tau}{2}\alpha_e k^2}\left(2-\frac{\tau\alpha_e}{2A_e}\sqrt{\alpha_e^2-4A_e}\,\eta_0\right)+\cdots\\
&=\frac{V_2}{4\pi}\left[\frac{4}{\alpha_e\tau}-\frac{1}{A_e}\sqrt{\alpha_e^2-4A_e}\,\eta_0\right]+\cdots\ .
\end{aligned}\tag{3.23}
$$

We have left out the terms which, after the integration over τ, will vanish when the limit $s\to 0$ is taken. From Eq. (3.20),

$$
\eta_0=\frac{\alpha_e C_e-2A_e\gamma_e}{\alpha_e\sqrt{\alpha_e^2-4A_e}}\ . \tag{3.24}
$$

Using this result and Eq. (3.23), the sum of the ζ-functions for the LLT and the TR parts for $N=2$ is

$$
\begin{aligned}
\zeta_2^{LLT+TR}(0) &= \lim_{s\to 0}\frac{V_2}{(4\pi)\Gamma(s)}\int_0^\infty d\tau\tau^{s-1}e^{-\frac{\tau}{2}\gamma_e}\left[\frac{4}{\alpha_e\tau}-\frac{1}{A_e}\sqrt{\alpha_e^2-4A_e}\,\eta_0\right]\ ,\\
&= \frac{V_2}{4\pi}\left[\left(\frac{\alpha_e}{A_e}\right)P-\left(\frac{a_1-2}{A_e}\right)Q\right]\ .
\end{aligned}\tag{3.25}
$$

Finally the ζ-function for the tensor operator on a two-dimensional Euclidean space is,

$$\begin{aligned}\zeta_2^T(0) &= \zeta_2^{TT}(0) + \zeta_2^{LTT}(0) + \zeta_2^{LLT+TR}(0) \ , \\ &= -\frac{V_2}{4\pi}\left[\left(\frac{2}{a_1-2} - \frac{\alpha_e}{A_e}\right)P + \left(\frac{a_1-2}{A_e}\right)Q\right] \ ,\end{aligned} \tag{3.26}$$

where α_e and A_e are given in Eqs. (3.13) and (3.15) with $N = 2$.

One can extend this procedure to higher even-dimensions. However, the number of terms involved increases very quickly and the answers are too lengthy to be written down in any simple way. We choose to list results in the Appendix for only the special case in which $a_1 = -2a_2$, $a_3 = 0$, and with dimensions up to ten. This case is of special interest because the tensor operator with these parametrization corresponds to the graviton operator in Einstein gravity with the covariant gauge fixing Lagragian,

$$\mathcal{L}_{gf} = -\frac{1}{2}\sqrt{-g}(1+a_2^2)\left(\nabla^\rho h^\mu_\rho - \frac{1}{2}\nabla^\mu h^\rho_\rho\right)\left(\nabla^\sigma h_{\mu\sigma} - \frac{1}{2}\nabla_\mu h^\sigma_\sigma\right) \ . \tag{3.27}$$

Here $h_{\mu\nu}$ is the graviton field, and $a_2 \to \infty$ gives the Landau-DeWitt gauge that we have mentioned in Section I.

IV. ζ-FUNCTIONS ON N-SPHERES

In this section we extend the considerations of the last section to N-spheres. We use the eigenvalues and the degeneracies for the Laplacian given in [17]. For spheres,

$$\kappa = \frac{1}{r^2} \ , \tag{4.1}$$

where r is the radius of the sphere.

A. Vector Case

From [17], the eigenvalues and the degeneracies for the Laplacian of the transverse part of the vector operator are,

$$\Lambda_l^T(N) = -\frac{l(l+N-1)-1}{r^2} , \tag{4.2}$$

$$D_l^T(N) = \frac{l(l+N-1)(2l+N-1)(l+N-3)!}{(N-2)!(l+1)!} , \tag{4.3}$$

where $l = 1, 2, 3, \cdots$. For the longitudinal part, they are

$$\Lambda_l^L(N) = -\frac{l(l+N-1)-(N-1)}{r^2} , \tag{4.4}$$

$$D_l^L(N) = \frac{(2l+N-1)(l+N-2)!}{l!(N-1)!} , \tag{4.5}$$

where $l = 1, 2, 3, \cdots$. Putting these into Eqs. (2.9) and (2.14), we obtain the eigenvalues for the transverse part of the vector operator,

$$\lambda_l^T(N) = \frac{l(l+N-1)-1}{r^2} - P , \tag{4.6}$$

with degeneracies $D_l^T(N)$, and for the longitudinal part the eigenvalues,

$$\lambda_l^L(N) = \frac{(1-a)[l(l+N-1)-(N-1)]}{r^2} - P - \frac{a(N-1)}{r^2} , \tag{4.7}$$

with degeneracies $D_l^L(N)$. The ζ-function is thus given by,

$$\begin{aligned} \zeta_N^V(s) &= \sum_{l=1}^{\infty}[D_l^T(N)(\lambda_l^T(N))^{-s} + D_l^L(N)(\lambda_l^L(N))^{-s}] , \\ &\equiv \zeta_N^T(s) + \zeta_N^L(s) . \end{aligned} \tag{4.8}$$

