The Effects of Inhomogeneities on Evaluating the mass parameter Ω_m and the cosmological constant Λ

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ABSTRACT

Analytic expressions for distance-redshift relations which have been corrected for the effects of inhomogeneities in the Friedmann-Lemaître-Robertson-Walker (FLRW) mass density are given in terms of Heun functions and are used to illustrate the significance of inhomogeneities on a determination of the mass parameter Ω_m and the cosmological constant Λ. The values of these parameters inferred from a given set of observations depend on the fractional amount of matter in inhomogeneities and can significantly differ from those obtained by using the standard magnitude-redshift (m-z) result for pure dust FLRW models. As an example a determination of Ω_m made by applying the homogeneous distance-redshift relation to SN 1997ap at $z = 0.83$ could be as much as 50% lower than its true value.

Subject headings: cosmology: theory – large-scale structure of universe

1. Introduction

When attempting to evaluate the mass parameter Ω_m and/or the cosmological constant Λ, observations of quantities such as magnitude, angular separation, and redshift are made on objects distant enough for curvature effects to be detected. As an example, for Type Ia supernovae (SNe Ia) corrected magnitudes and redshifts (m-z) are measured, plotted, and compared with theoretical $m(\Omega_m, \Lambda; z)$ curves computed for the FLRW models (Perlmutter et al. 1997, Perlmutter et al. 1998, Garnavich et al. 1998). In spite of the fact that the FLRW models contain only homogeneously and isotropically distributed perfect fluid gravity sources, one of these models is assumed to represent the "large scale" geometry of the universe. Relations like $m(\Omega_m, \Lambda; z)$ are also commonly assumed to be valid, on average. This latter assumption may well be incorrect for some distant observations including SNe Ia, but even if technically correct may not be useful in determining Ω_m and Λ. In particular if the underlying mass density approximately follows luminous matter (*i.e.,* associated with bounded galaxies) then effects of inhomogeneities on relations like $m(\Omega_m, \Lambda; z)$ must be taken into account. The majority of currently observed SNe Ia are not being seen through foreground galaxies and whether or not this is due entirely to selection (rather than statistics) is not important. If the objects observed do not have the average FLRW mass density ρ_0 in their foregrounds then the FLRW m-z relation does not apply to them. Ultimately some SNe Ia should exist behind foreground galaxies (Rauch 1991) and for these, m-z should be computed using the lensing formulas. These formulas (Bourassa & Kantowski 1975 and Cooke & Kantowski 1975) contain source-observer, deflector-observer, and source-deflector distances, respectively D_s, D_d, and D_{ds}, all of which depend on the mass density in the observing beam, <u>excluding</u> the deflector. These distances will not be given by the standard FLRW result if the observing beam contains less than the average FLRW mass density.

In §2 the average area-redshift equation (1) for a light beam traveling through a FLRW Swiss cheese universe is given and its solution is related to the luminosity distance-redshift relation $D_\ell(z)$. In §3 the solution of this equation is given for the case where gravitational lensing can be neglected. The new result of this paper, $D_\ell(z)$ without lensing for FLRW Swiss cheese can be found in equations (42) and (47), and for the special case $\Omega_0 = 1$ in equations (A2) and (A3) of Appendix A. In §4 numerous m-z plots are given to illustrate the effects of inhomogeneities and some conclusions are drawn. It is argued that if homogeneities are not taken into account when attempting to determine Ω_m and Λ, errors as large as 50% could be made. Even though the $D_\ell(z)$ given here has been derived using the exact Swiss cheese cosmologies, the result are valid for observations in essentially any perturbed pressure-free FLRW models in which lensing can be neglected.

Inhomogeneous models of the Swiss cheese type and their associated optical equations discussed here are often mistakenly attributed to Dyer and Roeder (see Appendix B).

Appendix C contains some useful simplifications for evaluating the real-valued Huen functions needed in the analytic m-z relations given here. Appendix C also contains six useful lines of Mathematica code which numerically evaluates and plot these same m-z relations.

2. Swiss Cheese Optics

Some years ago the author (Kantowski 1969), used the "Swiss cheese" cosmologies to study the effects of local inhomogeneities in the FRW mass density on the propagation of light through an otherwise homogeneous and isotropic universe. That analytic work was undertaken because prior results computed in perturbed FRW models were suspect. In particular, results were inconsistent with weak lensing results, *e.g.,* on average, the luminosity of a distant object was not given by the FRW result (Bertotti 1966). Numerous sources of error were suggested but particularly mistakes inherent in using perturbative gravity were suggested. For example the FRW relations between radius, redshift, and affine parameters were (and still are for approximate GR solutions) assumed valid in the presence of mass perturbations. At that time Kantowski 1969 put to rest any question of the possible existence of an effect on the mean luminosity; theoretically it could exist! Because the Swiss cheese models are exact solutions to the Einstein equations the accuracy of the FRW relations between radius, redshift, and affine parameters could be directly established. Today Swiss cheese itself is under attack as the source of the 'erroneous' prediction (see Frieman 1997 and Wambsganss et al. 1997, and additionally see the related work of Premadi et al. 1997). The intent of this particular paper is not to defend Swiss cheese predictions against these attacks, that can be done elsewhere. The reader should not dismiss distance-redshift predictions made by Swiss cheese models because of their non-physical distribution of matter, *e.g.,* cheese and holes. Because these models contain the only two types of gravitational curvature (Ricci and Weyl) that affect optical observations, and because they are fairly flexible in including density perturbations, they should adequately describe optical observations at least as far as $z = 1$ (a distance to which galaxies and other inhomogeneities are thought to have undergone only minor changes). The conjectured extension of the validity of the optical equation (1) used here beyond the Swiss cheese models has been made frequently since it was first derived and is argued by Schneider et al. 1992 in their Sections 4.5.2 and 4.5.3.

The purpose of this paper is to extend analytic results for distance-redshift relations in inhomogeneous FRW models to FLRW, *i.e.,* to include the cosmological constant (see Kantowski et al. 1995 and Seitz & Schneider 1994). Kantowski 1969 derived a 2nd order intergal-differential equation [see (43a) of that paper or (1) of Kantowski et al. 1995 as well as the equivalent 3rd order differential equation, (43b) of that paper or (6) of Kantowski et al. 1995] for the average cross sectional area A of a beam of light starting from a distant source and propagating through a $\Lambda = 0$ Swiss cheese universe (see Fig. 1). The solution of this equation, with appropriate boundary conditions, gives all average quantities relating to distance-redshift. This equation and its derivation were easily extended to include a cosmological constant by Dyer & Roeder 1974 five years after it was first written down. However, as will be seen below, extension of the equation's analytic solutions of Kantowski et al. 1995 and Seitz & Schneider 1994 is somewhat involved and the special functions required are much less familiar to the math/physics community. Weinberg sign conventions will be again used (Misner et al. 1973).

As a light beam from a distant SN Ia propagates through the universe (Fig. 1) the cheese of the model produces the same focusing effect as does the transparent material actually appearing within the beam. The holes in the cheese with their condensed central masses reproduce the optical effects of the remaining Friedmann matter that has been condensed into clumps, *e.g.,* galaxies. The extended equation for the average area A traversing the universe, that is randomly focused by numerous clouds of transparent matter and lensed by numerous clumps is:

$$\frac{\sqrt{A}''}{\sqrt{A}} + \frac{\langle \xi^2 \rangle}{A^2} = -\frac{3}{2}\,\frac{\rho_D}{\rho_0}\,\Omega_m\,(1+z)^5\;, \qquad (1)$$

where prime ($'$) is differentiation with respect to an affine parameter,

$$' \;\equiv -(1+z)^3\sqrt{1+\Omega_m z + \Omega_\Lambda[(1+z)^{-2}-1]}\;\frac{d}{dz}\;, \qquad (2)$$

and $\langle \xi^2 \rangle$ is the average of $(\sigma/A)^2$, the square of the wavefront's shear over its area,[2]

$$\langle \xi^2 \rangle = \frac{15}{2}\,\frac{\rho_I}{\rho_0}\,B_0\,\Omega_m \int \frac{A^2(1+z)^6}{z'}dz\,. \qquad (3)$$

[2]The form of $\langle \xi^2 \rangle$ depends on structure details of the clumps. What is given in (3) is for objects completely condensed into opaque spherical masses. This particular type of Swiss cheese clumping is expected to produce maximum lensing.

In (1) ρ_D (D is for dust) is the average mass density of all transparent material interior to light beams used to observe the given objects and ρ_I (I is for inhomogeneous) is the average mass density of all types of clumpy material systematically or statistically excluded from the light beams. The average shear term in (1) comes from the Weyl (conformal) curvature tensor of inhomogeneous material exterior to the beams. The ρ_D/ρ_0 term comes from the Ricci tensor of transparent material within the beams. For the SNe Ia observations, ρ_D would certainly include those ubiquitous low mass neutrinos (if they exist) as well as other transparent material not confined to galaxies, while ρ_I would contain all matter clumped with galaxies. If there is no correlation of mass and light, deciding what goes in ρ_D and what goes in ρ_I is problematic, and the relative value becomes another unknown parameter of the theory. The current Friedmann mass density is the total $\rho_0 = \rho_D + \rho_I$ and the curvature parameter $\Omega_0 \equiv \Omega_m + \Omega_\Lambda$ consists of a mass part and a cosmological constant part:

$$\Omega_m \equiv \frac{8\pi G\rho_0}{3H_0^2} \quad \text{and} \quad \Omega_\Lambda \equiv \frac{\Lambda c^2}{3H_0^2}. \tag{4}$$

Inclusion of the cosmological constant Λ (using FLRW rather than FRW) only modifies the functional relationship between redshift and affine parameter (2), see Dyer & Roeder 1974. The unitless gravitational lensing parameter B_0 is defined in equation (A2) of Kantowski et al. 1995 and its effects on the solution of equation (1) are described in Dyer & Roeder 1974. In this paper analytic solutions to equation (1) will be given for $B_0 = 0$, *i.e.,* for distance-redshift in any Swiss cheese model where $< \xi^2 >$ is neglected. In Kantowski et al. 1995 it was argued that even maximal lensing effects ($B_0 \neq 0$) are not expected to be significant when observing SN Ia's at $z \leq 1$ and, as pointed out above, when lensing events do occur, the observed magnitudes should be analyzed by using the lensing formula and not by incorporation into the m-z relation. The resulting $B_0 = 0$ equation (10) represents the equation for the average area of a light beam only Ricci focused by part of the mass density, $\rho_D(\leq \rho_0)$. Such a light beam is not conformally lensed by inhomogeneities ($\rho_I = \rho_0 - \rho_D$)

that remain exterior to the beam. If Weyl lensing is infrequent, as apparently is the case with SNe Ia beams passing exterior to galaxies, a distribution of areas will occur for which the $B_0 = 0$ equation gives the 'mode' value (the most likely).[3] This 'most likely' m-z relation should be more useful in determining Ω_m and Ω_Λ from SNe Ia observations than is the mean m-z relation (standard FLRW relation).

