Distance-Redshift in Inhomogeneous FLRW

R. Kantowski

University of Oklahoma, Department of Physics and Astronomy,
Norman, OK 73019, USA

kantowski@mail.nhn.ou.edu

J. K. Kao

Tamkang University, Department of Physics,
Tamsui, Taipei, Taiwan 25137 R.O.C.

g3180011@tkgis.tku.edu.tw

R. C. Thomas

University of Oklahoma, Department of Physics and Astronomy,
Norman, OK 73019, USA

thomas@mail.nhn.ou.edu

ABSTRACT

We give distance–redshift relations in terms of elliptic integrals for three different mass distributions of the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology. These models are dynamically pressure free FLRW on large scales but, due to mass inhomogeneities, differ in their optical properties. They are the filled-beam model (standard FLRW), the empty-beam model (no mass density exists in the observing beams) and the 2/3 filled-beam model. For special Ω_m– Ω_Λ values the elliptic integrals reduce to more familiar functions. These new expressions for distance-redshift significantly reduce computer evaluation times.

Subject headings: cosmology: theory – large-scale structure of universe

1. INTRODUCTION

As limits on the global cosmological parameters Ω_m and Λ have been refined, Schmidt et al. (1998) and Perlmutter et al. (1999), the optical inadequacy of the standard distance-redshift relation (D-z) of FLRW has become more apparent. The problem was first recognized long ago by Zel'dovich

(1964), Bertotti (1966), and Kantowski (1969) but the lack of relevant data limited its significance. Even though the average mass density parameter Ω_m (along with H_0 and Λ) determines the large scale dynamic behavior of the pressure free universe, knowledge of the actual mass inhomogeneity is necessary to accurately determine these parameters from most observations. Most observations determine Ω_m and Λ by (indirectly) comparing theoretical D-z curves to observed data. However, D-z depends on more than the average mass density. It can depend significantly on details of how the mass is distributed, i.e., on how inhomogeneous the mass is on the scale of the widths of the observing beams. If some significant fraction ($\rho_I/\rho_0 \leq 1$) of the total mass density is in the form of inhomogeneities and is excluded from the lines of sight to the distant objects observed, a modified, i.e., a partially filled-beam D-z is required.

The necessity of taking into account the effect of inhomogeneities on observations is relatively easy to understand. Homogeneous matter inside an observing beam of light gravitationally focuses the beam much differently than does an equal-mass clump of externally lensing matter. The simplest correction for this gravity-light effect requires the introduction of another parameter ν, $0 \leq \nu \leq 2$, which gives the fraction $\rho_I/\rho_0 = \nu(\nu+1)/6$ of the mass density of the universe removed from the observing beams as inhomogeneities. Using ν rather than ρ_I/ρ_0 or some other parameter is dictated by the mathematics of special functions. A reduced mass density in an observing beam causes it to diverge relative to a standard FLRW beam. For an observed object in such a universe to have the standard FLRW angular size it would thus have to be moved to a smaller z; i.e., objects will appear less bright than in the standard FLRW universe. A reasonable application of this model to SNe Ia observations takes ρ_I as the galactic contribution to the total mass density ρ_0 and the remaining contribution as a smooth intergalactic medium. Galaxies are easily excluded from SNe Ia foregrounds by selection (intended or not) and if galaxy mass roughly follows light, including their mass in ρ_I is appropriate. In the partially filled-beam model where the additional parameter $\nu \neq 0$ has been introduced, only lensing by mass clumps external to the beam has been neglected. To compare individual observations to D-z of this model requires only an occasional lensing correction; however, comparison with the standard FLRW D-z ($\nu = 0$) model requires a defocusing correction for the partially empty-beam of every observation, as well as the occasional lensing correction. If only weak and transparent lensing occurs (to the z_{max} being observed) the standard FLRW D-z ($\nu = 0$) should give the mean D-z curve. Wang (1999) argues that by using flux-averaging the mean can be accurately obtained. Kantowski (1998a) and Kantowski (1998b) claims that determining cosmological parameters from data compared with the partially filled Hubble curves given here is likely to be easier. Beyond selection effects, unknown lensing probabilities can be highly non-Gaussian and should make the mean more difficult to observationally determine, i.e., should require more data if a given accuracy of the cosmic parameters is to be obtained, Bertotti (1966); Holz & Wald (1998); Holz (1998). The down side for partially filled-beam models is that you must select against lensing and must determine the additional parameter ν.

In Sec. 2 we outline the procedure required to obtain D-z for partially filled-beam FLRW observations and how the result simplifies for the three special cases of $\nu = 0$, 1, and 2. In Sec. 3

we give the new results for these three special cases. Some concluding remarks are given in Sec. 4 and in the Appendix we discuss our fortran implementation of these results.

2. The Luminosity Distance-redshift Relation

For models being discussed here (and for most cosmological models), angular or apparent size distance is related to luminosity distance by $D_<(z) = D_\ell(z)/(1+z)^2$. Hence we need to give only one or the other, and we have chosen to give luminosity distances. The $D_\ell(z)$ which accounts for a partially depleted mass density in the observing beam but neglects lensing by external masses is found by integrating the second order differential equation for the cross sectional area $A(z)$ of an observing beam from source ($z = z_s$) to observer ($z = 0$), see Kantowski (1998a) for some history of this equation:

$$
\begin{aligned}
&(1+z)^3\sqrt{1+\Omega_m z+\Omega_\Lambda[(1+z)^{-2}-1]}\times\\
&\qquad\frac{d}{dz}(1+z)^3\sqrt{1+\Omega_m z+\Omega_\Lambda[(1+z)^{-2}-1]}\,\frac{d}{dz}\sqrt{A(z)}\\
&\qquad\qquad+\frac{(3+\nu)(2-\nu)}{4}\Omega_m(1+z)^5\sqrt{A(z)}=0.
\end{aligned}
\tag{1}
$$

The required boundary conditions are:

$$
\begin{aligned}
\sqrt{A}|_s &= 0,\\
\frac{d\sqrt{A}|_s}{dz} &= -\sqrt{\delta\Omega}\frac{c}{H_s(1+z_s)},
\end{aligned}
\tag{2}
$$

where $\delta\Omega$ is the solid angle of the beam at the source and the FLRW value of the Hubble parameter at z_s is related to the current value H_0 at $z = 0$ by:

$$
H_s = H_0(1+z_s)\ \sqrt{1+\Omega_m z_s+\Omega_\Lambda[(1+z_s)^{-2}-1]}. \tag{3}
$$

The luminosity distance is then simply related to the area $A|_0$ of the beam at the observer by:

$$
D_\ell^2 \equiv \frac{A|_0}{\delta\Omega}(1+z_s)^2. \tag{4}
$$

Equation (1) can be put into the form of a Lamé equation and its solution has been given in terms of Heun functions in Kantowski (1998a). Solutions can also be given in terms of Lamé functions but neither Heun nor Lamé functions are currently available in standard computer libraries. Consequently, such expressions are not particularly useful for comparison with data, at this time. For the special case where $\Lambda = 0$ the Lamé functions reduce to associated Legendre functions and these expressions are useful. Other special cases also exist as is pointed out in Kantowski (1998a).