Consider first $\zeta_N^T(s)$,

$$\begin{aligned} \zeta_N^T(s) &= \sum_{l=1}^{\infty} D_l^T(N)[\lambda_l^T(N)]^{-s} , \\ &= \sum_{l=0}^{\infty} D_{l+1}^T(N)[\lambda_{l+1}^T(N)]^{-s} , \\ &= \frac{1}{\Gamma(s)}\int_0^{\infty} d\tau\,\tau^{s-1} \sum_{l=0}^{\infty} D_{l+1}^T(N) e^{-\tau\lambda_{l+1}^T(N)} . \end{aligned} \tag{4.9}$$

For example, for $N = 2$, we have

$$\begin{aligned} \zeta_2^T(s) &= \frac{1}{\Gamma(s)}\int_0^{\infty} d\tau\,\tau^{s-1} \sum_{l=0}^{\infty}(2l+3)e^{-\tau[\frac{(l+1)(l+2)-1}{r^2}-P]} , \\ &= \frac{r^{2s}}{\Gamma(s)}\int_0^{\infty} d\tau\,\tau^{s-1} e^{-\tau(-Pr^2+1)} \sum_{l=0}^{\infty}(2l+3)e^{-\tau(l^2+3l)} . \end{aligned} \tag{4.10}$$

Since we just want to evaluate ζ-functions at $s=0$, we can concern ourselves with the small τ behavior of the integrand above. It is sufficient to have an asymptotic expansion of the l-sum for small τ to evaluate Eq. (4.10). This can be achieved using the Euler-Maclaurin formula [18],

$$\sum_{l=0}^{\infty} f(l) = \int_0^{\infty} dl\ f(l) + \frac{1}{2}[f(\infty)+f(0)] + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}[f^{(2k-1)}(\infty) - f^{(2k-1)}(0)] \ , \tag{4.11}$$

where B_{2k} are the Bernoulli numbers. Using this formula, the sum in $\zeta_2^T(s)$ can be expanded into,

$$\sum_{l=0}^{\infty}(2l+3)e^{-\tau(l^2+3l)} = \frac{1}{\tau} + \frac{4}{3} + O(\tau) \ . \tag{4.12}$$

When this asymptotic expansion is put back into Eq. (4.10), the terms with order τ or higher in the expansion will vanish as $s \to 0$ (because $\Gamma(s) \sim 1/s$). Therefore,

$$\begin{aligned} \zeta_2^T(0) &= \lim_{s\to 0} \frac{1}{\Gamma(s)} \int_0^{\infty} d\tau\ \tau^{s-1} e^{-\tau(-Pr^2+1)} \left(\frac{1}{\tau} + \frac{4}{3}\right) \ , \\ &= (Pr^2) + \frac{1}{3} \ . \end{aligned} \tag{4.13}$$

Similarly, for $\zeta_2^L(s)$, we have

$$\zeta_2^L(0) = \left(\frac{1}{1-a}\right)(Pr^2) + \left(\frac{1}{1-a} - \frac{2}{3}\right) \ . \tag{4.14}$$

Hence,

$$\begin{aligned} \zeta_2^V(0) &= \zeta_2^T(0) + \zeta_2^L(0) \ , \\ &= \left(\frac{1}{1-a}+1\right)(Pr^2) + \left(\frac{1}{1-a} - \frac{1}{3}\right) \ . \end{aligned} \tag{4.15}$$

We have extended this procedure up to $N = 10$, and the result is summarized in the Appendix.

B. Tensor Case

From [17], the eigenvalues and degeneracies for the Laplacian of the TT part of the tensor operator are,

$$\Lambda_l^{TT}(N) = -\frac{l(l+N-1)-2}{r^2} \ ,$$
$$D_l^{TT}(N) = \frac{(N+1)(N-2)(l+N)(l-1)(2l+N-1)(l+N-3)!}{2(N-1)!(l+1)!} \ , \tag{4.16}$$

with $l = 2, 3, \cdots$. For the LTT part, they are

$$\Lambda_l^{LTT}(N) = -\frac{l(l+N-1)-(N+2)}{r^2} \ ,$$
$$D_l^{LTT}(N) = \frac{l(l+N-1)(2l+N-1)(l+N-3)!}{(N-2)!(l+1)!} \ , \tag{4.17}$$

with $l = 2, 3, \cdots$. For the LLT part, they are

$$\Lambda_l^{LLT}(N) = -\frac{l(l+N-1)-2N}{r^2} \ ,$$
$$D_l^{LLT}(N) = \frac{(2l+N-1)(l+N-2)!}{l!(N-1)!} \ , \tag{4.18}$$