To relate the differential equation (1) to observations, consider a source at redshift z_s radiating power $\delta\mathcal{P}$ into solid angle $\delta\Omega$. The flux received by an observer at $z = 0$ in area $A|_0$ is given by $\mathcal{F} = \delta\mathcal{P}/A|_0(1 + z_s)^2$. The two factors of $(1 + z_s)$ can be thought of as coming separately from the redshift of the observed photons and their decreased rates of reception. The definition of luminosity distance is motivated by this result, *i.e.,*

$$D_\ell^2 \equiv \frac{A|_0}{\delta\Omega}(1 + z_s)^2. \tag{5}$$

The observed area $A|_0$ is evaluated by integrating equation (1) from the source $z = z_s$ to the observer $z = 0$ with initial data which makes the wave front satisfy Euclidean geometry when leaving the source (area=radius$^2\times$ solid angle):

$$\begin{aligned} \sqrt{A}|_s &= 0, \\ \frac{d\sqrt{A}|_s}{dz} &= -\sqrt{\delta\Omega}\frac{c}{H_s(1 + z_s)}, \end{aligned} \tag{6}$$

where in FLRW the value of the Hubble parameter at z_s is related to the current value H_0 at $z = 0$ by

$$H_s = H_0(1 + z_s)\sqrt{1 + \Omega_m z_s + \Omega_\Lambda[(1 + z_s)^{-2} - 1]}. \tag{7}$$

The series solution of equation (1), combined with (5) and (6) is:

$$D_\ell(\Omega_m, \Omega_\Lambda, \nu, B_0; z) = \sqrt{\frac{A}{\delta\Omega}}\bigg|_0 (1 + z) = \frac{c}{H_0}\bigg\{z + \frac{1}{2}\left[1 + \Omega_\Lambda - \frac{1}{2}\Omega_m\right] z^2$$

---

[3]This assertion is supported by numerous plots found in the numerical work of Holz & Wald 1997.

$$
\begin{aligned}
&+ \quad \frac{1}{2}\left[\frac{1}{2}\Omega_m\left(\frac{1}{2}\Omega_m + \frac{\nu(\nu+1)}{6} - 1\right) - \Omega_\Lambda\Big(1 + \Omega_m - \Omega_\Lambda\Big)\right] z^3 \\
&+ \quad \frac{1}{8}\Bigg[\Omega_m\left(\frac{1}{8}\Omega_m\Big[10 - 2\nu(\nu+1) - 5\Omega_m\Big] - B_0\frac{\nu(\nu+1)}{6}\right) + \\
&\qquad\qquad \Omega_\Lambda\left(5 + \frac{1}{2}\Big[5 + \nu(\nu+1)\Big]\Omega_m + \frac{15}{4}\Omega_m^2 + 5\Omega_\Lambda^2 - \frac{5}{2}\Omega_\Lambda\Big[4 + 3\Omega_m\Big]\right)\Bigg] z^4 \\
&+ \quad O[z^5]\Bigg\}, \qquad (8)
\end{aligned}
$$

where the source redshift z_s has been simplified to z and ρ_I/ρ_0 has been replaced for later convenience by a clumping parameter ν, $0 \le \nu \le 2$,

$$\nu \equiv \frac{\sqrt{1 + 24(\rho_I/\rho_0)} - 1}{2} \Rightarrow \frac{\rho_I}{\rho_0} = \frac{\nu(\nu+1)}{6} \qquad (9)$$

This series is useful for understanding the low-redshift sensitivity of D_ℓ to the various parameters; *e.g.,* Ω_m and Ω_Λ appear in the z^2 term, the clumping parameter ν first appears in the z^3 term, whereas the lensing parameter B_0 doesn't appear until the z^4 term. Additionally, analytic results computed in the next section can be checked by comparison with this series.

3. The Analytic Solution For $D_\ell(z)$ when Lensing can be Neglected

In this section the general $B_0 = 0$ solution of (1) will be given for boundary conditions appropriate for $D_\ell(z)$. If apparent-size (angular) distances are desired the reader has only to compute $D_<(z) = D_\ell(z)/(1+z)^2$. The new solution appears in (42), (47), (A2), and (A3) expressed in terms of Heun functions Hl. All previously known special solutions are limiting cases of the general solution (47) [see (18) and Appendix B]. To solve equation (1) it is first rewritten as:

$$
\begin{aligned}
&(1+z)^3\sqrt{1 + \Omega_m z + \Omega_\Lambda[(1+z)^{-2} - 1]} \times \\
&\qquad \frac{d}{dz}(1+z)^3\sqrt{1 + \Omega_m z + \Omega_\Lambda[(1+z)^{-2} - 1]}\,\frac{d}{dz}\sqrt{A(z)}
\end{aligned}
$$

$$+\frac{(3+\nu)(2-\nu)}{4}\Omega_m(1+z)^5\sqrt{A(z)} = 0. \tag{10}$$

This equation is often attributed to Dyer-Roeder in the literature (see Appendix B for some history of this equation). To date only numerical solutions have been obtained when $\Omega_\Lambda \neq 0$, *e.g.,* see Asada 1996, Suto & Matsubara 1996, Kayser et al. 1997. It can be put into a recognizable form by changing the independent variable from z to y and the dependent variable from $\sqrt{A(z)}$ to h,

$$\begin{aligned} y &= y_0(1+z) = \frac{\Omega_m}{1-\Omega_m-\Omega_\Lambda}(1+z), \\ h &= (1+z)\sqrt{\frac{A}{\delta\Omega}}. \end{aligned} \tag{11}$$

The resulting equation is

$$\frac{d^2h}{dy^2} + \frac{\left(1+\frac{3}{2}y\right)y}{y^3+y^2-b_\Omega}\,\frac{dh}{dy} - \frac{\frac{1}{4}\nu(\nu+1)y+1}{y^3+y^2-b_\Omega}\;h = 0\ , \tag{12}$$

where $b_\Omega \equiv -\Omega_m^2\Omega_\Lambda/(1-\Omega_m-\Omega_\Lambda)^3$. When the cubic $y^3+y^2-b_\Omega = (y-y_1)(y-y_2)(y-y_3)$ is factored, (12) simplifies to a recognizable form of the Heun equation (see Ronveaux 1995, Erdélyi 1955, Whittaker & Watson 1927, Heun 1889):

$$\frac{d^2h}{dy^2} + \left(\frac{\gamma}{y-y_1} + \frac{\delta}{y-y_2} + \frac{\epsilon}{y-y_3}\right)\frac{dh}{dy} + \frac{\alpha\beta\; y-q}{(y-y_1)(y-y_2)(y-y_3)}\;h = 0\ , \tag{13}$$

where (12) requires

$$\begin{aligned} &\gamma = \delta = \epsilon = \frac{1}{2}, \\ &\alpha = -\frac{1}{2}\nu, \\ &\beta = \frac{1}{2}(\nu+1), \\ &q = 1, \end{aligned} \tag{14}$$

and additionally the three roots to be constrained by:

$$y_1y_2y_3 = b_\Omega = -\Omega_m^2\Omega_\Lambda/(1-\Omega_m-\Omega_\Lambda)^3,$$

$$y_1 + y_2 + y_3 = -1,$$
$$y_1 y_2 + y_1 y_3 + y_2 y_3 = 0. \tag{15}$$

The Heun equation is slightly more complicated than the hypergeometric equation; it possesses four regular singular points in the entire complex plane rather than just three. In the form given by (13) one of the two exponents of each finite singular point (y_1, y_2, y_3) vanishes and the other exponent is given respectively by $(1-\gamma, 1-\delta, 1-\epsilon)$.[4] The point at ∞ is the fourth singular point and its exponents are α and β. For the point at ∞ to also be regular (*i.e.,* for this to be a Heun equation) all exponents must sum to a value of 2, equivalently

$$\alpha + \beta + 1 = \gamma + \delta + \epsilon. \tag{16}$$

For (12) this necessary constraint is satisfied. From (15) it follows that at least one root has to be real and complex roots must come in conjugate pairs. For convenience y_1 will be chosen as real throughout. This Heun equation (13) is conveniently expressed in terms of a Riemann P-symbol as:

$$P\left\{\begin{array}{cccccc} y_1 & y_2 & y_3 & \infty & & \\ 0 & 0 & 0 & \alpha & y & q \\ 1-\gamma & 1-\delta & 1-\epsilon & \beta & & \end{array}\right\}. \tag{17}$$

The first 4 columns of (17) are the 4 regular singular points and their 2 exponents, and the 5th column is the independent variable, all analogous to the Riemann P-symbol for the hypergeometric equation. The 6th column is the constant q from the numerator of the coefficient of h in the Heun equation [when put into the standard form of (13)]. The hypergeometric equation is uniquely specified by information about its regular singular

[4]Recall that an exponent gives the analytic behavior of a solution within the neighborhood of a regular singular point, *e.g.,* $h = (y - y_1)^{1-\gamma}(1 + c_1(y - y_1) + \cdots)$.

points but Heun requires the extra parameter q. Because the three finite singular points of this Heun equation have values of $1/2$ for their nonvanishing exponents (12) can be transformed into the Lame′ equation. In this paper, solutions of (12) will be given as local Heun functions and in a following paper they will be expressed as Lame′ functions. When $\Lambda = 0$ ($\Rightarrow b_\Omega = 0$) equation (12) has only three regular singular points *i.e.,* this Heun equation simplifies to an equation of the hypergeometric type. Additionally the corresponding Lame′ equation reduces to the associated Legendre equation. Solutions for $\Lambda = 0$ can be written either as combinations of hypergeometric functions or as associated Legendre functions [see (18) and (B7) below]. To motivate the form of the $\Omega_\Lambda \neq 0$ solution, the $\Omega_\Lambda = 0$ solution will be given first [Kantowski et al. 1995 and Seitz & Schneider 1994],

$$
\begin{aligned}
&D_\ell(\Omega_m = \Omega_0, \Omega_\Lambda = 0, \nu; z) \\
&= \frac{c}{H_0} \frac{1}{(\nu + \frac{1}{2})} \times \\
&\Bigg[\Bigg[(1+\Omega_0 z)^{1+\nu/2} \; {}_2F_1\left(\frac{\nu}{2}+2, \frac{\nu}{2}+\frac{3}{2}; \nu+\frac{3}{2}; 1-\Omega_0\right) {}_2F_1\left(-\frac{\nu}{2}-1, -\frac{\nu}{2}-\frac{1}{2}; \frac{1}{2}-\nu; \frac{1-\Omega_0}{1+\Omega_0 z}\right) \\
- &\frac{(1+z)^2}{(1+\Omega_0 z)^{3/2+\nu/2}} \; {}_2F_1\left(-\frac{\nu}{2}-1, -\frac{\nu}{2}-\frac{1}{2}; \frac{1}{2}-\nu; 1-\Omega_0\right) {}_2F_1\left(\frac{\nu}{2}+2, \frac{\nu}{2}+\frac{3}{2}; \nu+\frac{3}{2}; \frac{1-\Omega_0}{1+\Omega_0 z}\right)\Bigg]\Bigg].
\end{aligned}
\tag{18}
$$

The expected form of the $\Omega_\Lambda \neq 0$ solution follows the above where the hypergeometric functions ${}_2F_1$ are replaced by local Heun functions Hl [see (42) and (47)].

From (15) the three singular points (y_1, y_2, y_3) are chosen from the six permutations of the three roots:

$$
\begin{aligned}
Y_1 &= -\frac{1}{3}\left[1 - \frac{1}{v_+} - v_+\right], \\
Y_2 &= -\frac{1}{3}\left[1 + \frac{1}{v_-} + v_-\right], \\
Y_3 &= -\frac{1}{3}\left[1 + \frac{e^{-i\frac{\pi}{3}}}{v_+} + e^{i\frac{\pi}{3}} v_+\right],
\end{aligned}
\tag{19}
$$

where

$$v_+ \equiv \left[-1+b+\sqrt{b(b-2)}\right]^{\frac{1}{3}},$$
$$v_- \equiv \left[1-b+\sqrt{b(b-2)}\right]^{\frac{1}{3}}, \tag{20}$$

with

$$b \equiv \frac{27}{2} b_\Omega = -\frac{27}{2} \Omega_m^2 \Omega_\Lambda / (1-\Omega_m-\Omega_\Lambda)^3. \tag{21}$$

The locations of the three finite singular points (y_1, y_2, y_3) are determined by the value of the single parameter b, $(-\infty < b < \infty)$. Some important values are shown as contours in Fig. 2.