In the next section we give useful expressions for D_ℓ for three special cases where Λ is arbitrary but where the filling parameter ν is restricted to values 0, 1, and 2. For these three cases we can write

D_ℓ as an elliptic integral and hence we can give D_ℓ in terms of the three fundamental incomplete Legendre elliptic integrals $F(\phi, \mathrm{k})$, $E(\phi, \mathrm{k})$, and $\Pi(\phi, \alpha^2, \mathrm{k})$. These functions are universally available and these new expressions significantly speed up the evaluation of D_ℓ (see the Appendix). Distance-redshift for $\Omega_0 = 1$ can be given in terms of hypergeometric functions, see (21) and (53), or associated Legendre functions, see (22) and (54); however, we also give D_ℓ as more complicated expressions involving Legendre elliptic integrals, (23) and (55), because these expressions evaluate more rapidly using currently available Fortran routines.

It is not at all clear that the solution of (1) can be written as elliptic integrals for the special cases of $\nu = 0$, 1 and 2. However, the steps required to arrive at this conclusion can be found in Whittaker & Watson (1927) under integral functions for Lamé and Matthew equations (see especially Sec. 19.53). The authors have carried out the conversion directly for all three cases; however, the $\nu = 0$ and 2 conversions can be reached by simpler means. The integral for $\nu = 0$, the standard FLRW filled-beam case, is given in (5) and is well known. The $\nu = 2$ (empty-beam) integral given in (46) is easy to obtain because the coefficient of $\sqrt{A}$ vanishes in (1). The first integral is trivial and the second is elliptic resulting in (46). For $\nu = 1$, the 66% filled-beam model, the integral is given in (30); however, no simple way of getting this from (1) seems to exist.

In Sec. 3. we outline results for all big bang models in the first quadrant of the Ω_m– Ω_Λ plane (see Fig. 1), hoping to facilitate their usage. Luminosity distances for the three large open domains are given in subsections A, and for the boundaries of these domains in subsections B.

3. Luminosity Distances as Legendre Elliptic Integrals

I. $\nu = 0$, Completely Filled-Beam Observations (Standard FLRW)

A. Three Open Big Bang Domains

Kaufman & Schucking (1971) and Kaufman (1971) gave magnitude-redshift relations for standard pressure-free FLRW models as inverse Weierstrass functions and more recently Feige (1992) gave comoving distances and light travel times for these models using Legendre elliptic integrals. In this section we give simpler and more useful results which are directly comparable with Edwards (1972) who used Jacobi elliptic functions. The well known and often used integral form for luminosity distance in standard FLRW is:

$$D_\ell(\Omega_m, \Omega_\Lambda, \nu = 0; z) = \frac{c}{H_0} \frac{1+z}{\sqrt{|1-\Omega_0|}}\, S_\kappa \left[\sqrt{|1-\Omega_0|} \int_0^z \frac{dz}{\sqrt{(1+z)^2(1+\Omega_m z) - z(z+2)\Omega_\Lambda}} \right] \quad (5)$$

which we integrate using Byrd & Friedman (1971) to obtain,

$$D_\ell(\Omega_m, \Omega_\Lambda, \nu = 0; z) = \frac{c}{H_0} \frac{1+z}{\sqrt{|1-\Omega_0|}}\, S_\kappa \Big[-g \Big\{ F(\phi_z, \mathrm{k}) - F(\phi_0, \mathrm{k}) \Big\} \Big], \quad (6)$$

or equivalently using an addition formula for $F(\phi, \mathrm{k})$, i.e., $F(\phi_z, \mathrm{k}) - F(\phi_0, \mathrm{k}) = F(\Delta\phi_z, \mathrm{k})$ we get:

$$D_\ell(\Omega_m, \Omega_\Lambda, \nu = 0; z) = \frac{c}{H_0} \frac{1+z}{\sqrt{|1-\Omega_0|}} \; S_\kappa\Big[-g \; F(\Delta\phi_z, \mathrm{k})\Big]. \tag{7}$$

The parameter $\kappa \equiv (\Omega_0 - 1)/|\Omega_0 - 1|$ is determined by the sign of the 3-curvature and $S_\kappa[\;]$ is one of two functions:

$$S_\kappa[\;] = \begin{cases} \sinh[\;] & : \;\; \kappa = -1, \\ \sin[\;] & : \;\; \kappa = +1. \end{cases}$$

Constants g and k depend on the cosmic parameters Ω_m & Ω_Λ, and $F(\phi, \mathrm{k})$ is the incomplete Legendre elliptic integral of the first kind.[1] The constants g and k depend on Ω_m & Ω_Λ only through a combination called b defined by:

$$b \equiv -(27/2) \frac{\Omega_m^2 \Omega_\Lambda}{(1-\Omega_0)^3}, \qquad\qquad -\infty \leq b \leq \infty, \tag{8}$$

$$\begin{aligned} b < 0 \;\; &\Leftrightarrow \kappa = -1, \\ b > 0 \;\; &\Leftrightarrow \kappa = +1. \end{aligned}$$

The functions ϕ_z and $\Delta\phi_z$ depend on the redshift z and the cosmic parameters Ω_m & Ω_Λ (not just on the combination b). Domains for the various b values in the Ω_m– Ω_Λ plane are shown in Fig. 1.