with $l = 2, 3, \cdots$. For the TR part, they are

$$\Lambda_l^{TR}(N) = -\frac{l(l+N-1)}{r^2} \ ,$$
$$D_l^{TR}(N) = \frac{(2l+N-1)(l+N-2)!}{l!(N-1)!} \ , \tag{4.19}$$

with $l = 0, 1, 2, \cdots$. Thus, from Eqs. (2.23) and (2.24), the eigenvalues for the TT part of the tensor operator are,

$$\lambda_l^{TT}(N) = \frac{l(l+N-1)-2}{r^2} - P \ , \tag{4.20}$$

with degeneracies $D_l^{TT}(N)$, and for the LTT part the eigenvalues are,

$$\lambda_l^{LTT}(N) = \left(1 - \frac{a_1}{2}\right)\left(\frac{l(l+N-1)-(N+2)}{r^2}\right) - P - \frac{a_1}{r^2} \ , \tag{4.21}$$

with degeneracies $D_l^{LTT}(N)$. One can see that they are very similar to the ones in the vector case, so it is straight forward to obtain the ζ-function corresponding to these two parts, $\zeta_N^{TT}(0)$ and $\zeta_N^{LLT}(0)$, of the tensor operator using the same method as in the case of the vector operator. For example, for $N = 2$ we have

$$\zeta_2^{TT}(0) = 0 \ ,$$
$$\zeta_2^{LLT}(0) = -\frac{2}{(a_1-2)}(Pr^2) - \left(\frac{4}{a_1-2} + \frac{5}{3}\right) \ . \tag{4.22}$$

For the LLT and the TR parts, the situation is more complicated because the functions are coupled together. As in the Euclidean case, we can obtain the following relations for the eigenvalues from Eqs. (2.25) and (2.26),

$$\lambda_1 + \lambda_2 = \frac{1}{r^2}[\alpha l^2 + \beta l + \gamma] \ , \tag{4.23}$$

$$\lambda_1 \lambda_2 = \frac{1}{r^4}(Al^4 + Bl^3 + Cl^2 + Dl + E) \ , \tag{4.24}$$

where

$$\alpha = 2 - (a_1 + a_2 + a_3) \ , \tag{4.25}$$

$$\beta = (N-1)\alpha \ , \tag{4.26}$$

$$\gamma = -2(Pr^2) - NQr^2 + (N-1)(a_1 - 2) - 2 \ , \tag{4.27}$$

$$A = 1 - (a_1 + a_2 + a_3) - (N-1)a_2 a_3 \ , \tag{4.28}$$

$$B = 2(N-1)A \ , \tag{4.29}$$

$$C = \frac{D}{N-1} + (N-1)^2 A \ , \tag{4.30}$$

$$\begin{aligned} D = &-(N-1)\alpha(Pr^2) + (N-1)^2(a_1 - 2)Qr^2 - (N-1)(2-N)Qr^2 \\ &- N(N-1)A - N(N-1)\alpha - (N-1)^2(a_1 - 2) + (N-1)(2-N) \ , \end{aligned} \tag{4.31}$$

$$E = -[\gamma + (Pr^2 + NQr^2)][Pr^2 + NQr^2] \ . \tag{4.32}$$

Note that the degeneracies $D_l^{LLT}(N)$ and $D_l^{TR}(N)$ are the same because they are both concerned with scalar functions, L and $H^\mu{}_\mu$. However, l starts from 2 in the LLT part, while l starts from 0 in the TR part. Thus the cases with $l = 0$ and 1 in the TR part have to be separated out. The ζ-function for these two coupled parts becomes,

$$\begin{aligned} \zeta_N^{LLT+TR}(s) = &\sum_{l=2}^{\infty} D_l^{TR}(N)\left[\lambda_1^{-s} + \lambda_2^{-s}\right] + D_0^{TR}(N)(-P - NQ)^{-s} \\ &+ D_1^{TR}(N)\left[\left(1 - \left(\frac{a_1}{N} + a_2 + a_3\right)\right)\frac{N}{r^2} - P - NQ\right]^{-s} \ . \end{aligned} \tag{4.33}$$