The standard form for the Heun equation ordinarily has its singularities at $(0, 1, a, \infty)$. The simple linear transformation

$$\zeta = \frac{y-y_1}{y_2-y_1}, \tag{22}$$

moves

$$y_1 \rightarrow 0,$$
$$y_2 \rightarrow 1,$$
$$y_3 \rightarrow a = \frac{y_3-y_1}{y_2-y_1} = \frac{y_1(2+3y_1)}{(y_2-y_1)^2} = \frac{(y_3-y_1)^2}{y_1(2+3y_1)}. \tag{23}$$

The latter two equalities are consequences of the useful identity:

$$(y_2-y_1)(y_3-y_1) = y_1(2+3y_1), \tag{24}$$

which results from (15). In terms of the new variable ζ, (12) becomes

$$\frac{d^2h}{d\zeta^2} + \frac{1}{2}\left(\frac{1}{\zeta} + \frac{1}{\zeta-1} + \frac{1}{\zeta-a}\right)\frac{dh}{d\zeta} + \frac{(-\frac{1}{2}\nu)\frac{1}{2}(\nu+1)\zeta - q}{\zeta(\zeta-1)(\zeta-a)}\, h = 0\ , \tag{25}$$

and the value of q changes to:

$$q = \frac{1+\frac{1}{4}\nu(\nu+1)y_1}{y_2-y_1}. \tag{26}$$

The new Riemann P-symbol is:

$$P\left\{\begin{array}{cccccc} 0 & 1 & a & \infty & & \\ 0 & 0 & 0 & -\frac{\nu}{2} & \zeta & \frac{1+\frac{1}{4}\nu(\nu+1)y_1}{y_2-y_1} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{\nu+1}{2} & & \end{array}\right\}. \tag{27}$$

See the figures in Fig. 3 for locations of a in the complex plane and the trajectories of $\zeta(z)$ starting with ζ_0 (the value of ζ at zero redshift [see (11) and (22)]) for the following three cases:

$$\begin{aligned} b < 0 &\longrightarrow y_1 = \text{real},\ y_2 = \bar{y}_3,\ \text{ and } |a| = 1, \\ 0 \le b \le 2 &\longrightarrow y_1, y_2, y_3, a \text{ are all real}, \\ 2 < b &\longrightarrow y_1 = \text{real},\ y_2 = \bar{y}_3,\ \text{ and } |a| = 1. \end{aligned} \tag{28}$$

Fig. 3 gives the proper choices for the three roots (y_1, y_2, y_3) from the six possible orderings of (Y_1, Y_2, Y_3) in each of the three b domains. It also contains values for a, q, the new variable ζ, and ζ_0. Hyperbolic and trigonometric variables, ξ and ϕ, can be used to parameterize the values of the three roots (rather than b) and they are also given in Fig. 3.

Boundary conditions on h come directly from its definition (11) and the desired boundary conditions on $\sqrt{A}$ [see (6)],

$$\begin{aligned} \sqrt{A}|_s &= 0 \Longrightarrow h(\zeta_s) = 0, \\ \left.\frac{d\sqrt{A}}{dz}\right|_s &= -\sqrt{\delta\Omega}\frac{c}{H_s} \Longrightarrow \left.\frac{dh}{dz}\right|_s = -\frac{c}{H_s}. \end{aligned} \tag{29}$$

Equation (5) then relates D_ℓ to the value of h at the observer,

$$D_\ell(z_s) = (1+z_s)h(z=0). \tag{30}$$

Using these boundary conditions on two independent solutions h_1 & h_2 of (25) gives

$$h(\zeta) = -\frac{h_1(\zeta_s)h_2(\zeta) - h_2(\zeta_s)h_1(\zeta)}{h_1(\zeta_s)\dot{h}_2(\zeta_s) - h_2(\zeta_s)\dot{h}_1(\zeta_s)}\left[\left(\frac{c}{H_s}\right)\Big/\left.\frac{d\zeta}{dz}\right|_{z_s}\right], \tag{31}$$

where $\dot{h} \equiv \frac{dh}{d\zeta}$. From (22) and (11)

$$\begin{aligned}
\zeta &= \frac{y_0(1+z)-y_1}{y_2-y_1}, \\
\zeta_0 &= \frac{y_0-y_1}{y_2-y_1}, \\
\frac{d\zeta}{dz} &= \frac{y_0}{y_2-y_1}.
\end{aligned} \tag{32}$$

The denominator of $h(\zeta)$ in (31) can be evaluated using the Wronskian of (25),

$$h_1(\zeta)\dot{h}_2(\zeta) - h_2(\zeta)\dot{h}_1(\zeta) = (C_W)\frac{1}{\sqrt{\zeta(\zeta-1)(\zeta-a)}}, \tag{33}$$

where C_W is a constant. The square root in this term can be evaluated using

$$\begin{aligned}
\sqrt{\zeta(\zeta-1)(\zeta-a)} &= \frac{\sqrt{y^3+y^2-b_\Omega}}{(y_2-y_1)^{3/2}}, \\
\sqrt{y^3+y^2-b_\Omega} &= (1+z)\frac{y_0^{3/2}}{\sqrt{\Omega_m}}\sqrt{1+\Omega_m z+\Omega_\Lambda[(1+z)^{-2}-1]}.
\end{aligned} \tag{34}$$

With D_ℓ from (30) and H_s from (7), equations (31) and (33) give the desired result:

$$D_\ell(z_s) = -\frac{c(1+z_s)y_0^{1/2}}{H_0\sqrt{\Omega_m}(y_2-y_1)^{1/2}(C_W)}\left[h_1(\zeta_s)h_2(\zeta_0) - h_2(\zeta_s)h_1(\zeta_0)\right]. \tag{35}$$

Fig. 2 shows domains in the $(\Omega_m, \Omega_\Lambda)$ plane separated by $b=2$, $b=\infty$, and $|\zeta_0|=1$. The $b=\infty$ contour is equivalent to $\Omega_0=1$. These contours are important because they separate domains for which different choices of the two independent solutions $h_1(\zeta)$ and $h_2(\zeta)$ must be taken. The $|\zeta_0|=1$ contour divides the $|\zeta_0|<1$ domain where solutions about the singular point $\zeta=0$ are chosen from the $|\zeta_0|>1$ domain where solutions about ∞ are chosen. These choices are necessary for convergence of the local Heun functions. For the $\Omega_\Lambda=0$ case, only expansions about ∞ were required [see (18)].

The Heun Function $Hl(a,q;\alpha,\beta,\gamma,\delta;\zeta)$ is the analytic solution of (39) defined by the infinite series (36), see Ronveaux 1995. It converges in a circle centered on $\zeta=0$ which extends to the nearest singular point 1 or a. This solution is analogous to the ${}_2F_1$ solution of the hypergeometric equation but unfortunately does not appear in any of the common computer libraries. When $c_0 = 1$ is chosen (as will be done here) the series is:

$$Hl(a,q;\alpha,\beta,\gamma,\delta;\zeta) \equiv 1 + \sum_{r=1}^{\infty} c_r \zeta^r, \tag{36}$$

where the c_r are constrained by a three term recursion relation (take $c_{-1} = 0$):

$$P_r c_{r-1} - (Q_r + q)c_r + R_r c_{r+1} = 0, \tag{37}$$

with

$$\begin{aligned} P_r &\equiv (r-1+\alpha)(r-1+\beta), \\ Q_r &\equiv r[(r-1+\gamma)(1+a) + a\delta + \epsilon], \\ R_r &\equiv (r+1)(r+\gamma)a. \end{aligned} \tag{38}$$

The ϵ parameter is not included as an argument in $Hl(a,q;\alpha,\beta,\gamma,\delta;\zeta)$ because of the constraint (16). This series corresponds to the zero exponent for the regular singular point $\zeta = 0$ and will be taken as $h_1(\zeta)$ in (35) when $|\zeta_0| < 1$. The second independent solution is

$$h_2(\zeta) = \zeta^{1-\gamma} Hl(a, q_{II}; \alpha_{II}, \beta_{II}, \gamma_{II}, \delta; \zeta), \tag{39}$$

where four parameters have changed,

$$\begin{aligned} q_{II} &\equiv (a\delta + \epsilon)(1-\gamma) + q, \\ \alpha_{II} &\equiv \alpha + 1 - \gamma, \\ \beta_{II} &\equiv \beta + 1 - \gamma, \\ \gamma_{II} &\equiv 2 - \gamma. \end{aligned} \tag{40}$$

The constant C_W in the Wronskian can be evaluated for the $\zeta \sim 0$ expansion using h_1 and h_2 above,

$$C_W = \frac{1}{2}\sqrt{a}. \tag{41}$$

This gives an expression for the luminosity distance appropriate for $|\zeta_0| < 1$,

$$
\begin{aligned}
&D_\ell(\Omega_m, \Omega_\Lambda, \nu; z) = -\frac{c(1+z)}{H_0 \frac{1}{2}\sqrt{\Omega_m}} \sqrt{\frac{y_0(y_0-y_1)}{y_1(2+3y_1)}} \times \\
&\left[\!\left[Hl\left(a, \frac{1+\frac{1}{4}\nu(\nu+1)y_1}{\sqrt{y_1(2+3y_1)}}\sqrt{a}; -\frac{\nu}{2}, \frac{\nu+1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{y_0(1+z)-y_1}{\sqrt{y_1(2+3y_1)}}\sqrt{a}\right)\right.\right. \\
&\times Hl\left(a, \frac{3+(\nu^2+\nu-3)y_1}{4\sqrt{y_1(2+3y_1)}}\sqrt{a}; -\frac{\nu-1}{2}, \frac{\nu+2}{2}, \frac{3}{2}, \frac{1}{2}; \frac{y_0-y_1}{\sqrt{y_1(2+3y_1)}}\sqrt{a}\right) \\
&-\sqrt{\frac{y_0(1+z)-y_1}{y_0-y_1}} Hl\left(a, \frac{3+(\nu^2+\nu-3)y_1}{4\sqrt{y_1(2+3y_1)}}\sqrt{a}; -\frac{\nu-1}{2}, \frac{\nu+2}{2}, \frac{3}{2}, \frac{1}{2}; \frac{y_0(1+z)-y_1}{\sqrt{y_1(2+3y_1)}}\sqrt{a}\right) \\
&\left.\left.\times Hl\left(a, \frac{1+\frac{1}{4}\nu(\nu+1)y_1}{\sqrt{y_1(2+3y_1)}}\sqrt{a}; -\frac{\nu}{2}, \frac{\nu+1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{y_0-y_1}{\sqrt{y_1(2+3y_1)}}\sqrt{a}\right)\right]\!\right], \qquad (42)
\end{aligned}
$$

where the source redshift z_s has again been replaced by z. The required values of y_1 and a can be found in Fig. 3 for all three b domains and $y_0 = \Omega_m/(1-\Omega_0)$ is the value of y at $z = 0$ [see (11)]. Even though the above Heun functions contain complex arguments and parameters, they are real valued functions of the real redshift varable z. As soon as these functions become available in Mathematica, expressions (42) and (47) will be immediately useful. Untill then, simpler expansions suitable for $|a| = 1$ are indicated in Appendix C.