1. For the two open domains defined by $b < 0$ and $2 < b$, quantities g, k, ϕ_z, and $\Delta\phi_z$ are conveniently written in terms of intermediate constants v_κ, y_1 and A defined by:

$$v_\kappa \equiv \Big[\kappa(b-1) + \sqrt{b(b-2)}\Big]^{1/3}, \qquad\qquad v_\kappa \geq 1. \tag{9}$$

$$y_1 \equiv \frac{-1 + \kappa(v_\kappa + v_\kappa^{-1})}{3}, \tag{10}$$

$$A = A(\Omega_m, \Omega_\Lambda) \equiv \sqrt{y_1(3y_1 + 2)} = \sqrt{\frac{v_\kappa^2 + v_\kappa^{-2} + 1}{3}} \geq 1. \tag{11}$$

Parameters g and k are then given by:

$$g = g(\Omega_m, \Omega_\Lambda) = 1/\sqrt{A(\Omega_m, \Omega_\Lambda)} = \left[\frac{3}{v_\kappa^2 + v_\kappa^{-2} + 1}\right]^{1/4} \leq 1, \tag{12}$$

and

$$\mathrm{k}^2 = \mathrm{k}^2(\Omega_m, \Omega_\Lambda) = \frac{2A + \kappa(1 + 3y_1)}{4A} = \left[\frac{1}{2} + \frac{1}{4} g^2 (v_\kappa + v_\kappa^{-1})\right] \leq 1. \tag{13}$$

---

[1] $F(\phi, \mathrm{k}) \equiv \int_0^\phi 1/\sqrt{1 - \mathrm{k}^2 \sin^2\phi} \; d\phi$

Functions ϕ_z, and $\Delta\phi_z$ are given by:

$$\phi_z = \phi(\Omega_m, \Omega_\Lambda; z) = \cos^{-1}\left[\frac{(1+z)\Omega_m/|1-\Omega_0| + \kappa y_1 - A}{(1+z)\Omega_m/|1-\Omega_0| + \kappa y_1 + A}\right], \tag{14}$$

and

$$\Delta\phi_z = \Delta\phi(\Omega_m, \Omega_\Lambda; z) = 2\ \tan^{-1}\left[\frac{-z\ \sqrt{A}\sqrt{|1-\Omega_0|}\sqrt{1+z[1-(1-\Omega_0)\Omega_m^{-1}y_1]^{-1}}}{1+z[1-(1-\Omega_0)\Omega_m^{-1}y_1]^{-1} + \sqrt{(1+z)^2(1+\Omega_m z) - z(z+2)\Omega_\Lambda}}\right]. \tag{15}$$

2. For the domain $0 < b < 2$ ($\Rightarrow \kappa = 1$) three intermediate parameter $y_1,\ y_2$ and y_3 are convenient to use, although none are really necessary. In this domain of b, intermediate parameters $y_1,\ y_2$ and y_2 are related to the cosmic parameters Ω_m& Ω_Λ through b by:

$$\begin{aligned}
y_1 &\equiv \frac{1}{3}\left(-1+\cos\left[\frac{\cos^{-1}(1-b)}{3}\right] + \sqrt{3}\ \sin\left[\frac{\cos^{-1}(1-b)}{3}\right]\right), \quad 0 \le y_1 \le 1/3, \\
y_2 &\equiv \frac{1}{3}\left(-1-2\ \cos\left[\frac{\cos^{-1}(1-b)}{3}\right]\right), \quad -1 \le y_2 \le -2/3, \\
y_3 &\equiv \frac{1}{3}\left(-1+\cos\left[\frac{\cos^{-1}(1-b)}{3}\right] - \sqrt{3}\ \sin\left[\frac{\cos^{-1}(1-b)}{3}\right]\right), \quad -2/3 \le y_3 \le 0.
\end{aligned} \tag{16}$$

The following expressions are valid **only** in the lower right part of the Ω_m– Ω_Λ plane. In the upper left domain where b also satisfies $0 \le b \le 2$, expressions can be given, but there a big bang doesn't occur. The parameters g and k and functions ϕ_z and $\Delta\phi_z$ needed to evaluate (6) and (7) are:

$$g = g(\Omega_m, \Omega_\Lambda) \equiv \frac{2}{\sqrt{y_1 - y_2}}, \tag{17}$$

$$\mathrm{k}^2 = \mathrm{k}^2(\Omega_m, \Omega_\Lambda) \equiv \frac{y_1 - y_3}{y_1 - y_2} \le 1, \tag{18}$$

$$\phi_z = \phi(\Omega_m, \Omega_\Lambda; z) = \sin^{-1}\sqrt{\frac{y_1 - y_2}{(1+z)\Omega_m/|1-\Omega_0| + y_1}}, \tag{19}$$

$$\begin{aligned}
\Delta\phi_z &= \Delta\phi(\Omega_m, \Omega_\Lambda; z) \\
&= 2\ \tan^{-1}\left[\frac{\sqrt{y_1 - y_2}\ \left[\sqrt{y_3 - \Omega_m/(1-\Omega_0)} - \sqrt{y_3 - (1+z)\Omega_m/(1-\Omega_0)}\right]}{\sqrt{[y_1 - \Omega_m/(1-\Omega_0)][y_2 - (1+z)\Omega_m/(1-\Omega_0)]} + \sqrt{[y_1 \longleftrightarrow y_2]}}\right],
\end{aligned} \tag{20}$$

where $y_1 \longleftrightarrow y_2$ means repeat the previous term with y_1 and y_2 exchanged.

B. Boundaries

1. $\Omega_0 \equiv \Omega_m + \Omega_\Lambda = 1$

For the spatially flat model ($b \to \pm\infty$) a much simpler expression involving hypergeometric functions results:

$$
\begin{aligned}
D_\ell(\Omega_m, \Omega_\Lambda = 1-\Omega_m, \nu = 0; z) &= \frac{c}{H_0}(1+z)\int_0^z \frac{dz}{\sqrt{1+\Omega_m z(3+3z+z^2)}} \\
= \frac{c}{H_0}\frac{2(1+z)}{\Omega_m^{1/3}} & \Bigg[{}_2F_1\left(\frac{1}{6},\frac{2}{3};\frac{7}{6};1-\Omega_m\right) \\
- \left(\frac{1}{[1+\Omega_m z(3+3z+z^2)]^{1/6}}\right) & {}_2F_1\left(\frac{1}{6},\frac{2}{3};\frac{7}{6};\frac{1-\Omega_m}{1+\Omega_m z(3+3z+z^2)}\right)\Bigg]. \qquad (21)
\end{aligned}
$$

When $\Omega_m \neq 1$ (21) can be expressed as associated Legendre functions,

$$
\begin{aligned}
D_\ell(\Omega_m, \Omega_\Lambda = 1-\Omega_m, \nu = 0; z) &= \frac{c}{H_0}\frac{2^{1/6}\Gamma\,(1/6)\,(1+z)}{3[\Omega_m^5(1-\Omega_m)]^{1/12}} \\
\times \left[\mathrm{P}_{-1/6}^{-1/6}\left(\frac{1}{\sqrt{\Omega_m}}\right) - \frac{1}{(1+z)^{(1/4)}}\right. & \left.\mathrm{P}_{-1/6}^{-1/6}\left(\sqrt{\frac{1+\Omega_m z(3+3z+z^2)}{\Omega_m(1+z)^3}}\right)\right]. \qquad (22)
\end{aligned}
$$