As $s \to 0$, which is the limit we shall ultimately take,

$$\begin{aligned} \zeta_N^{LLT+TR}(0) &= \lim_{s\to 0}\sum_{l=2}^{\infty} D_l^{TR}(N)[\lambda_1^{-s} + \lambda_2^{-s}] + D_0^{TR}(N) + D_1^{TR}(N) \ , \\ &= \lim_{s\to 0}\sum_{l=0}^{\infty} D_l^{TR}(N)\left[\lambda_1^{-s} + \lambda_2^{-s}\right] - (N+2) \ , \end{aligned} \tag{4.34}$$

because

$$D_0^{LLT}(N) = D_0^{TR}(N) = 1 \ , \tag{4.35}$$

$$D_1^{LLT}(N) = D_1^{TR}(N) = N+1 \ . \tag{4.36}$$

For λ_1 and λ_2, we have

$$\lambda_1 = \frac{1}{2r^2}[(\alpha l^2 + \beta l + \gamma) + \sqrt{(\alpha l^2 + \beta l + \gamma)^2 - 4(Al^4 + Bl^3 + Cl^2 + Dl + E)}] \ , \tag{4.37}$$

$$\lambda_2 = \frac{1}{2r^2}[(\alpha l^2 + \beta l + \gamma) - \sqrt{(\alpha l^2 + \beta l + \gamma)^2 - 4(Al^4 + Bl^3 + Cl^2 + Dl + E)}] \ . \tag{4.38}$$

Since the eigenvalues are not polynomials in l, we cannot (as in the vector case) apply the Euler-Maclaurin formula directly to obtain the asymptotic series in τ. As was done in the Euclidean case, we first expand λ_1 and λ_2 as power series in $1/l$, and then use the Euler-Maclaurin formula to evaluate the sums over l. Suppose that,

$$\begin{aligned} -\frac{1}{2}l^2 &\left[\sqrt{\left(\alpha + \frac{\beta}{l} + \frac{\gamma}{l^2}\right)^2 - 4\left(A + \frac{B}{l} + \frac{C}{l^2} + \frac{D}{l^3} + \frac{E}{l^4}\right)} - \left(\alpha + \frac{\beta}{l} + \frac{\gamma}{l^2}\right)\sqrt{1 - \frac{4A}{\alpha^2}}\right] \\ &\equiv \xi_0 + \frac{\xi_1}{l} + \frac{\xi_2}{l^2} + \cdots \ . \end{aligned} \tag{4.39}$$

There is no l^2 term in this expansion because we have explicitly subtracted it out. The l term also vanishes as a result of the form of β and B in Eqs. (4.26) and (4.29). With this expansion we can evaluate the sum in Eq. (4.34),

$$\begin{aligned} &\lim_{s\to 0} \sum_{l=0}^{\infty} D_l^{TR}(N)[\lambda_1^{-s} + \lambda_2^{-s}] \\ &= \lim_{s\to 0} \sum_{l=0}^{\infty} D_l^{TR}(N) \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \left[e^{-\tau\lambda_1} + e^{-\tau\lambda_2}\right] \ , \\ &= \lim_{s\to 0} \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} e^{-\frac{\tau}{2}\gamma} \sum_{l=0}^{\infty} D_l^{TR}(N) e^{-\frac{\tau}{2}(\alpha l^2 + \beta l)} \times \\ &\quad \left[2 - \frac{\tau\alpha}{2A}\sqrt{\alpha^2 - 4A}\left(\xi_0 + \frac{\xi_1}{l} + \frac{\xi_2}{l^2} + \cdots\right) \right. \\ &\quad \left. + \frac{\tau^2}{2}\left(\frac{\alpha^2(\alpha^2 - 2A)}{4A^2}\right)\left(\xi_0^2 + \frac{2\xi_0\xi_1}{l} + \cdots\right) + \cdots\right] \ . \end{aligned} \tag{4.40}$$

This last step involves a scaling

$$
\frac{\tau}{r^2}\left[1 \pm \sqrt{1-\frac{4A}{\alpha^2}}\right] \rightarrow \tau \ . \tag{4.41}
$$

In this form, one can now apply the Euler-Maclaurin formula to obtain an asymptotic series for small τ for the sum. To be explicit, we consider the $N = 2$ case, where the sum in Eq. (4.40) becomes,

$$
\begin{aligned}
&\sum_{l=0}^{\infty}(2l+1)e^{-\frac{\tau}{2}(\alpha l^2+\beta l)}\left(2-\frac{\tau\alpha}{2A}\sqrt{\alpha^2-4A}\xi_0\right)+\cdots \\
&=\frac{4}{\alpha\tau}+\left(\frac{2}{3}-\frac{1}{A}\sqrt{\alpha^2-4A}\,\xi_0\right)+\cdots \ .
\end{aligned} \tag{4.42}
$$

Again we have left out the terms which, after the integration over τ, will vanish when the limit $s \rightarrow 0$ is taken. Putting the expressions from Eq. (4.25) to Eq. (4.32), with $N = 2$, into the expansion in Eq. (4.39), we have

$$
\xi_0 = \frac{\alpha D - 2A\gamma}{\alpha\sqrt{\alpha^2-4A}} \ . \tag{4.43}
$$