The solution similar to (42) but suitable for $|\zeta_0| > 1$ is given by choosing

$$h_1(\zeta) = \zeta^{-\alpha} Hl\left(\frac{1}{a}, \underline{q}; \alpha, \underline{\beta}, \underline{\gamma}, \delta; \frac{1}{\zeta}\right), \qquad (43)$$

and

$$h_2(\zeta) = \zeta^{-\beta} Hl\left(\frac{1}{a}, \underline{q}_{II}; \alpha_{II}, \underline{\beta}_{II}, \underline{\gamma}_{II}, \delta; \frac{1}{\zeta}\right), \qquad (44)$$

where seven parameters have now changed:

$$
\begin{aligned}
&\underline{q} \equiv \frac{q}{a} - \alpha\left[\beta\left(1+\frac{1}{a}\right) - \frac{\delta}{a} - \epsilon\right] \\
&\underline{\beta} \equiv -\beta + \delta + \epsilon \\
&\underline{\gamma} \equiv 1 + \alpha - \beta
\end{aligned}
$$

$$
\begin{aligned}
\underline{q}_{II} &\equiv \underline{q} + \left(\frac{\delta}{a} + \epsilon\right)\left(1 - \underline{\gamma}\right), \\
\underline{\alpha}_{II} &\equiv \alpha + 1 - \underline{\gamma}, \\
\underline{\beta}_{II} &\equiv \underline{\beta} + 1 - \underline{\gamma}, \\
\underline{\gamma}_{II} &\equiv 2 - \underline{\gamma}.
\end{aligned}
\tag{45}
$$

For this choice of h_1 and h_2 the constant in the Wronskian (33) becomes

$$
C_W = -\beta + \alpha = -(\nu + \frac{1}{2}). \tag{46}
$$

From (35) the resulting expression for the luminosity distance is

$$
\begin{aligned}
&D_\ell(\Omega_m, \Omega_\Lambda, \nu; z) = \frac{c(1+z)}{H_0(\nu+\frac{1}{2})\sqrt{\Omega_m}\sqrt{(1+z) - y_1/y_0}} \times \\
&\Bigg[\!\Bigg[\left(\frac{y_0(1+z) - y_1}{y_0 - y_1}\right)^{\frac{\nu+1}{2}} \\
&\quad \times Hl\left(\frac{1}{a}, \frac{4 - \nu^2 + \nu(1-2\nu)y_1}{4\sqrt{y_1(2+3y_1)}}\frac{1}{\sqrt{a}}; -\frac{\nu}{2}, \frac{1-\nu}{2}, \frac{1-2\nu}{2}, \frac{1}{2}; \frac{\sqrt{y_1(2+3y_1)}}{y_0(1+z) - y_1}\frac{1}{\sqrt{a}}\right) \\
&\quad \times Hl\left(\frac{1}{a}, \frac{(3+\nu)(1-\nu) - (3+2\nu)(1+\nu)y_1}{4\sqrt{y_1(2+3y_1)}}\frac{1}{\sqrt{a}}; \frac{1+\nu}{2}, \frac{\nu+2}{2}, \frac{3+2\nu}{2}, \frac{1}{2}; \frac{\sqrt{y_1(2+3y_1)}}{y_0 - y_1}\frac{1}{\sqrt{a}}\right) \\
&- \left(\frac{y_0 - y_1}{y_0(1+z) - y_1}\right)^{\frac{\nu}{2}} \\
&\quad \times Hl\left(\frac{1}{a}, \frac{(3+\nu)(1-\nu) - (3+2\nu)(1+\nu)y_1}{4\sqrt{y_1(2+3y_1)}}\frac{1}{\sqrt{a}}; \frac{1+\nu}{2}, \frac{\nu+2}{2}, \frac{3+2\nu}{2}, \frac{1}{2}; \frac{\sqrt{y_1(2+3y_1)}}{y_0(1+z) - y_1}\frac{1}{\sqrt{a}}\right) \\
&\quad \times Hl\left(\frac{1}{a}, \frac{4 - \nu^2 + \nu(1-2\nu)y_1}{4\sqrt{y_1(2+3y_1)}}\frac{1}{\sqrt{a}}; -\frac{\nu}{2}, \frac{1-\nu}{2}, \frac{1-2\nu}{2}, \frac{1}{2}; \frac{\sqrt{y_1(2+3y_1)}}{y_0 - y_1}\frac{1}{\sqrt{a}}\right)\Bigg]\!\Bigg].
\end{aligned}
\tag{47}
$$

The special case of $\Omega_0 \equiv \Omega_m + \Omega_\Lambda = 1$ can be obtained from (42) and (47) by taking the appropriate limits. Some details of this process along with the resulting luminosity distance is given in Appendix A. For those values of Ω_m and Ω_Λ where $|\zeta_0| < 1$ it is clear that for large enough values of z, $|\zeta| > 1$ and hence (42) is no longer valid (Hl no longer converges). For some values in the $(\Omega_m, \Omega_\Lambda)$ plane above the $|\zeta_0| = 1$ contour in Fig. 2, (42) will not converge for a SNe Ia range of $z \sim 0.5$, but for most values it does.

In the next section several plots of magnitude vs. redshift are made to illustrate the importantce of take clumping into account when attempting to determine Ω_m and Ω_Λ.

4. m-z Plots for Clumpy Universes & Conclusions

In this section several magnitude-redshift plots are given to illustrate the effects that density clumps can have on the m-z relation and consequently on a determination of Ω_m and Λ made by using this relation. Because m-z depends differently on Ω_m and Ω_Λ as a function of redshift for the FLRW models, both parameters could in principle be determined from a sufficient quantity of accurate SNe Ia data. Clumping provides an additional parameter ν which complicates any such determination. As can be seen from (8) the dependence of m-z on this additional parameter could also be determined by enough data. However, such a triple determination is certainly more complicated. What will be done here to illustrate the effects of the ν parameter is to plot multiple m-z curves for various values of all three parameters ν, Ω_m, and Ω_Λ. In all plots the unit of distance is taken to be c/H_0. In these figures D_ℓ is plotted on a magnitude scale, 5 Log D_ℓ (*i.e.,* the distance modulus plus 5 Log $10pc\,H_0/c$). In Fig. 4, Ω_Λ is held fixed while ν and Ω_m are varied and in Fig. 5, Ω_m is held fixed while ν and Ω_Λ are varied. In Fig. 6, $\Omega_0 = \Omega_m + \Omega_\Lambda = 1$ is fixed while all three parameters vary.

In Fig. 7 the sensitivity of observed magnitudes to variations of Ω_m is illustrated by fixing $z = 0.83$ and $\Omega_\Lambda = 0.1$. In Fig. 8 a similar plot is given showing the sensitivity to variations of Ω_Λ. The importance of the clumping parameter is easily seen from these last two figures. If the distance modulus of a source such as SN 1997ap at $z = 0.83$ were precisely known (*e.g.,* see the two sample horizontal lines in Fig. 7) then a determination of Ω_m could be made, assuming Ω_Λ were somehow known. Likewise, from Fig. 8, a determination of Ω_Λ could be made if Ω_m were somehow known. From Fig. 7 the reader can easily see that the determined value of Ω_m depends on the clumping parameter ν. The Ω_m value will be about 95% larger for a $\nu = 2$ completely clumpy universe than it will be for a $\nu = 0$ completely smooth FLRW universe. Equivalently, Ω_m could be underestimated by as much as 50% if the FLRW is used. The maximum underestimate is reduced to 33% at the smaller redshift of $z = 0.5$ (see a similar result for $\Omega_\Lambda = 0$ in Kantowski et al. 1995).

Slightly different conclusions follow from Fig. 8 about Ω_Λ. The discrepancy in the determined value of Ω_Λ is $\Delta\Omega_\Lambda \sim -0.14$ for $\nu = 2$ compared to $\nu = 0$, and is not sensitive to the distance modulus. The discrepancy is halved, $\Delta\Omega_\Lambda \sim -0.07$, at a smaller redshift of $z = 0.5$.

A minimal estimate of the quantity of data required to begin distinguishing between the various ν values can be easily be made. At $z = 0.5$ the differences in observed magnitudes of a SN Ia in a $\nu = 0$ (100% smooth FLRW) and a $\nu = 2$ (100% clumpy) universe is about $\Delta m \sim 0.02$ if $\Omega_m \sim 0.2$, and $\Delta m \sim 0.09$ if $\Omega_m \sim 0.8$. These differences are not sensitive to Λ. With corrected-intrinsic and observed magnitude uncertainties of ± 0.2, Branch 1998, data on over 200 SNe Ia will be required if we live in a light universe and over a dozen if we live in a heavy one.

The results presented here (42),(47),(A2), and (A3) for the 'most likely' distance-redshift relation are quite general. They contain corrections (for mass inhomogeneities) to the standard FLRW result, applicable to observations where gravitational lensing can be neglected, *i.e.,* observations where the conformal (Weyl) curvature doesn't produce significant average shear in (1). Even though the original area equation (1) was rigorously established for a particular type of Swiss cheese model, the resulting equation which neglects lensing (10) is expected to be widely applicable to observations at redshifts of $z = 1$ and less. Application of its solution to a given set of observations requires that the average fraction of the mass density contained in the observing beams (*i.e.,* the ν parameter) be determined. This fraction obviously depends on the number as well as the type of object observed. Collecting CMB radiation at wide angles is likely to produce a $\nu = 0$ value but observing a few dozen SNe Ia might well result in a value close to $\nu = 2$ (*i.e.,* we might in fact live in a universe where mass, dark or otherwise, is primarily associated with galaxies). If a significant fraction of the universe's mass density is clumped on galactic scales, then the effects of these clumps on SNe Ia observations should be taken into account by using the lensing formulas rather than by decreasing ν to zero. Recent numerical work by Holz & Wald 1997 confirms the assertion that, given galaxy clumping, the cross sectional area of a typical light beam will not follow the FLRW area-redshift relation. Instead the area will follow more closely one of the m-z relations given here, until a lensing event occurs. The new luminosity distances presented here and those in Kantowski et al. 1995 represent the theoretical 'mode' of the observed magnitudes (*i.e.,* the most probable) and are applicable to situations where lensing is infrequent. Because Holz & Wald 1997 did not include any diffuse transparent matter, the applicable 'most likely' m-z relations given here are those with $\nu = 2$. For $\Lambda = 0$ and $\Omega_0 < 1$ it is Dyer-Roder solution (B3) and for $\Lambda = 0$ and $\Omega_0 = 1$ its the $\nu = 2$ solution of Dashevskii & Slysh (B2).

The $\nu = 0$ (standard FLRW) result represents the theoretical 'mean' for m-z for a universe in which only weak-lensing events occur. For extremely non-symmetric probability distributions, the "mean" is not likely the best estimator - in this case the "most probable" is likely better, Sivia 1996.

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A. The Special Case: $\Omega_0 = \Omega_m + \Omega_\Lambda = 1$

A complete derivation of the $\Omega_0 = \Omega_m + \Omega_\Lambda = 1$ case can be done by introducing a new independent variable in (12), $y \to -y/\Delta$ where $\Delta \equiv \Omega_m + \Omega_\Lambda - 1$ and then taking the limit $\Delta \to 0$. The resulting differential equation which replaces (12) has the same exponents given in (14) but has $q = 0$. The three finite regular singular points are now located at $(y_1, y_2, y_3) = (\Omega_m^2\Omega_\Lambda)^{1/3}(-1, e^{-i\pi/3}, e^{i\pi/3})$. When the modified (12) equation is transformed by (22) a modified (25) results which is described by the Riemann P-symbol:

$$P\left\{\begin{array}{cccccc} 0 & 1 & e^{i\pi/3} & \infty & & \\ 0 & 0 & 0 & -\frac{\nu}{2} & \left[\left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{\frac{1}{3}}(1+z)+1\right]\frac{e^{i\pi/6}}{\sqrt{3}}, & -\frac{\nu(\nu+1)}{4\sqrt{3}}e^{i\pi/6} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{\nu+1}{2} & & \end{array}\right\}. \tag{A1}$$