If (22) is given in terms of Legendre elliptic integrals the result is more complicated:

$$
\begin{aligned}
D_\ell(\Omega_m, \Omega_\Lambda = 1-\Omega_m, \nu = 0; z) &= \frac{c}{H_0}\frac{1+z}{(3)^{1/4}\sqrt{\Omega_m}(\Omega_m^{-1}-1)^{1/6}}\Big[-\{F(\phi_z,\mathrm{k}) - F(\phi_0,\mathrm{k})\}\Big], \\
&= \frac{c}{H_0}\frac{1+z}{(3)^{1/4}\sqrt{\Omega_m}(\Omega_m^{-1}-1)^{1/6}}\Big[-F(\Delta\phi_z,\mathrm{k})\Big], \qquad (23)
\end{aligned}
$$

where

$$
\mathrm{k}^2 = \left[\frac{1}{2} + \frac{\sqrt{3}}{4}\right], \qquad (24)
$$

$$
\phi_z = \phi(\Omega_m; z) = \cos^{-1}\left[\frac{1+z+(1-\sqrt{3})(\Omega_m^{-1}-1)^{1/3}}{1+z+(1+\sqrt{3})(\Omega_m^{-1}-1)^{1/3}}\right], \qquad (25)
$$

and

$$
\Delta\phi_z = \Delta\phi(\Omega_m, \Omega_\Lambda; z) \equiv 2\ \tan^{-1}\left[\frac{-z\ \sqrt{\sqrt{3}\Omega_m(1/\Omega_m - 1)^{1/3}}\sqrt{1+z[1+(1/\Omega_m-1)^{1/3}]^{-1}}}{1+z[1+(1/\Omega_m-1)^{1/3}]^{-1}+\sqrt{1+\Omega_m z(3+3z+z^2)}}\right]. \qquad (26)
$$

2. $b = 2$

This value of b can be identified with "critical" values of the cosmic parameters, Felten & Isaacman (1986). We give a result good only for the lower $b = 2$ curve, see (44). These models start with a big bang and expand to the the finite Einstein radius at $t = \infty$, see A3(vii-b) in the appendix of McVittie (1965):

$$D_\ell(\Omega_m, \Omega_\Lambda(\Omega_m), \nu = 0; z) = \frac{c}{H_0} \frac{1+z}{\sqrt{|1-\Omega_0|}}$$
$$\times \sin\left\{ \ln\left(\frac{\left[\sqrt{1/3 - \Omega_m/(1-\Omega_0)} + 1\right]\left[\sqrt{1/3 - (1+z)\Omega_m/(1-\Omega_0)} - 1\right]}{\left[\sqrt{1/3 - \Omega_m/(1-\Omega_0)} - 1\right]\left[\sqrt{1/3 - (1+z)\Omega_m/(1-\Omega_0)} + 1\right]} \right) \right\}. \tag{27}$$

3. $\Omega_\Lambda = 0$

This result is due to Mattig (1958), we include it for completeness:

$$D_\ell(\Omega_m, \Omega_\Lambda = 0, \nu = 0; z) = \frac{2c}{H_0 \Omega_m^2} \left\{ \Omega_m z + (\Omega_m - 2)\left(\sqrt{1 + \Omega_m z} - 1\right) \right\}. \tag{28}$$

4. $\Omega_m = 0$

These are massless big bang models, $\Omega_\Lambda < 1$, discussed by Robertson (1933):

$$D_\ell(\Omega_m = 0, \Omega_\Lambda, \nu = 0; z) = \frac{c(1+z)}{H_0 \Omega_\Lambda} \left\{ 1 + z - \sqrt{\Omega_\Lambda + (1+z)^2(1-\Omega_\Lambda)} \right\}. \tag{29}$$

II. $\nu = 1$, 66% Filled-Beam Observations

A. Four Open Big Bang Domains

$$D_\ell(\Omega_m, \Omega_\Lambda, \nu = 1; z) =$$
$$\frac{c}{H_0} 2\,(1+z) \mathrm{Sign}\,[3 - \Omega_m/(1-\Omega_0)] \sqrt{\left| \frac{[3 - \Omega_m(1+z)/(1-\Omega_0)]\,[3 - \Omega_m/(1-\Omega_0)]}{(1-\Omega_0)[36 + \Omega_m^2 \Omega_\Lambda/(1-\Omega_0)^3]} \right|} \times$$
$$S_{(\Omega_m, \Omega_\Lambda, z)}\left[\sqrt{|(1-\Omega_0)[36 + \Omega_m^2\Omega_\Lambda/(1-\Omega_0)^3]|} \times \right.$$
$$\left. P \int_0^z \frac{dz}{2\,[3 - \Omega_m(1+z)/(1-\Omega_0)]\,\sqrt{(1+z)^2(1+\Omega_m z) - z(z+2)\Omega_\Lambda}} \right], \tag{30}$$

where

$$S_{(\Omega_m, \Omega_\Lambda, z)}[\,] = \begin{cases} \cosh[\,] & : \; b < 0\,\&\; [3 - \Omega_m(1+z)/(1-\Omega_0)]\,[3 - \Omega_m/(1-\Omega_0)] < 0, \\ \sinh[\,] & : \; b < 0\,\&\; [3 - \Omega_m(1+z)/(1-\Omega_0)]\,[3 - \Omega_m/(1-\Omega_0)] > 0, \\ \sin[\,] & : \; 0 < b < 486, \\ \sinh[\,] & : \; 486 < b. \end{cases}$$

Only the principal value of the integral (P) is needed and unlike the $\nu = 0$ case, this integral takes on different forms when evaluated using Legendre elliptic integrals, depending on the value of the parameter b. Parts of the analytic result (31) sometimes diverge even though the total expression remains finite. For example when $b = 486$, i.e., when $\sqrt{36 + \Omega_m^2 \Omega_\Lambda / (1 - \Omega_0)^3} = 0$ or equivalently $y_1 = 3$, a limit must be taken. The resulting D_ℓ on this new boundary can be found in II.B.5 below. This new boundary splits the one open domain $2 < b < \infty$ into two parts, see Fig. 2. Consequently, the Ω_m– Ω_Λ plane is more complicated for $\nu = 1$ than for either $\nu = 0$ or $\nu = 2$. See A1 below for additional trouble points that occur.