Therefore,

$$
\begin{aligned}
&\lim_{s\to 0}\sum_{l=0}^{\infty} D_l^{TR}(2)\left[\lambda_1^{-s}+\lambda_2^{-s}\right] \\
&=\lim_{s\to 0}\frac{r^{2s}}{\Gamma(s)}\int_0^{\infty} d\tau\tau^{s-1}e^{-\frac{\tau}{2}\gamma}\left[\frac{4}{\alpha\tau}+\frac{2}{3}-\frac{1}{A}\left(\sqrt{\alpha^2-4A}\,\xi_0\right)\right] \ , \\
&=\left(\frac{\alpha}{A}\right)(Pr^2)-\left(\frac{a_1-2}{A}\right)(Qr^2)+\left[\frac{2}{3A}(3\alpha+4A)+\frac{a_1-2}{A}\right] \ .
\end{aligned} \tag{4.44}
$$

Putting this result into Eq. (4.34), the ζ-function for the LLT and the TR parts for $N = 2$ is

$$
\zeta_2^{LLT+TR}(0)=\left(\frac{\alpha}{A}\right)(Pr^2)-\left(\frac{a_1-2}{A}\right)(Qr^2)+\left[\frac{2}{3A}(3\alpha-2A)+\frac{a_1-2}{A}\right] \ . \tag{4.45}
$$

Finally the ζ-function for the tensor operator on 2-sphere is,

$$
\begin{aligned}
\zeta_2^T(0) &= \zeta_2^{TT}(0)+\zeta_2^{LTT}(0)+\zeta_2^{LLT+TR}(0) \ , \\
&= -\left(\frac{2}{a_1-2}-\frac{\alpha}{A}\right)(Pr^2)-\left(\frac{a_1-2}{A}\right)(Qr^2) \\
&\quad +\left[-\frac{4}{a_1-2}+\frac{1}{A}(2\alpha-3A)+\frac{a_1-2}{A}\right] \ ,
\end{aligned} \tag{4.46}
$$

where α and A are given in Eqs. (4.25) and (4.28) with $N = 2$. One can extend this procedure to higher even-dimensions. The results for $N = 2$ through 10 with $a_1 = -2a_2$ and $a_3 = 0$ are listed in the Appendix.

V. CONCLUSIONS

We have shown how to evaluate the zeta-function at zero, $\zeta(0)$, for certain non-minimal vector and tensor operators. The procedure is to first decompose the vector and tensor functions into their irreducible parts. For vectors there are the transverse (T) and the longitudinal (L) parts. For tensors there are the TT, the LTT, the LLT and the TR parts. Then evaluate the eigenvalues for the various parts of each operator. Due to the fact that the tensor operator is non-minimal, the LLT and the TR parts are in fact coupled together and the eigenvalues for these two parts are complicated. However, we have shown that it is still possible to obtain $\zeta(0)$ by use of the appropriate series expansion for the eigenvalues. Using this procedure we explicitly evaluated $\zeta(0)$ for Euclidean spaces and N-spheres for even dimensions up to ten, and summarized the results in the Appendix. Other techniques have been developed to successfully deal with flat backgrounds [14]; however, our use of the Euler-Maclaurin formula [18] has allowed us to now deal with more interesting backgrounds.

Although the above procedure gets more tedious as one goes to higher dimensions, there is no conceptual difficulty in doing so. One can extend this method to dimensions higher than ten, to Kaluza-Klein spacetimes like $M^4 \times S^N$ [14], and to more general coset spaces for which eigenvalues of the corresponding Laplacians are known.

The method developed here is general enough to be useful in many circumstances when one-loop quantum effects are calculated in gauge theories with general gauge conditions. What we have in mind in particular is the calculation of the VD effective potentials in Kaluza-Klein spaces. We are also interested in using the VD formalism to evaluate the gauge-independent trace anomaly for gravitons [19]. The explicit evaluation of this trace anomaly in different spacetimes will be possible by making use of the ζ-functions derived here.

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APPENDIX

In this appendix we summarize the ζ-function values that we have obtained for the non-minimal vector operator,

$$M_V{}^{\alpha}{}_{\beta} = \delta^{\alpha}_{\beta}(-\Box - P) + a\nabla^{\alpha}\nabla_{\beta} \tag{A1}$$

and the non-minimal tensor operator,

$$M_T{}^{\alpha\beta}{}_{\rho\sigma} = \delta^{(\alpha}_{\rho}\delta^{\beta)}_{\sigma}(-\Box - P) - g^{\alpha\beta}g_{\rho\sigma}Q - 2a_2\delta^{(\alpha}_{(\rho}\nabla^{\beta)}\nabla_{\sigma)} + a_2 g_{\rho\sigma}\nabla^{(\alpha}\nabla^{\beta)} \tag{A2}$$

1. Vector Case

For N-dimensional Euclidean spaces, the ζ-function values for M_V in Eq. (A1) are,

$$\zeta^V_N(0) = \frac{V_N}{(4\pi)^{N/2}(N/2)!}\left[(1-a)^{-N/2} + (N-1)\right]P^{N/2} \tag{A3}$$

where V_N is the volume of the N-dimensional space.