The solutions to this simpler Heun equation with boundary conditions appropriate for luminosity distance D_ℓ can be obtained directly from (42) and (47) by simply substituting $y_0 = \Omega_m/\Delta,\ y_1 = -(\Omega_m^2\Omega_\Lambda)^{1/3}/\Delta, a = e^{i\pi/3}$, and then taking the limit $\Delta \to 0$. This gives an expression for the luminosity distance, appropriate for $|\zeta_0| < 1$,

$$\begin{aligned} D_\ell(\Omega_m, \Omega_\Lambda, \nu; z) = -&\frac{c(1+z)\sqrt{1+(\frac{\Omega_\Lambda}{\Omega_m})^{\frac{1}{3}}}}{H_0\frac{1}{2}\sqrt{3}\Omega_m^{\frac{1}{6}}\Omega_\Lambda^{\frac{1}{3}}} \times \\ \Bigg[\Bigg[&Hl\left(e^{i\frac{\pi}{3}}, -\frac{\nu(\nu+1)}{4\sqrt{3}}e^{i\pi/6}; -\frac{\nu}{2}, \frac{\nu+1}{2}, \frac{1}{2}, \frac{1}{2}; \left[\left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{\frac{1}{3}}(1+z)+1\right]\frac{e^{i\pi/6}}{\sqrt{3}}\right) \\ \times &Hl\left(e^{i\frac{\pi}{3}}, -\frac{(\nu^2+\nu-3)}{4\sqrt{3}}e^{i\pi/6}; -\frac{\nu-1}{2}, \frac{\nu+2}{2}, \frac{3}{2}, \frac{1}{2}; \left[\left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{\frac{1}{3}}+1\right]\frac{e^{i\pi/6}}{\sqrt{3}}\right) \\ -&\sqrt{\frac{(1+z)+(\frac{\Omega_\Lambda}{\Omega_m})^{\frac{1}{3}}}{1+(\frac{\Omega_\Lambda}{\Omega_m})^{\frac{1}{3}}}} \end{aligned}$$

$$\times Hl\left(e^{i\frac{\pi}{3}},-\frac{(\nu^2+\nu-3)}{4\sqrt{3}}e^{i\pi/6};-\frac{\nu-1}{2},\frac{\nu+2}{2},\frac{3}{2},\frac{1}{2};\left[\left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{\frac{1}{3}}(1+z)+1\right]\frac{e^{i\pi/6}}{\sqrt{3}}\right)$$
$$\times Hl\left(e^{i\frac{\pi}{3}},-\frac{\nu(\nu+1)}{4\sqrt{3}}e^{i\pi/6};-\frac{\nu}{2},\frac{\nu+1}{2},\frac{1}{2},\frac{1}{2};\left[\left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{\frac{1}{3}}+1\right]\frac{e^{i\pi/6}}{\sqrt{3}}\right)\Bigg]\Bigg], \qquad \text{(A2)}$$

The expression for the luminosity distance appropriate for $|\zeta_0|>1$ is obtained by applying the limiting proceedure to (47),

$$D_\ell(\Omega_m,\Omega_\Lambda,\nu;z)=\frac{c(1+z)}{H_0(\nu+\frac{1}{2})\sqrt{\Omega_m}\sqrt{(1+z)+(\frac{\Omega_\Lambda}{\Omega_m})^{\frac{1}{3}}}}\times$$
$$\Bigg[\Bigg[\left(\frac{(1+z)+(\frac{\Omega_\Lambda}{\Omega_m})^{\frac{1}{3}}}{1+(\frac{\Omega_\Lambda}{\Omega_m})^{\frac{1}{3}}}\right)^{\frac{\nu+1}{2}}$$
$$\times Hl\left(e^{-i\frac{\pi}{3}},\frac{\nu(2\nu-1)}{4\sqrt{3}}e^{-i\pi/6};-\frac{\nu}{2},\frac{1-\nu}{2},\frac{1-2\nu}{2},\frac{1}{2};\sqrt{3}\left[\left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{\frac{1}{3}}(1+z)+1\right]^{-1}e^{-i\pi/6}\right)$$
$$\times Hl\left(e^{-i\frac{\pi}{3}},\frac{(\nu+1)(2\nu+3)}{4\sqrt{3}}e^{-i\pi/6};\frac{1+\nu}{2},\frac{\nu+2}{2},\frac{3+2\nu}{2},\frac{1}{2};\sqrt{3}\left[\left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{\frac{1}{3}}+1\right]^{-1}e^{-i\pi/6}\right)$$
$$-\left(\frac{1+(\frac{\Omega_\Lambda}{\Omega_m})^{\frac{1}{3}}}{(1+z)+(\frac{\Omega_\Lambda}{\Omega_m})^{\frac{1}{3}}}\right)^{\frac{\nu}{2}}$$
$$\times Hl\left(e^{-i\frac{\pi}{3}},\frac{(\nu+1)(2\nu+3)}{4\sqrt{3}}e^{-i\pi/6};\frac{1+\nu}{2},\frac{\nu+2}{2},\frac{3+2\nu}{2},\frac{1}{2};\sqrt{3}\left[\left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{\frac{1}{3}}(1+z)+1\right]^{-1}e^{-i\pi/6}\right)$$
$$\times Hl\left(e^{-i\frac{\pi}{3}},\frac{\nu(2\nu-1)}{4\sqrt{3}}e^{-i\pi/6};-\frac{\nu}{2},\frac{1-\nu}{2},\frac{1-2\nu}{2},\frac{1}{2};\sqrt{3}\left[\left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{\frac{1}{3}}+1\right]^{-1}e^{-i\pi/6}\right)\Bigg]\Bigg]. \qquad \text{(A3)}$$

These expressions were used to produce Fig. 6 of §4. The $\Omega_\Lambda \to 0$ limit of (A3) results in the solution (B2) below, first given by Dashevskii & Slysh 1966.

B. Previously Known Solutions For $D_\ell(z)$ when $B_0 = 0$

Until now, analytic solutions to the average area equation (1), neglecting lensing (*i.e.*, putting $\xi^2 = 0$), have been found only for $\Lambda = 0$. The earliest solutions were written down before the equation was formulated by Kantowski 1969. The standard homogeneous FRW solution was given by Mattig 1958. It is the $\rho_I = 0$ (*i.e.*, $\nu = 0$) solution of (18) and (B7),

$$D_\ell(\Omega_m = \Omega_0, \Omega_\Lambda = 0, \nu = 0; z) = \frac{2c}{H_0\Omega_0^2}\left\{\Omega_0 z + (\Omega_0 - 2)\left(\sqrt{1+\Omega_0 z} - 1\right)\right\}. \qquad \text{(B1)}$$

The $\Omega_0 = 1$ solution was given by Dashevskii & Slysh 1966,

$$D_\ell(\Omega_m = 1, \Omega_\Lambda = 0, \nu; z) = \frac{c}{H_0}\frac{1}{(\nu + \frac{1}{2})}\left[(1+z)^{\frac{\nu}{2}+1} - (1+z)^{-\frac{\nu}{2}+\frac{1}{2}}\right]. \qquad \text{(B2)}$$

The $\nu = 2$ (*i.e.*, $\rho_D = 0$) solution is due to Dyer & Roeder 1972 :

$$\begin{aligned} &D_\ell(\Omega_m, \Omega_\Lambda = 0, \nu = 2; z) \\ &= \frac{c}{H_0}\frac{\Omega_0(1+z)^2}{4(1-\Omega_0)^{3/2}}\left[\frac{3\Omega_0}{2(1-\Omega_0)}\ln\left\{\left(\frac{1+\sqrt{1-\Omega_0}}{1-\sqrt{1-\Omega_0}}\right)\left(\frac{\sqrt{1+\Omega_0 z} - \sqrt{1-\Omega_0}}{\sqrt{1+\Omega_0 z} + \sqrt{1-\Omega_0}}\right)\right\}\right. \\ &\left. + \frac{3}{\sqrt{1-\Omega_0}}\left(\frac{\sqrt{1+\Omega_0 z}}{1+z} - 1\right) + \frac{2\sqrt{1-\Omega_0}}{\Omega_0}\left(1 - \frac{\sqrt{1+\Omega_0 z}}{(1+z)^2}\right)\right]. \end{aligned} \qquad \text{(B3)}$$

This result can be rewritten using the identity

$$\sinh^{-1}\sqrt{\frac{1-\Omega_0}{\Omega_0(1+z)}} = \frac{1}{2}\ln\left(\frac{\sqrt{1+\Omega_0 z} + \sqrt{1-\Omega_0}}{\sqrt{1+\Omega_0 z} - \sqrt{1-\Omega_0}}\right), \qquad \text{(B4)}$$

in the form actually given by Dyer & Roeder 1972. When $\Omega_0 > 1$ equation (B3) is analytically continued using $\sqrt{1-\Omega_0} \longrightarrow \pm i\sqrt{\Omega_0 - 1}$, which simplifies by using, $\sinh^{-1}(ix) = i\sin^{-1}(x)$ to give a form containing only real variables,

$$\begin{aligned} &D_\ell(\Omega_m, \Omega_\Lambda = 0, \nu = 2; z) \\ &= \frac{c}{H_0}\frac{\Omega_0(1+z)^2}{(\Omega_0 - 1)^{3/2}}\left[\frac{3\Omega_0}{\Omega_0 - 1}\left\{\sin^{-1}\sqrt{\frac{2\Omega_0 - 1}{2\Omega_0}} - \sin^{-1}\sqrt{\frac{2\Omega_0 - 1}{2\Omega_0(1+z)}}\right\}\right. \\ &\left. + \frac{3}{2\sqrt{\Omega_0 - 1}}\left(\frac{\sqrt{1+\Omega_0 z}}{1+z} - 1\right) - \frac{\sqrt{\Omega_0 - 1}}{\Omega_0}\left(1 - \frac{\sqrt{1+\Omega_0 z}}{(1+z)^2}\right)\right]. \end{aligned} \qquad \text{(B5)}$$

The above solution was the first analytic solution to the $\Lambda = 0$ version of (1) and cited Kantowski 1969 as the origin of the equation. This solution represents the most extreme effects that clumps perhaps have on the standard Mattig result (B1) and is therefore one of the more interesting and useful solutions. For this solution no matter converges the light beam. The following solution (B6) appeared in Dyer & Roeder 1973 and after the above solution but did not cite Kantowski 1969. Instead, the derivation was essentially repeated neglecting lensing (*i.e.,* putting the shear term to zero) and assuming, without justification, that the FRW relation between redshift and affine parameter was valid. Many authors have subsequently dubbed (1) with $B_0 = 0$ as the Dyer-Roder equation. Besides making frequent and successful use of the equation, their contribution to its development was to give two special solutions to it and to extend the equation to include $\Lambda \neq 0$. If anyone deserves to have their name attached to a curved-space optics equation it's R. K. Sachs. Equations such as (1) are direct applications of Sachs 1961. The $\nu = 1$ (*i.e.,* $\rho_D = 2\rho_I$) solution of (1) that appeared in (Dyer & Roeder 1973) was:

$$D_\ell(\Omega_m, \Omega_\Lambda = 0, \nu = 1; z) = \frac{4}{3\Omega_0^2}\left[\frac{3}{2}\Omega_0 - 1 + \frac{1}{2}\Omega_0 z\sqrt{1+\Omega_0 z} - \frac{3}{2}\Omega_0 - 1\right]. \tag{B6}$$

The general $\Lambda = 0$ solution (18) was only recently obtained and can be written using associated Legendre functions (Kantowski et al. 1995 and Seitz & Schneider 1994) as

$$D_\ell(\Omega_m, \Omega_\Lambda = 0, \nu; z) = \frac{c}{H_0}\frac{1}{\sqrt{1-\Omega_0}}\frac{2(1+z)}{(\nu+2)(\nu+1)(\nu)(\nu-1)} \times$$
$$\left[Q_\nu^2\left(\frac{1}{\sqrt{1-\Omega_0}}\right)P_\nu^2\left(\frac{\sqrt{1+\Omega_0 z}}{\sqrt{1-\Omega_0}}\right) - P_\nu^2\left(\frac{1}{\sqrt{1-\Omega_0}}\right)Q_\nu^2\left(\frac{\sqrt{1+\Omega_0 z}}{\sqrt{1-\Omega_0}}\right)\right]. \tag{B7}$$

The general $\Lambda \neq 0$ solutions (42),(47), (A2),and (A3) take on a form similar to this except the associated Legendre functions P_ν^2 & Q_ν^2 are replaced by Lame$'$ functions.

C. A Simplification for Hl and Mathematica Code for $D_\ell(\Omega_m, \Omega_\Lambda, \nu; z)$

Because the Huen functions are not yet available as standard computer routines, some programming skills are required if expressions (42),(47),(A2), and (A3) are to be made used of. For those interested in computing the Heun functions used in these expressions the modified form of (36) used to produce Fig's. 4-8 is given and for those simply interested in plots of the luminosity distance, a few lines of Mathematica code are also given.