1. For the three open domains defined by $b < 0, 2 < b < 486$, and $486 < b$ the luminosity distance D_ℓ takes the form:

$$
\begin{aligned}
&D_\ell(\Omega_m, \Omega_\Lambda, \nu = 1; z) = \\
&\frac{c}{H_0} 2\,(1+z) \mathrm{Sign}\left[3 - \Omega_m/(1-\Omega_0)\right] \sqrt{\left| \frac{\left[3 - \Omega_m(1+z)/(1-\Omega_0)\right]\left[3 - \Omega_m/(1-\Omega_0)\right]}{(1-\Omega_0)[36 + \Omega_m^2 \Omega_\Lambda/(1-\Omega_0)^3]} \right|} \times \\
&S_{(\Omega_m, \Omega_\Lambda, z)} \left[\frac{\kappa\, \sqrt{|36 + \Omega_m^2 \Omega_\Lambda/(1-\Omega_0)^3|}}{2\sqrt{A}\ [A + \kappa(y_1 - 3)]} \left\{ [F(\phi_z, \mathrm{k}) - F(\phi_0, \mathrm{k})] \right.\right. \\
&\left.\left. + \frac{A - \kappa(y_1 - 3)}{2\kappa(y_1 - 3)} \left[\mathrm{P}\,\Pi(\phi_z, \hat{\alpha}^2, \mathrm{k}) - \mathrm{P}\,\Pi(\phi_0, \hat{\alpha}^2, \mathrm{k})\right] \right\} + f_b \right], \qquad (31)
\end{aligned}
$$

where y_1, A, k, and ϕ_z are defined in (10)-(14) and the additional constant $\hat{\alpha}^2$ is:

$$
\hat{\alpha}^2 \equiv \frac{(A + \kappa(y_1 - 3))^2}{4A\kappa(y_1 - 3)}. \qquad (32)
$$

$\Pi(\phi, \alpha^2, \mathrm{k})$ is the incomplete Legendre elliptic integral of the third kind[2] and $\mathrm{P}\,\Pi(\phi, \alpha^2, \mathrm{k})$ is the principal part of that integral. The function f_b is one of,

$$
f_b = \begin{cases} \frac{1}{4} \ln \left|\{[1 + h(z)][1 - h(0)]\}/\{[1 + h(0)][1 - h(z)]\}\right| & : \quad b < 0 \text{ or } 486 < b, \\ \frac{1}{2}\left[\tan^{-1} h(z) - \tan^{-1} h(0)\right] & : \quad 2 < b < 486, \end{cases}
$$

where $h(z)$ is defined by:

$$
h(z) \equiv \frac{\sqrt{|36 + \Omega_m^2 \Omega_\Lambda/(1-\Omega_0)^3|}\sqrt{(1+z)\Omega_m/|1-\Omega_0| + \kappa y_1}}{(3 - y_1)\sqrt{[(1+z)\Omega_m/|1-\Omega_0| - \kappa(1+y_1)/2]^2 - (1+y_1)(1-3y_1)/4}}. \qquad (33)
$$

Some care has to be taken when using these expressions. Divergences in the function f_b necessarily occur and cancel divergences in $\Pi(\phi, \alpha^2, \mathrm{k})$. Divergences in f_b also occur which add to divergences in $\Pi(\phi, \alpha^2, \mathrm{k})$ and cancel zeros in the multiplicative factor $\sqrt{|[3 - \Omega_m(1+z)/(1-\Omega_0)]\,[3 - \Omega_m/(1-\Omega_0)]|}$ of (31). Redshift independent divergences occur when $\Omega_m/(1-\Omega_0) = 3$ and when $\Omega_m(3 - y_1)/(1-$

[2] $\Pi(\phi, \alpha^2, \mathrm{k}) \equiv \int_0^\phi 1/\left[(1 - \alpha^2 \sin^2\phi)\sqrt{1 - \mathrm{k}^2 \sin^2\phi}\,\right]\, d\phi$. In arriving at the results for the two-thirds filled beam model we discovered that equation 361.54 of Byrd & Friedman (1971) has the two square-root terms interchanged for the case $\alpha^2/(\alpha^2 - 1) > \mathrm{k}^2$.

$\Omega_0) = y_1(2y_1+5)$. These points are plotted in Figure 2. Redshift dependent divergences occur at $(1+z) = 3(1-\Omega_0)/\Omega_m$ and at $(1+z)\Omega_m(3-y_1)/(1-\Omega_0) = y_1(5+2y_1)$. These points appear in the Ω_m– Ω_Λ plane respectively to the left of the $\Omega_m/(1-\Omega_0) = 3$ line and between the $\Omega_m(3-y_1)/(1-\Omega_0) = y_1(5+2y_1)$ and $b = 486$ curves.

Computer evaluation of (31) can be speeded up by reducing the number of Legendre elliptic integrals that must be evaluated. As in (7) we can use the addition formula for $F(\phi, \mathrm{k})$, i.e., $F(\phi_z, \mathrm{k}) - F(\phi_0, \mathrm{k}) = F(\Delta\phi_z, \mathrm{k})$ and an addition formula for $\Pi(\phi, \alpha^2, \mathrm{k})$,[3]

$$\Pi(\phi_z, \alpha^2, \mathrm{k}) - \Pi(\phi_0, \alpha^2, \mathrm{k}) = \Pi(\Delta\phi_z, \alpha^2, \mathrm{k}) + \frac{1}{2}\sqrt{\frac{\alpha^2}{(\alpha^2-1)(\alpha^2-\mathrm{k}^2)}}\log\left(\frac{1+\xi}{1-\xi}\right), \tag{34}$$

where

$$\xi \equiv \frac{\sin\phi_z \sin\phi_0 \sin\Delta\phi_z \sqrt{\alpha^2(\alpha^2-1)(\alpha^2-\mathrm{k}^2)}}{1-\alpha^2\sin^2\Delta\phi_z - \alpha^2 \sin\phi_z \sin\phi_0 \cos\Delta\phi_z \sqrt{1-\mathrm{k}^2\sin^2\Delta\phi_z}}, \tag{35}$$

to cut the number of elliptic functions from four to two. We were not able to simplify this expression enough to justify inclusion of a rewritten version of (31). However, it was used in our Fortran implementation (see Appendix).