While for N-spheres with radius r, where $N = 2, 4, 6, 8, 10$, we have,

$$\zeta^V_2(0) = \frac{1}{3(1-a)}\left[3(2-a)(Pr^2) + (2+a)\right] \tag{A4}$$

$$\zeta^V_4(0) = \frac{1}{180(1-a)^2}\times$$
$$\Big[15(4-6a+3a^2)(Pr^2)^2 + 30(8-8a+3a^2)(Pr^2)$$

$$+(172+106a-143a^2)\Big] \tag{A5}$$

$$\zeta_6^V(0)=\frac{1}{2520(1-a)^3}\times$$
$$\Big[7(6-15a+15a^2-5a^3)(Pr^2)^3+105(6-13a+12a^2-4a^3)(Pr^2)^2$$
$$+21(134-203a+129a^2-35a^3)(Pr^2)$$
$$+(3394+213a-5358a^2+2626a^3)\Big] \tag{A6}$$

$$\zeta_8^V(0)=\frac{1}{907200(1-a)^4}\times$$
$$\Big[45(8-28a+42a^2-28a^3+7a^4)(Pr^2)^4$$
$$+420(32-104a+150a^2-100a^3+25a^4)(Pr^2)^3$$
$$+630(280-796a+1038a^2-668a^3+167a^4)(Pr^2)^2$$
$$+180(5200-11064a+9930a^2-4724a^3+1001a^4)(Pr^2)$$
$$+(1592968-1081132a-2826102a^2+3532148a^3-1109837a^4)\Big] \tag{A7}$$

$$\zeta_{10}^V(0)=\frac{1}{59875200(1-a)^5}\times$$
$$\Big[33(10-45a+90a^2-90a^3+45a^4-9a^5)(Pr^2)^5$$
$$+495(50-215a+420a^2-420a^3+210a^4-42a^5)(Pr^2)^4$$
$$+330(2150-8611a+16028a^2-15810a^3+7905a^4-1581a^5)(Pr^2)^3$$
$$+330(29018-103119a+172158a^2-160184a^3+78570a^4-15714a^5)(Pr^2)^2$$
$$+1485(40002-112777a+135030a^2-89946a^3+34405a^4-5985a^5)(Pr^2)$$
$$+(129517198-192948785a-167760920a^2+502687820a^3$$
$$-351902170a^4+82355474a^5)\Big] \tag{A8}$$

2. Tensor Case

For the N-dimensional Euclidean spaces, where $N=2,4,6,8,10$, the ζ-function values for M_T in Eq. (A2) are,

$$\zeta_2^T(0)=\frac{V_2}{4\pi(1+a_2)}\left[(3+a_2)P+2(1+a_2)Q\right] \tag{A9}$$

$$\zeta_4^T(0) = \frac{V_4}{(4\pi)^2(1+a_2)^2} \times$$
$$\Big[(5+6a_2+3a_2^2)P^2 + 2(2+6a_2+3a_2^2)PQ$$
$$+2(2+3a_2)^2Q^2\Big] \tag{A10}$$
$$\zeta_6^T(0) = \frac{V_6}{6(4\pi)^3(1+a_2)^3} \times$$
$$\Big[3(7+15a_2+15a_2^2+5a_2^3)P^3 + 6(3+15a_2+15a_2^2+5a_2^3)P^2Q$$
$$+12(9+45a_2+65a_2^2+25a_2^3)PQ^2 + 8(3+5a_2)^3Q^3\Big] \tag{A11}$$
$$\zeta_8^T(0) = \frac{V_8}{6(4\pi)^4(1+a_2)^4} \times$$
$$\Big[(9+28a_2+42a_2^2+28a_2^3+7a_2^4)P^4$$
$$+2(4+28a_2+42a_2^2+28a_2^3+7a_2^4)P^3Q$$
$$+6(16+112a_2+238a_2^2+182a_2^3+49a_2^4)P^2Q^2$$
$$+8(4+7a_2)^2(4+14a_2+7a_2^2)PQ^3$$
$$+4(4+7a_2)^4Q^4\Big] \tag{A12}$$
$$\zeta_{10}^T(0) = \frac{V_{10}}{120(4\pi)^5(1+a_2)^5} \times$$
$$\Big[5(11+45a_2+90a_2^2+90a_2^3+45a_2^4+9a_2^5)P^5$$
$$+10(5+45a_2+90a_2^2+90a_2^3+45a_2^4+9a_2^5)P^4Q$$
$$+40(25+225a_2+630a_2^2+720a_2^3+387a_2^4+81a_2^5)P^3Q^2$$
$$+80(125+1125a_2+3600a_2^2+5094a_2^3+3159a_2^4+729a_2^5)P^2Q^3$$
$$+80(5+9a_2)^3(5+18a_2+9a_2^2)PQ^4$$
$$+32(5+9a_2)^5Q^5\Big] \tag{A13}$$