For most points in the Ω_m-Ω_Λ plane of Fig. 2 Hl as given by (36) contains complex coefficients C_r as well as a complex variable ζ. Because the needed Hl are in fact real functions of a real variable z it is convenient to modify this series to make its coeffecients real. The modification is slightly different for expansions about $\zeta = 0$ and $\zeta = \infty$. For the expansions about $\zeta = 0$ a convenient coefficient to iterate is $\hat{C}_r$ defined by:

$$C_r \zeta^r = \frac{\hat{C}_r}{r!(\gamma)_r 2^{2r}[(y_3 - y_1)(y_2 - y_1)]}(y - y_1)^r, \tag{C1}$$

and for the expansion about ∞,

$$C_r \left(\frac{1}{\zeta}\right)^r = \frac{\hat{C}_r}{r!(\gamma)_r 2^{2r}} \left(\frac{1}{y - y_1}\right)^r. \tag{C2}$$

The new $\hat{C}_r$ are real polynomials and hence easily evaluated and simplified.

For those who want only to plot m-z relations a Mathematica routine which numerically integrates (12) with boundary conditions (6) appropriate for obtaining $D_\ell(\Omega_m, \Omega_\Lambda, \nu; z)$ follows. As input the program requires values for Om, Ol, and nu which stand for Ω_m, Ω_Λ, and ν respectively.

```
y0= Om/(1−Om−Ol)
bO= −Om^2*Ol/(1−Om−Ol)^3
P= (1+3/2y)y/(y^3+y^2−bO)
Q= −(1+nu*(nu+1)*y/4)/(y^3+y^2−bO)
distance= NDSolve[{d''[y]+P*d'[y]+Q*d[y] == 0,
        d[y0]==0, d'[y0]==1/y0}, d, {y, y0, 2*y0}]
Plot[Evaluate[(1+z)*d[y0*(1+z)] /. distance], {z, 0, 1}]
```

This routine was used to check for errors in (42),(47),(A2), and (A3).

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Fig. 1.— Radiation beam of cross sectional area A propagating through a Swiss cheese universe from distant source to observer.

Fig. 2.— The Ω_m-Ω_Λ plane and various domains required for the needed Heun functions.

Fig. 3.— This table/figure contains various parameters needed for evaluating the luminosity distance $D_\ell(\Omega_m, \Omega_\Lambda, \nu; z)$ as given by (42) and (47). All parameters are functions of the cosmic parameters Ω_m, Ω_Λ, and the clumping parameter ν. Three columns are given for the three domains of the parameter b separated by contours in Fig. 2. The figure in each respective column gives the locations of the regular singular point a in the complex ζ-plane, the orbit of ζ in (32) as a function of redshift z, including the starting point ζ_0. Domains in the Ω_m-Ω_Λ plane that correspond to $|\zeta_0| < 1$ and $|\zeta_0| > 1$ are shown in the small reproductions of Fig. 2 included in each column.

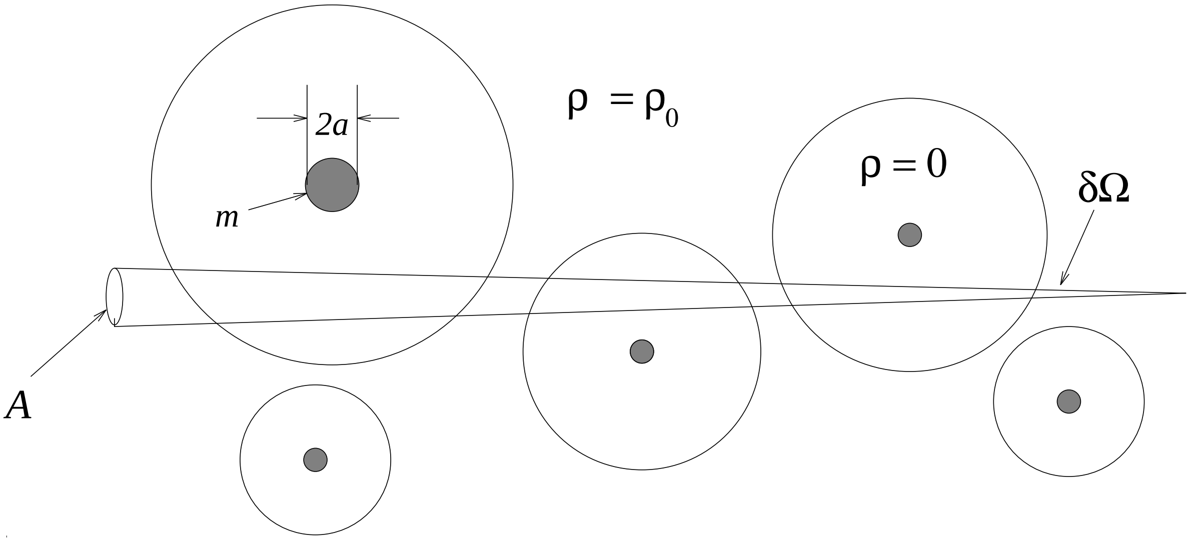
Fig. 4.— Magnitude-redshift relation, $5\log_{10}\frac{H_0}{c}D_\ell(\Omega_m, \Omega_\Lambda = 0.1, \nu; z)$, as a function of redshift z for three values of Ω_m and three values of ν.

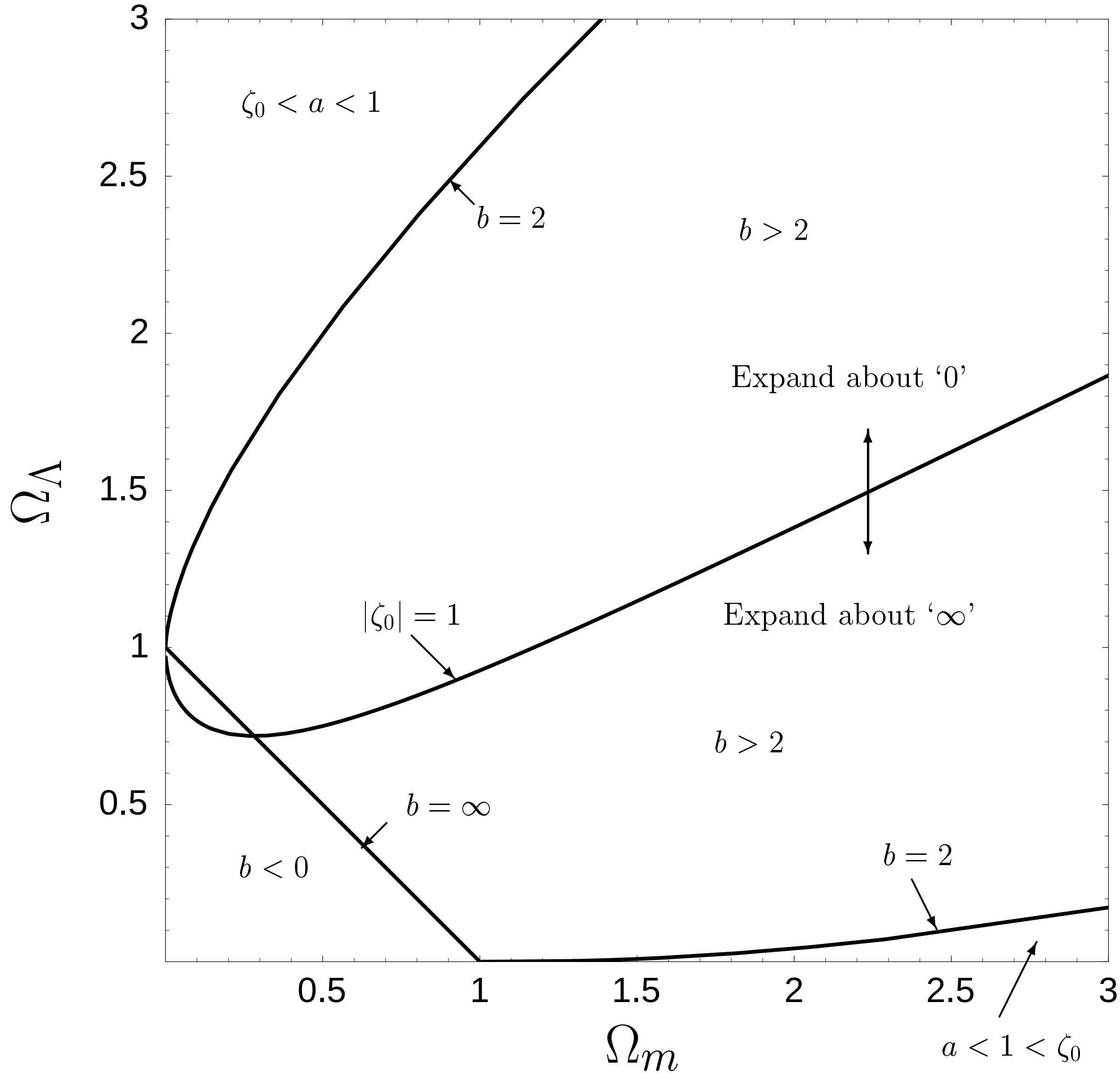
Fig. 5.— Magnitude-redshift relation, $5\log_{10}\frac{H_0}{c}D_\ell(\Omega_m = 0.2, \Omega_\Lambda, \nu; z)$, as a function of redshift z for three values of Ω_Λ and three values of ν.

Fig. 6.— Magnitude-redshift relation, $5\log_{10}\frac{H_0}{c}D_\ell(\Omega_m, \Omega_\Lambda = 1 - \Omega_m, \nu; z)$, as a function of redshift z for three values of Ω_m and Ω_Λ ($\Omega_0 \equiv \Omega_m + \Omega_\Lambda = 1$) and three values of ν.

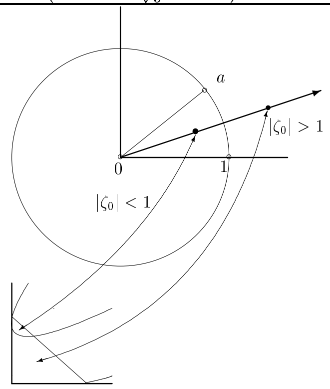
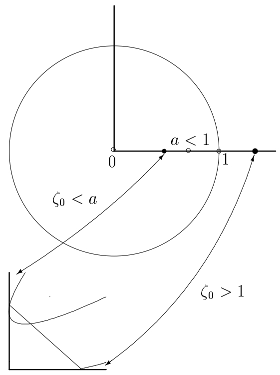
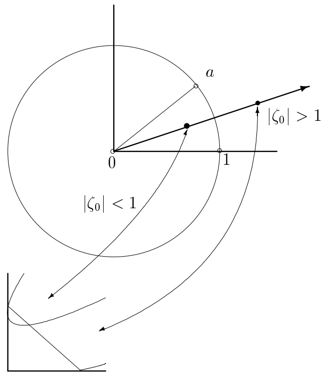
Fig. 7.— Magnitude, $5\log_{10}\frac{H_0}{c}D_\ell(\Omega_m, \Omega_\Lambda = 0.1, \nu; z = 0.83)$, as a function of Ω_m for three values of ν.