2. For the open domain defined by $0 < b < 2$ the luminosity distance D_ℓ has a somewhat simpler form:

$$\begin{aligned} D_\ell(\Omega_m, \Omega_\Lambda, \nu=1; z) &= \frac{c}{H_0}\frac{2\,(1+z)}{\sqrt{|1-\Omega_0|}}\sqrt{\frac{[3-\Omega_m(1+z)/(1-\Omega_0)]\,[3-\Omega_m/(1-\Omega_0)]}{[36+\Omega_m^2\Omega_\Lambda/(1-\Omega_0)^3]}} \times \\ \sin&\left[\frac{\sqrt{36+\Omega_m^2\Omega_\Lambda/(1-\Omega_0)^3}}{(3-y_1)\sqrt{y_1-y_2}}\left\{-\left[F(\phi_z, \mathrm{k}) - F(\phi_0, \mathrm{k})\right]\right.\right. \\ &\left.\left. + \left[\Pi\left(\phi_z, \frac{y_1-3}{y_1-y_2}, \mathrm{k}\right) - \Pi\left(\phi_0, \frac{y_1-3}{y_1-y_2}, \mathrm{k}\right)\right]\right\}\right]. \end{aligned} \tag{36}$$

The constants y_1, y_2 and k, and the function ϕ_z are as defined in I.A.2 above [see (16)-(19)]. Just as in the previous case, the number of Legendre elliptic functions in (36) can be reduced from four to two by using the appropriate addition formulas. For $F(\phi, \mathrm{k})$ the formula is always the same, see (6) and (7), but because α^2 is negative (34) changes to:

$$\Pi(\phi_z, \alpha^2, \mathrm{k}) - \Pi(\phi_0, \alpha^2, \mathrm{k}) = \Pi(\Delta\phi_z, \alpha^2, \mathrm{k}) - \frac{1}{2}\sqrt{\frac{\alpha^2}{(1-\alpha^2)(\alpha^2-\mathrm{k}^2)}}\tan^{-1}(\xi), \tag{37}$$

where

$$\xi \equiv \frac{\sin\phi_z \sin\phi_0 \sin\Delta\phi_z \sqrt{\alpha^2(1-\alpha^2)(\alpha^2-\mathrm{k}^2)}}{1-\alpha^2\sin^2\Delta\phi_z - \alpha^2 \sin\phi_z \sin\phi_0 \cos\Delta\phi_z \sqrt{1-\mathrm{k}^2\sin^2\Delta\phi_z}}. \tag{38}$$

---

[3]This equation is 116.03 of Byrd & Friedman (1971), corrected for two sign errors.

This is 116.02 of Byrd & Friedman (1971) with one sign error corrected.

B. Boundaries

1. $\Omega_0 \equiv \Omega_m + \Omega_\Lambda = 1$

For these models $b \to \pm\infty$ and a much simpler expression results:

$$\begin{aligned} D_\ell(\Omega_m, \Omega_\Lambda = 1-\Omega_m, \nu = 1; z) \\ &= \frac{c}{H_0}\frac{2(1+z)^{3/2}}{\sqrt{1-\Omega_m}}\sinh\left[\frac{\sqrt{1-\Omega_m}}{2}\int_0^z \frac{dz}{(1+z)\sqrt{1+\Omega_m z(3+3z+z^2)}}\right], \\ &= \frac{c}{H_0\sqrt{1-\Omega_m}}(1+z)^2\left[\left(\frac{1+\sqrt{1-\Omega_m}}{\sqrt{1+\Omega_m z(3+3z+z^2)}+\sqrt{1-\Omega_m}}\right)^{1/3}\right. \\ &\qquad \left. - \left(\frac{1-\sqrt{1-\Omega_m}}{\sqrt{1+\Omega_m z(3+3z+z^2)}-\sqrt{1-\Omega_m}}\right)^{1/3}\right]. \end{aligned} \tag{39}$$

This result can be given in terms of Legendre elliptic integrals $F(\phi, \mathrm{k})$ and $\Pi(\phi, \alpha^2, \mathrm{k})$; however, the authors can think of no useful purpose in doing so.

2. $b = 2$

See the description for the $\nu = 0$ case in section I.B.2 including (44) for this "critical" value of b:

$$\begin{aligned} D_\ell(\Omega_m, \Omega_\Lambda(\Omega_m), \nu = 1; z) = \frac{c}{H_0}\frac{(1+z)\sqrt{3/2}}{\sqrt{|1-\Omega_0|}\,(11)}\Bigg\{ \\ \sqrt{8}\left[\sqrt{1-3\,\frac{\Omega_m}{1-\Omega_0}} - \sqrt{1-3\,\frac{\Omega_m(1+z)}{1-\Omega_0}}\right]\cos\left(\frac{4}{\sqrt{6}}\log(h_z)\right) \\ + \left[8+\sqrt{\left(1-3\,\frac{\Omega_m}{1-\Omega_0}\right)\left(1-3\,\frac{(1+z)\Omega_m}{1-\Omega_0}\right)}\right]\sin\left(\frac{4}{\sqrt{6}}\log(h_z)\right)\Bigg\}, \end{aligned} \tag{40}$$

where h_z is defined by:

$$h_z = \left(\frac{1+\sqrt{1/3-\Omega_m/(1-\Omega_0)}}{1+\sqrt{1/3-(1+z)\Omega_m/(1-\Omega_0)}}\right)\sqrt{\frac{2/3+(1+z)\Omega_m/(1-\Omega_0)}{2/3+\Omega_m/(1-\Omega_0)}}. \tag{41}$$

3. $\Omega_\Lambda = 0$

This result was first given by Dyer & Roeder (1973),

$$D_\ell(\Omega_m, \Omega_\Lambda = 0, \nu = 1; z) = \frac{c}{H_0}\frac{4}{3\Omega_m^2}\left[\left(\frac{3}{2}\Omega_m - 1 + \frac{1}{2}\Omega_m z\right)\sqrt{1+\Omega_m z} - \left(\frac{3}{2}\Omega_m - 1\right)\right]. \tag{42}$$

4. $\Omega_m = 0$

This result is exactly the same as the $\nu = 0$ result (29). If there is no mass in the universe then removing 33% of no mass from the beam changes nothing.