While for the N-spheres, where again $N = 2, 4, 6, 8, 10$, we have,

$$\zeta_2^T(0) = \frac{1}{1+a_2}\Big[(3+a_2)Pr^2 + 2(1+a_2)Qr^2 + (1-3a_2)\Big] \tag{A14}$$
$$\zeta_4^T(0) = \frac{1}{18(1+a_2)^2} \times$$
$$\Big[3(5+6a_2+3a_2^2)(Pr^2)^2 + 6(2+6a_2+3a_2^2)(Pr^2)(Qr^2)$$
$$+6(2+3a_2)^2(Qr^2)^2 + 12(5-a_2-3a_2^2)(Pr^2)$$

$$+12(2+5a_2)(Qr^2)+2(11-122a_2-97a_2^2)\Big] \tag{A15}$$

$$\zeta_6^T(0) = \frac{1}{360(1+a_2)^3}\times$$

$$\Big[3(7+15a_2+15a_2^2+5a_2^3)(Pr^2)^3$$

$$+6(3+15a_2+15a_2^2+5a_2^3)(Pr^2)^2(Qr^2)$$

$$+12(9+45a_2+65a_2^2+25a_2^3)(Pr^2)(Qr^2)^2$$

$$+8(3+5a_2)^3(Qr^2)^3$$

$$+15(21+27a_2+15a_2^2+5a_2^3)(Pr^2)^2$$

$$+60(3+13a_2+5a_2^2-a_2^3)(Pr^2)(Qr^2)$$

$$+60(9+39a_2+43a_2^2+5a_2^3)(Qr^2)^2$$

$$+24(53-39a_2-142a_2^2-65a_2^3)(Pr^2)$$

$$+48(9+33a_2-16a_2^2-15a_2^3)(Qr^2)$$

$$+10(95-915a_2-1539a_2^2-633a_2^3)\Big] \tag{A16}$$

$$\zeta_8^T(0) = \frac{1}{75600(1+a_2)^4}\times$$

$$\Big[15(9+28a_2+42a_2^2+28a_2^3+7a_2^4)(Pr^2)^4$$

$$+30(4+28a_2+42a_2^2+28a_2^3+7a_2^4)(Pr^2)^3(Qr^2)$$

$$+90(16+112a_2+238a_2^2+182a_2^3+49a_2^4)(Pr^2)^2(Qr^2)^2$$

$$+120(4+7a_2)^2(4+14a_2+7a_2^2)(Pr^2)(Qr^2)^3$$

$$+60(4+7a_2)^4(Qr^2)^4$$

$$+140(36+90a_2+114a_2^2+76a_2^3+19a_2^4)(Pr^2)^3$$

$$+420(8+50a_2+42a_2^2+8a_2^3-a_2^4)(Pr^2)^2(Qr^2)$$

$$+840(32+200a_2+348a_2^2+167a_2^3+14a_2^4)(Pr^2)(Qr^2)^2$$

$$+560(4+7a_2)^2(8+22a_2+5a_2^2)(Qr^2)^3$$

$$+1680(38+54a_2+16a_2^2-3a_2^3)(Pr^2)^2$$

$$+2520(12+66a_2-a_2^2-52a_2^3-18a_2^4)(Pr^2)(Qr^2)$$

$$+2520(48+264a_2+355a_2^2+34a_2^3-42a_2^4)(Qr^2)^2$$

$$+240(1270-992a_2-6515a_2^2-6318a_2^3-1890a_2^4)(Pr^2)$$

$$+720(120+570a_2-683a_2^2-1126a_2^3-336a_2^4)(Qr^2)$$

$$+56(6903-48948a_2-134742a_2^2-113908a_2^3-32437a_2^4)\Big] \tag{A17}$$

$$\zeta_{10}^T(0) = \frac{1}{5443200(1+a_2)^5}\times$$

$$\Big[15(11+45a_2+90a_2^2+90a_2^3+45a_2^4+9a_2^5)(Pr^2)^5$$

$$+30(5+45a_2+90a_2^2+90a_2^3+45a_2^4+9a_2^5)(Pr^2)^4(Qr^2)$$

$$+120(25+225a_2+630a_2^2+720a_2^3+387a_2^4+81a_2^5)(Pr^2)^3(Qr^2)^2$$

$$+240(125+1125a_2+3600a_2^2+5094a_2^3+3159a_2^4+729a_2^5)(Pr^2)^2(Qr^2)^3$$

$$+240(5+9a_2)^3(5+18a_2+9a_2^2)(Pr^2)(Qr^2)^4$$

$$+96(5+9a_2)^5(Qr^2)^5$$

$$+45(275+995a_2+1830a_2^2+1830a_2^3+915a_2^4+183a_2^5)(Pr^2)^4$$

$$+360(25+205a_2+270a_2^2+150a_2^3+39a_2^4+3a_2^5)(Pr^2)^3(Qr^2)$$

$$+1080(125+1025a_2+2450a_2^2+2098a_2^3+725a_2^4+81a_2^5)(Pr^2)^2(Qr^2)^2$$

$$+1440(625+5125a_2+14700a_2^2+17820a_2^3+8451a_2^4+1215a_2^5)(Pr^2)(Qr^2)^3$$

$$+720(5+9a_2)^3(25+70a_2+21a_2^2)(Qr^2)^4$$

$$+60(5815+16805a_2+24400a_2^2+22290a_2^3+11361a_2^4+2301a_2^5)(Pr^2)^3$$

$$+360(545+4033a_2+1934a_2^2-2838a_2^3-2427a_2^4-495a_2^5)(Pr^2)^2(Qr^2)$$

$$+720(2725+20165a_2+39718a_2^2+19410a_2^3-2679a_2^4-2403a_2^5)(Pr^2)(Qr^2)^2$$