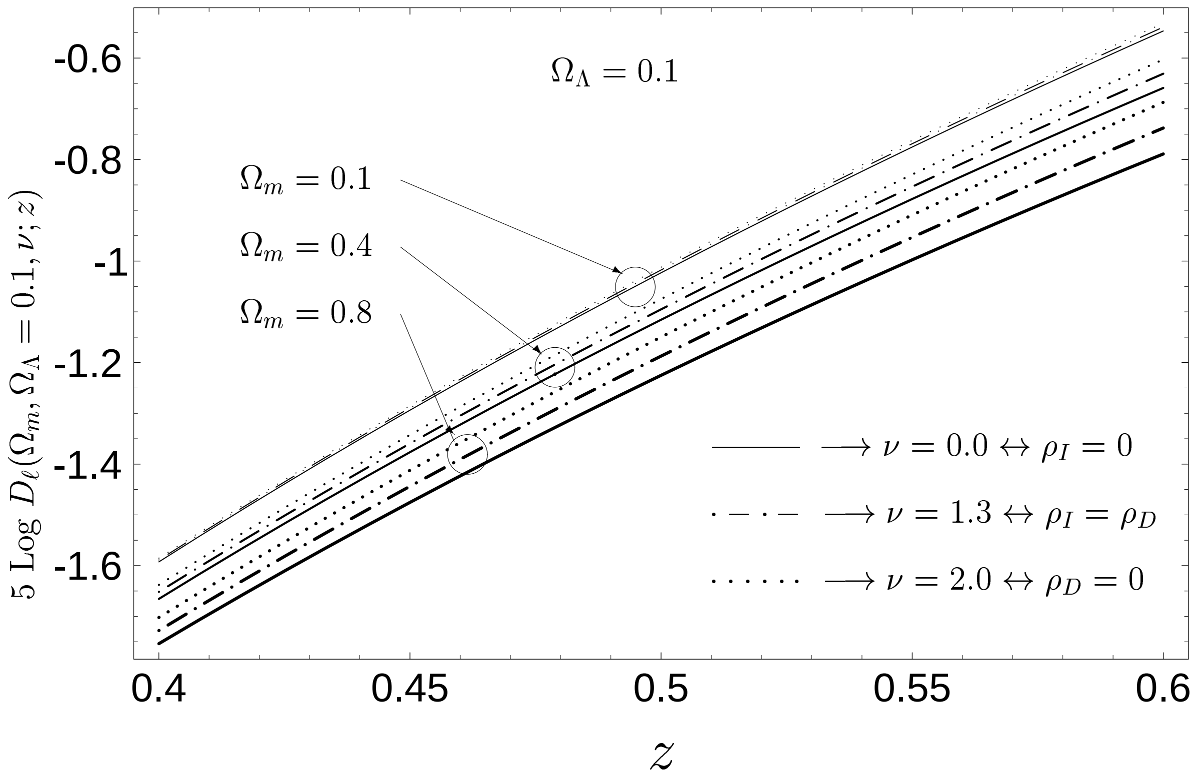
Fig. 8.— Magnitude, $5\log_{10}\frac{H_0}{c}D_\ell(\Omega_m = 0.2, \Omega_\Lambda, \nu; z = 0.83)$, as a function of Ω_Λ for three values of ν.

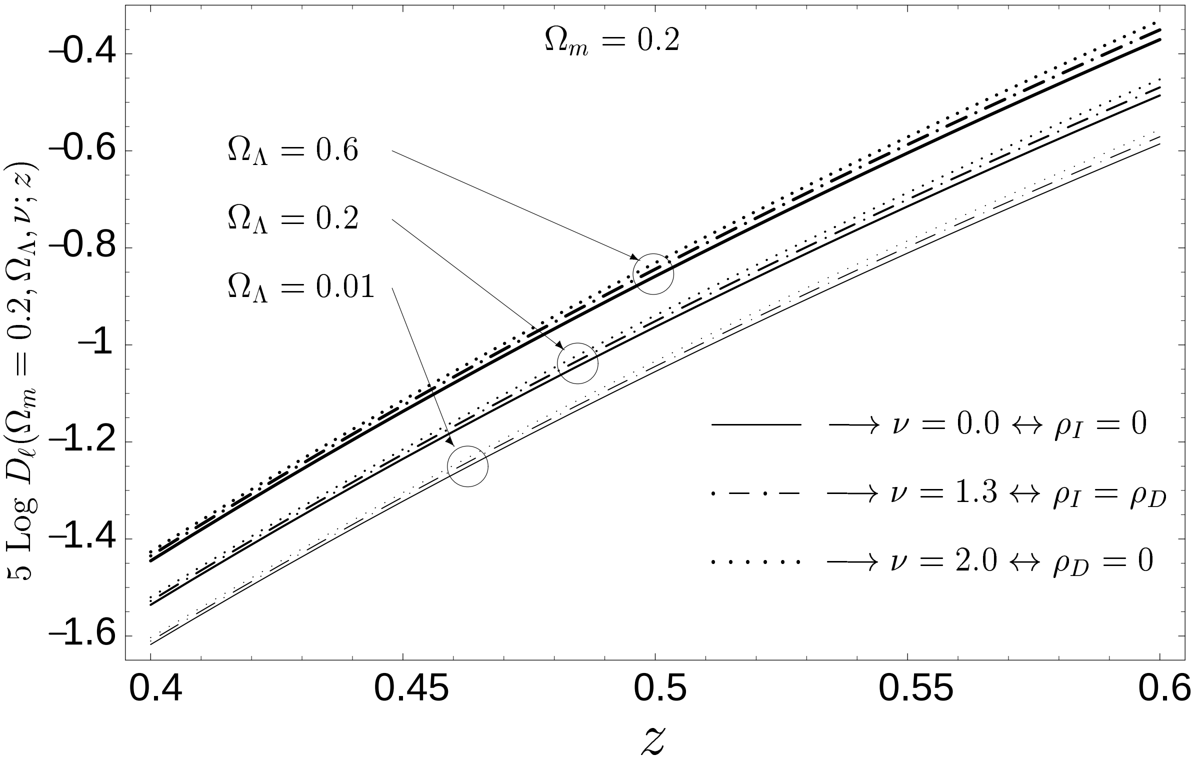

2a
m
ρ = ρ0
ρ = 0
δΩ
A



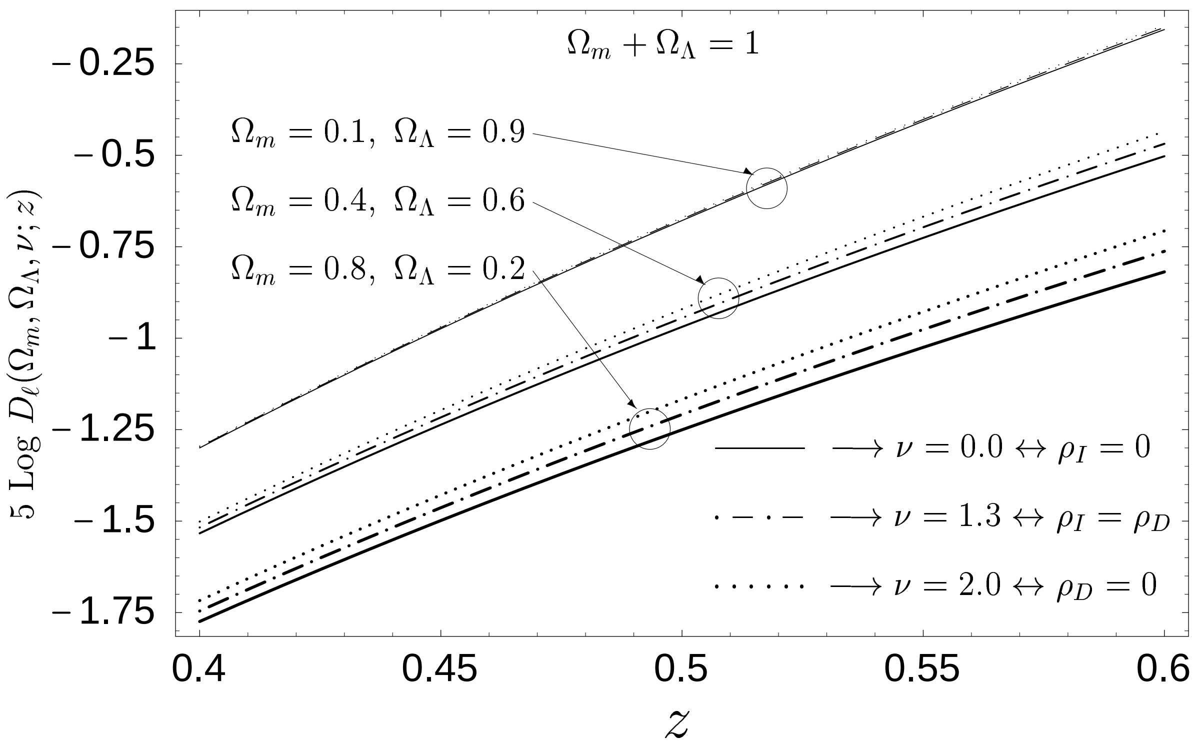
$\zeta_0 < a < 1$
$b = 2$
$b > 2$
Expand about '0'
$|\zeta_0| = 1$
Expand about '∞'
$b > 2$
$b = \infty$
$b < 0$
$b = 2$
$a < 1 < \zeta_0$
Ω_Λ
Ω_m
0.5
1
1.5
2
2.5
3

	$b \equiv -(27/2)\Omega_m^2\Omega_\Lambda/(1-\Omega_m-\Omega_\Lambda)$		
	$b \le 0$	$0 \le b \le 2$	$2 \le b$
	useful parameters for finding singular points		
	$\cosh\xi^- \equiv 1-b$ $v_- \equiv \left[1-b+\sqrt{b(b-2)}\right]^{\frac{1}{3}}$ $\Rightarrow \cosh\frac{\xi^-}{3} = \frac{1}{2}\left(v_- + v_-^{-1}\right)$	$\cos\phi \equiv 1-b$ $\Rightarrow \cos\frac{\phi}{3} = \cos\left[\frac{1}{3}\cos^{-1}(1-b)\right]$	$\cosh\xi^+ \equiv b-1$ $v_+ \equiv \left[-1+b+\sqrt{b(b-2)}\right]^{\frac{1}{3}}$ $\Rightarrow \cosh\frac{\xi^+}{3} = \frac{1}{2}\left(v_+ + v_+^{-1}\right)$
	good choices for order of roots		
y_1	$Y_2 = -\frac{1}{3} - \frac{2}{3}\cosh\frac{\xi^-}{3}$	$Y_1 = -\frac{1}{3} + \frac{1}{3}\cos\frac{\phi}{3} + \frac{1}{\sqrt{3}}\sin\frac{\phi}{3}$	$Y_1 = -\frac{1}{3} + \frac{2}{3}\cosh\frac{\xi^+}{3}$
y_2	$Y_1 = -\frac{1}{3} + \frac{1}{3}\cosh\frac{\xi^-}{3} - \frac{i}{\sqrt{3}}\sinh\frac{\xi^-}{3}$	$Y_2 = -\frac{1}{3} - \frac{2}{3}\cos\frac{\phi}{3}$	$Y_2 = -\frac{1}{3} - \frac{1}{3}\cosh\frac{\xi^+}{3} + \frac{i}{\sqrt{3}}\sinh\frac{\xi^+}{3}$
y_3	$Y_3 = -\frac{1}{3} + \frac{1}{3}\cosh\frac{\xi^-}{3} + \frac{i}{\sqrt{3}}\sinh\frac{\xi^-}{3}$	$Y_3 = -\frac{1}{3} + \frac{1}{3}\cos\frac{\phi}{3} - \frac{1}{\sqrt{3}}\sin\frac{\phi}{3}$	$Y_3 = -\frac{1}{3} - \frac{1}{3}\cosh\frac{\xi^+}{3} - \frac{i}{\sqrt{3}}\sinh\frac{\xi^+}{3}$
	parameters for the new variable $\zeta = (y-y_1)/(y_2-y_1)$		
	$y = y_0(1+z)$ & $y_0 \equiv \Omega_m/(1-\Omega_m-\Omega_\Lambda)$		
ζ_0	$(y_0-Y_2)/(Y_1-Y_2) =$ $\frac{\left(y_0+\frac{1}{3}+\frac{2}{3}\cosh\frac{\xi^-}{3}\right)}{\left(\cosh\frac{\xi^-}{3}-\frac{i}{\sqrt{3}}\sinh\frac{\xi^-}{3}\right)}$	$(y_0-Y_1)/(Y_2-Y_1) =$ $\frac{\left(-y_0-\frac{1}{3}+\frac{1}{3}\cos\frac{\phi}{3}+\frac{1}{\sqrt{3}}\sin\frac{\phi}{3}\right)}{\left(\cos\frac{\phi}{3}+\frac{1}{\sqrt{3}}\sin\frac{\phi}{3}\right)}$	$(y_0-Y_1)/(Y_2-Y_1) =$ $\frac{\left(-y_0-\frac{1}{3}+\frac{2}{3}\cosh\frac{\xi^+}{3}\right)}{\left(\cosh\frac{\xi^+}{3}-\frac{i}{\sqrt{3}}\sinh\frac{\xi^+}{3}\right)}$
a	$(Y_3-Y_2)\,/\,(Y_1-Y_2) =$ $\frac{\left(\cosh\frac{\xi^-}{3}+\frac{i}{\sqrt{3}}\sinh\frac{\xi^-}{3}\right)}{\left(\cosh\frac{\xi^-}{3}-\frac{i}{\sqrt{3}}\sinh\frac{\xi^-}{3}\right)}$	$0 \le (Y_3-Y_1)\,/\,(Y_2-Y_1) =$ $\frac{\left(\frac{2}{\sqrt{3}}\sin\frac{\phi}{3}\right)}{\left(\cos\frac{\phi}{3}+\frac{1}{\sqrt{3}}\sin\frac{\phi}{3}\right)} \le 1$	$(Y_3-Y_1)\,/\,(Y_2-Y_1) =$ $\frac{\left(\cosh\frac{\xi^+}{3}+\frac{i}{\sqrt{3}}\sinh\frac{\xi^+}{3}\right)}{\left(\cosh\frac{\xi^+}{3}-\frac{i}{\sqrt{3}}\sinh\frac{\xi^+}{3}\right)}$
q	$\left(1+\frac{1}{4}\nu(\nu+1)Y_2\right)\,/\,(Y_1-Y_2) =$ $\frac{1+\frac{1}{4}\nu(\nu+1)\left(-\frac{1}{3}-\frac{2}{3}\cosh\frac{\xi^-}{3}\right)}{\left(\cosh\frac{\xi^-}{3}-\frac{i}{\sqrt{3}}\sinh\frac{\xi^-}{3}\right)}$	$\left(1+\frac{1}{4}\nu(\nu+1)Y_1\right)\,/\,(Y_2-Y_1) =$ $\frac{1+\frac{1}{4}\nu(\nu+1)\left(\frac{1}{3}-\frac{1}{3}\cos\frac{\phi}{3}-\frac{1}{\sqrt{3}}\sin\frac{\phi}{3}\right)}{\left(\cos\frac{\phi}{3}+\frac{1}{\sqrt{3}}\sin\frac{\phi}{3}\right)}$	$\left(1+\frac{1}{4}\nu(\nu+1)Y_1\right)\,/\,(Y_2-Y_1) =$ $\frac{1+\frac{1}{4}\nu(\nu+1)\left(\frac{1}{3}-\frac{2}{3}\cosh\frac{\xi^+}{3}\right)}{\left(\cosh\frac{\xi^+}{3}-\frac{i}{\sqrt{3}}\sinh\frac{\xi^+}{3}\right)}$
			