5. $b = 486$

This result is equivalent to the $b \to 486$ limit of (31) but is simpler to use. Because $\Omega_\Lambda(\Omega_m)$ is double valued for $b =$ constant ≥ 2 , two expressions must be given to draw the $b = 486$ curve, see Fig. 2. For the upper part of the curve:

$$\Omega_\Lambda(\Omega_m) = 1 - \Omega_m + 3\sqrt{2/b}\ \Omega_m\ \cosh\left[\frac{\cosh^{-1}\left[\sqrt{b/2}\ (\Omega_m^{-1} - 1)\right]}{3}\right], \tag{43}$$

where $0 \leq \Omega_m \leq 1/(1 - \sqrt{2/b})$. In this expression hyperbolic cosine analytically becomes cosine for $\Omega_m \geq 1/(1 + \sqrt{2/b})$. For the lower part of the curve:

$$\Omega_\Lambda(\Omega_m) = 1 - \Omega_m + 3\sqrt{2/b}\ \Omega_m\ \cos\left[\frac{\cos^{-1}\left[\sqrt{b/2}\ (1 - \Omega_m^{-1})\right] + \pi}{3}\right], \tag{44}$$

where $1 \leq \Omega_m \leq 1/(1 - \sqrt{2/b})$. The simplified result is:

$$\begin{aligned}
D_\ell(\Omega_m, \Omega_\Lambda(\Omega_m), \nu = 1; z) = \frac{c}{H_0}\frac{(1+z)}{\sqrt{|1-\Omega_0|(33)^{(3/4)}}}\sqrt{\left[3 - \frac{\Omega_m(1+z)}{(1-\Omega_0)}\right]\left[3 - \frac{\Omega_m}{(1-\Omega_0)}\right]}\times \\
\left\{F\left(\phi_0, \frac{\sqrt{33}+5}{2\sqrt{33}}\right) - F\left(\phi_z, \frac{\sqrt{33}+5}{2\sqrt{33}}\right) - 2\left[E\left(\phi_0, \frac{\sqrt{33}+5}{2\sqrt{33}}\right) - E\left(\phi_z, \frac{\sqrt{33}+5}{2\sqrt{33}}\right)\right]\right. \\
+2\,(33)^{(1/4)}\left[\frac{\sqrt{8 + [2 + \Omega_m/(1-\Omega_0)]^2}}{\sqrt{3 - \Omega_m/(1-\Omega_0)}\,[3 + \sqrt{33} - \Omega_m/(1-\Omega_0)]}\right. \\
\left.\left. - \frac{\sqrt{8 + [2 + (1+z)\Omega_m/(1-\Omega_0)]^2}}{\sqrt{3 - (1+z)\Omega_m/(1-\Omega_0)}\,[3 + \sqrt{33} - (1+z)\Omega_m/(1-\Omega_0)]}\right]\right\}.
\end{aligned} \tag{45}$$

The arguments of the elliptic functions, ϕ_z and ϕ_0, can be calculated from (14) using $y_1 = 3$ and $A = \sqrt{33}$. To reduce the number of elliptic functions needed to evaluate (45), addition formulas for $F(\phi, \mathrm{k})$ and $E(\phi, \mathrm{k})$ can be used [see (7), (48), and (49)]. The value of $\Delta\phi_z$ is given by (15).

III. $\nu = 2$, Empty-Beam Observations

A. Three Open Big Bang Domains

$$D_\ell(\Omega_m, \Omega_\Lambda, \nu = 2; z) \quad = \quad \frac{c}{H_0}(1+z)^2\int_0^z \frac{dz}{(1+z)^2\sqrt{(1+z)^2(1+\Omega_m z) - z(z+2)\Omega_\Lambda}}. \tag{46}$$

Like the $\nu = 1$ case this integral takes on different forms when evaluated in terms of Legendre elliptic integrals, depending on the value of the parameter b.

1. For the two open domains defined by $b < 0$ and $2 < b$ the luminosity distance D_ℓ takes the form:

$$
\begin{aligned}
D_\ell(\Omega_m, \Omega_\Lambda, \nu = 2; z) \quad &= \quad \frac{c}{H_0}\frac{(1+z)^2}{\Omega_\Lambda}\Bigg\{ \\
-(A+\kappa y_1)&\left[\frac{\sqrt{(1+z)^2(1+\Omega_m z) - z(z+2)\Omega_\Lambda}}{(1+z)[(1+z)\Omega_m/|1-\Omega_0| + A + \kappa y_1]} - \frac{1}{\Omega_m/|1-\Omega_0| + A + \kappa y_1}\right] \\
-\frac{(A-\kappa y_1)\sqrt{|1-\Omega_0|}}{2\sqrt{A}}&\left[F(\phi_z, \mathrm{k}) - F(\phi_0, \mathrm{k})\right] + \sqrt{A}\sqrt{|1-\Omega_0|}\left[E(\phi_z, \mathrm{k}) - E(\phi_0, \mathrm{k})\right]\Bigg\} \qquad (47)
\end{aligned}
$$

where y_1, A, k, and ϕ_z are defined in (10)-(14).[4] Just as with the result for the $\nu = 1$ case, i.e., (31), the number of Legendre elliptic integrals required to evaluate (47) can be reduced from four to two by using addition formulas 116.01 of Byrd & Friedman (1971). The addition formula for $E(\phi, \mathrm{k})$ is:

$$E(\phi_z, \mathrm{k}) - E(\phi_0, \mathrm{k}) = E(\Delta\phi_z, \mathrm{k}) - \mathrm{k}^2 \sin\phi_z \sin\phi_0 \sin\Delta\phi_z. \qquad (48)$$

For this case

$$
\begin{aligned}
-\mathrm{k}^2 \sin\phi_z \sin\phi_0 \sin\Delta\phi_z = &-\frac{2\,[2A + \kappa(1+3y_1)]}{[(1+z)\Omega_m/(1-\Omega_0) - y_1 - \kappa A]} \\
&\times \frac{\sqrt{[(1+z)\Omega_m/(1-\Omega_0) - y_1]\,[\Omega_m/(1-\Omega_0) - y_1]}}{[\Omega_m/(1-\Omega_0) - y_1 - \kappa A]\,[\tan(\Delta\phi_z/2) + 1/\tan(\Delta\phi_z/2)]}, \qquad (49)
\end{aligned}
$$

where an expression for $\tan(\Delta\phi_z/2)$ is given by (15).