$$+480(13625+100825a_2+257210a_2^2+260406a_2^3+80757a_2^4+729a_2^5)(Qr^2)^3$$

$$+120(37589+63843a_2+3462a_2^2-53032a_2^3-34947a_2^4-6183a_2^5)(Pr^2)^2$$

$$+480(3805+25113a_2-15072a_2^2-63632a_2^3-40959a_2^4-7875a_2^5)(Pr^2)(Qr^2)$$

$$+480(19025+125565a_2+193326a_2^2-6850a_2^3-103923a_2^4-33615a_2^5)(Qr^2)^2$$

$$+4320(5920-2712a_2-42445a_2^2-65558a_2^3-40680a_2^4-9300a_2^5)(Pr^2)$$

$$+17280(350+2030a_2-4172a_2^2-10153a_2^3-6280a_2^4-1250a_2^5)(Qr^2)$$

$$+8(5745607-30218365a_2-125264810a_2^2-163829090a_2^3-94569445a_2^4$$

$$-20749889a_2^5)\Big] \tag{A18}$$

REFERENCES

[1] T. Appelquist and A. Chodos, Phys. Rev. Lett. **50**, 141 (1983); Phys. Rev. D**28**, 772 (1983).

[2] I. L. Buchbinder, S. D. Odintsov, and I. L. Shapiro, *Effective Action in Quantum Gravity*, (IOP Publishing Ltd., Bristol, 1992).

[3] S. Randjbar-Daemi and M. H. Sarmadi, Phys. Lett. **151B**, 343 (1985).

[4] G. Kunstatter and H. P. Leivo, Phys. Lett. B **166**, 321 (1986).

[5] I. L. Buchbinder and S. D. Odintsov, Yad. Phys. **47**, 598 (1988); Sov. J. Nucl. Phys.**47**, 377 (1988).

[6] S. R. Huggins, G. Kunstatter, H. P. Leivo, and D. J. Toms, Phys. Rev. Lett. **58**, 296 (1987); Nucl. Phys. **B301**, 627 (1988).

[7] G. Vilkovisky, Nucl. Phys. **B234**, 125 (1984); in *Quantum Theory of Gravity*, edited by S. M. Christensen (Hilger, Bristol, 1984), pp. 169-209.

[8] B. S. DeWitt, in *Architecture of Fundamental Interactions at Short Distances*, proceedings of the Les Houches Summer School, Les Houches, France, 1985, edited by P. Ramond and R. Stora (Les Houches Summer School Proceedings, Vol. 44) (North-Holland, Amsterdam, 1987).

[9] I. L. Buchbinder, P. M. Lavrov, and S. D. Odintsov, Nucl. Phys. **B308**, 191 (1988).

[10] Danny Birmingham, R. Kantowski, H. P. Leivo, and Kimball A. Milton, Nucl. Phys. (Proc. Suppl.) B **6**, 151 (1989).

[11] H. T. Cho and R. Kantowski, Phys. Rev. Lett. **67**, 422 (1991).

[12] A. O. Barvinsky and G. Vilkovisky, Phys. Rep. **119**, 1 (1985).

[13] E. S. Fradkin and A. A. Tseytlin, Nucl. Phys. **B234**, 509 (1984).

[14] P. M. Lavrov, S. D. Odintsov, and V. Tyutin, Sov. J. Nucl. Phys. **46**, 932 (1987).

[15] I. L. Buchbinder and S. D. Odintsov, Fortschr. Phys. . **37**, 225 (1989)

[16] I. L. Buchbinder, E. N. Kirillova, and S. D. Odintsov, Mod. Phys. Lett. **4**, 633 (1989).

[17] Mark A. Rubin and Carlos R. Ordóñez, J. Math. Phys. **25**, 2888 (1984); **26**, 65 (1985).

[18] Danny Birmingham, R. Kantowski, and Kimball A. Milton, Phys. Rev. D**38**, 1809 (1988).

[19] H. T. Cho and R. Kantowski, *Gauge Independent Trace Anomaly for Gravitons*, Preprint.