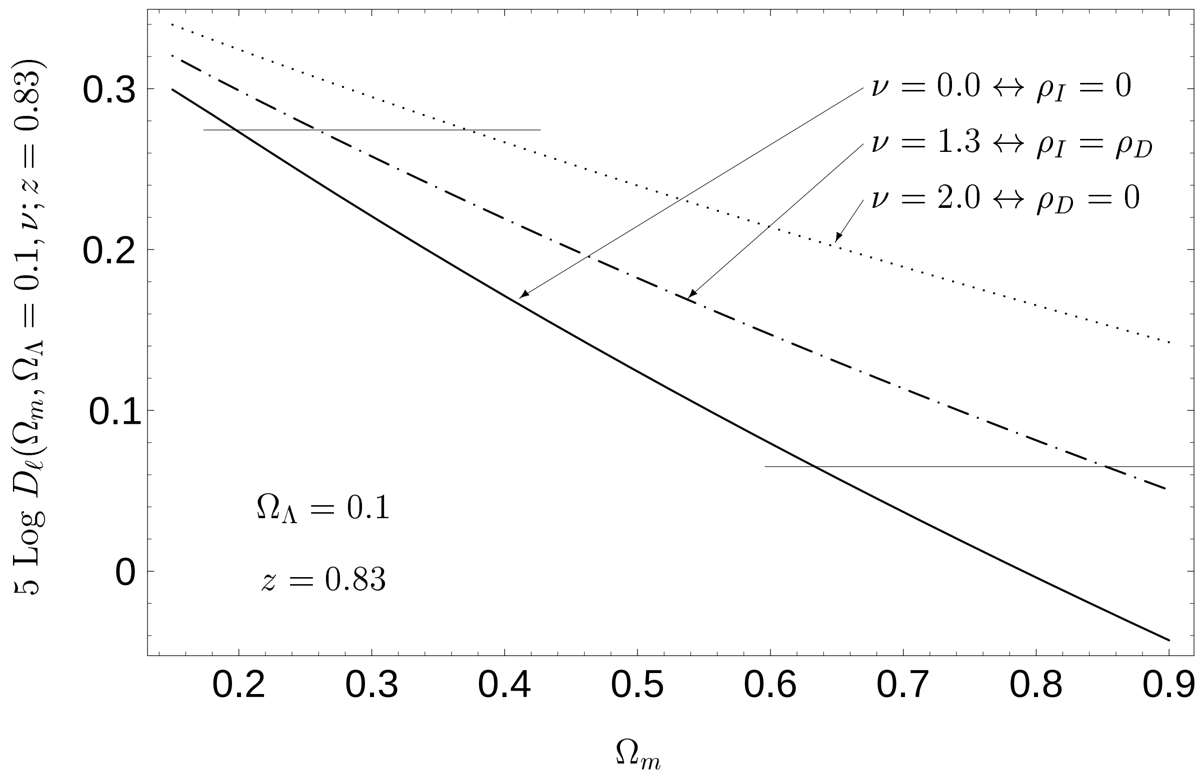

$\Omega_\Lambda = 0.1$
$\Omega_m = 0.1$
$\Omega_m = 0.4$
$\Omega_m = 0.8$
$\longrightarrow \nu = 0.0 \leftrightarrow \rho_I = 0$
$\longrightarrow \nu = 1.3 \leftrightarrow \rho_I = \rho_D$
$\longrightarrow \nu = 2.0 \leftrightarrow \rho_D = 0$
-0.6
-0.8
-1
-1.2
-1.4
-1.6
$5\ \mathrm{Log}\ D_\ell(\Omega_m, \Omega_\Lambda = 0.1, \nu; z)$
0.4
0.45
0.5
0.55
0.6
z

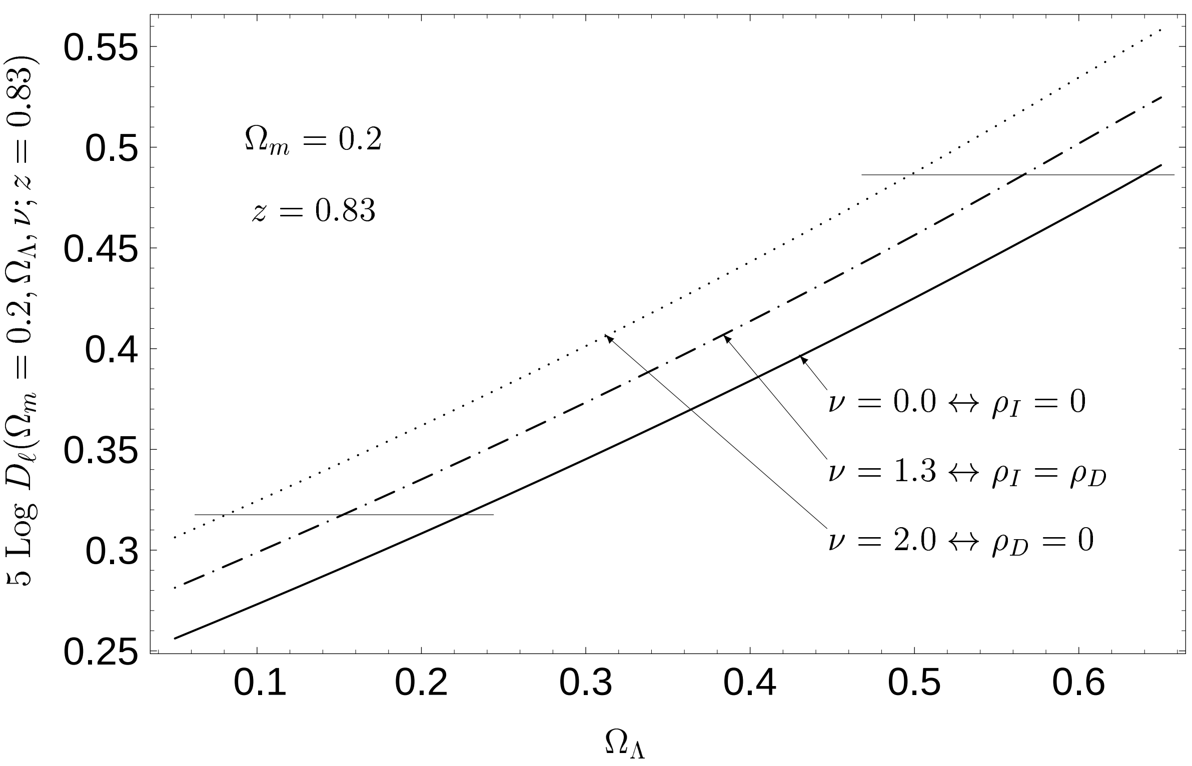


$\Omega_m = 0.2$
$\Omega_\Lambda = 0.6$
$\Omega_\Lambda = 0.2$
$\Omega_\Lambda = 0.01$
$\longrightarrow \nu = 0.0 \leftrightarrow \rho_I = 0$
$\longrightarrow \nu = 1.3 \leftrightarrow \rho_I = \rho_D$
$\longrightarrow \nu = 2.0 \leftrightarrow \rho_D = 0$
$5\ \mathrm{Log}\ D_\ell(\Omega_m = 0.2, \Omega_\Lambda, \nu; z)$
z
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
0.4
0.45
0.5
0.55
0.6



$\Omega_m + \Omega_\Lambda = 1$
$\Omega_m = 0.1,\ \Omega_\Lambda = 0.9$
$\Omega_m = 0.4,\ \Omega_\Lambda = 0.6$
$\Omega_m = 0.8,\ \Omega_\Lambda = 0.2$
$\longrightarrow \nu = 0.0 \leftrightarrow \rho_I = 0$
$\longrightarrow \nu = 1.3 \leftrightarrow \rho_I = \rho_D$
$\longrightarrow \nu = 2.0 \leftrightarrow \rho_D = 0$
$5\ \mathrm{Log}\ D_\ell(\Omega_m, \Omega_\Lambda, \nu; z)$
z
−0.25
−0.5
−0.75
−1
−1.25
−1.5
−1.75
0.4
0.45
0.5
0.55
0.6


$\nu = 0.0 \leftrightarrow \rho_I = 0$
$\nu = 1.3 \leftrightarrow \rho_I = \rho_D$
$\nu = 2.0 \leftrightarrow \rho_D = 0$
$\Omega_\Lambda = 0.1$
$z = 0.83$
0.3
0.2
0.1
0
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
Ω_m
5 Log $D_\ell(\Omega_m, \Omega_\Lambda = 0.1, \nu; z = 0.83)$



$\Omega_m = 0.2$
$z = 0.83$
$\nu = 0.0 \leftrightarrow \rho_I = 0$
$\nu = 1.3 \leftrightarrow \rho_I = \rho_D$
$\nu = 2.0 \leftrightarrow \rho_D = 0$
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0.1
0.2
0.3
0.4
0.5
0.6
$5 \text{ Log } D_\ell(\Omega_m = 0.2, \Omega_\Lambda, \nu; z = 0.83)$
Ω_Λ