2. For the domain $0 < b < 2$ the luminosity distance D_ℓ takes the form:

$$
\begin{aligned}
&D_\ell(\Omega_m, \Omega_\Lambda, \nu = 2; z) = \\
&\quad \frac{c}{H_0}\frac{(1+z)^2}{\Omega_\Lambda}\Bigg\{ -y_3\left[\frac{\sqrt{(1+z)^2(1+\Omega_m z) - z(z+2)\Omega_\Lambda}}{(1+z)[(1+z)\Omega_m/|1-\Omega_0| + y_3]} - \frac{1}{\Omega_m/|1-\Omega_0| + y_3}\right] \\
&\quad -\frac{y_2\sqrt{|1-\Omega_0|}}{\sqrt{y_1 - y_2}}\left[F(\phi_z, \mathrm{k}) - F(\phi_0, \mathrm{k})\right] - \sqrt{y_1 - y_2}\sqrt{|1-\Omega_0|}\left[E(\phi_z, \mathrm{k}) - E(\phi_0, \mathrm{k})\right]\Bigg\}, \qquad (50)
\end{aligned}
$$

where the constants y_1, y_2, y_3 and k are defined in (16)-(18) but the function ϕ_z is now defined as

$$\phi_z = \phi(\Omega_m, \Omega_\Lambda; z) \quad = \quad \sin^{-1}\sqrt{\frac{(1+z)\Omega_m/|1-\Omega_0| + y_2}{(1+z)\Omega_m/|1-\Omega_0| + y_3}}. \qquad (51)$$

For this case the value of $\Delta\phi_z$ needed to reduce the number of elliptic integrals is the NEGATIVE of that given by (20) for the $\nu = 0$ case. When the addition formula (48) is used, an additional

[4] $E(\phi, \mathrm{k}) \equiv \int_0^\phi \sqrt{1 - \mathrm{k}^2\sin^2\phi}\; d\phi$

term is contributed to (50) which can be evaluated using,

$$-\mathrm{k}^2 \sin\phi_z \sin\phi_0 \sin\Delta\phi_z = \frac{\Omega_m(y_1-y_3)|1-\Omega_0|^{(-3/2)}(y_1-y_2)^{(-1/2)}}{[(1+z)\Omega_m^2/(1-\Omega_0)^2-(2+z)y_1\Omega_m/(1-\Omega_0)-2y_1(1+y_1)]}$$
$$\times\left\{\frac{[y_2-\Omega_m/(1-\Omega_0)]\sqrt{(1+z)^2(1+\Omega_m z)-z(z+2)\Omega_\Lambda}}{[y_3-(1+z)\Omega_m/(1-\Omega_0)]}-\frac{[y_2-(1+z)\Omega_m/(1-\Omega_0)]}{[y_3-\Omega_m/(1-\Omega_0)]}\right\} \quad (52)$$

B. Boundaries

1. $\Omega_0 \equiv \Omega_m + \Omega_\Lambda = 1$

This case is the $b \to \pm\infty$ limit of (46) and a simpler expression containing hypergeometric functions results:

$$D_\ell(\Omega_m, \Omega_\Lambda = 1-\Omega_m, \nu = 2; z)$$
$$= \frac{c}{H_0}(1+z)^2\left\{1-\frac{1}{(1+z)\sqrt{1+\Omega_m z(3+3z+z^2)}}\right.$$
$$+\frac{3}{5}\,\Omega_m^{1/3}\left[\left(\frac{1}{[1+\Omega_m z(3+3z+z^2)]^{5/6}}\right)\,{}_2F_1\left(\frac{5}{6},\frac{1}{3};\frac{11}{6};\frac{1-\Omega_m}{1+\Omega_m z(3+3z+z^2)}\right)\right.$$
$$\left.\left.-{}_2F_1\left(\frac{5}{6},\frac{1}{3};\frac{11}{6};1-\Omega_m\right)\right]\right\}. \quad (53)$$

When $\Omega_m \neq 1$, (53) can be expressed in terms of associated Legendre functions as,

$$D_\ell(\Omega_m, \Omega_\Lambda = 1-\Omega_m, \nu = 2; z) =$$
$$\frac{c}{H_0}(1+z)^2\left\{1-\frac{1}{(1+z)\sqrt{1+\Omega_m z(3+3z+z^2)}}+\frac{\Gamma\,(5/6)}{2^{1/6}}\left[\frac{\Omega_m}{1-\Omega_m}\right]^{5/12}\right.$$
$$\left.\times\left[\frac{(1+z)^{1/4}}{\sqrt{1+\Omega_m z(3+3z+z^2)}}\mathrm{P}_{1/6}^{-5/6}\left(\sqrt{\frac{1+\Omega_m z(3+3z+z^2)}{\Omega_m(1+z)^3}}\right)-\mathrm{P}_{1/6}^{-5/6}\left(\frac{1}{\sqrt{\Omega_m}}\right)\right]\right\}. \quad (54)$$

When $\Omega_m \neq 1$, (53) can also be expressed in terms of Legendre elliptic integrals as,

$$D_\ell(\Omega_m, \Omega_\Lambda = 1-\Omega_m, \nu = 2; z) \quad = \quad \frac{c}{H_0}\frac{(1+z)^2}{1-\Omega_m}\left\{\right.$$
$$-(\sqrt{3}+1)\left(\Omega_m^{-1}-1\right)^{1/3}\left[\frac{\sqrt{1+\Omega_m z(3+3z+z^2)}}{(1+z)[1+z+(\sqrt{3}+1)\left(\Omega_m^{-1}-1\right)^{1/3}]}-\frac{1}{1+(\sqrt{3}+1)\left(\Omega_m^{-1}-1\right)^{1/3}}\right]$$
$$-\frac{1}{(\sqrt{3}+1)(3)^{1/4}}\sqrt{\Omega_m}\left(\Omega_m^{-1}-1\right)^{1/6}\left[F(\phi_z,\mathrm{k})-F(\phi_0,\mathrm{k})\right]$$
$$\left.+(3)^{1/4}\sqrt{\Omega_m}\left(\Omega_m^{-1}-1\right)^{1/6}\left[E(\phi_z,\mathrm{k})-E(\phi_0,\mathrm{k})\right]\right\}, \quad (55)$$

where the constant k is given by (24) and the functions ϕ_z and $\Delta\phi_z$ are given respectively by (25) and (26). For this case the additional term needed to use the addition formula (48) in (55) is:

$$
\begin{aligned}
&-\mathrm{k}^2 \sin\phi_z \sin\phi_0 \sin\Delta\phi_z \\
&= \frac{z\; 2(3)^{3/4}\left(2+\sqrt{3}\right)\sqrt{1-\Omega_m}}{\left[1+z+\left(1+\sqrt{3}\right)\left(\Omega_m^{-1}-1\right)^{1/3}\right]\left[1+\left(1+\sqrt{3}\right)\left(\Omega_m^{-1}-1\right)^{1/3}\right]} \\
&\times \frac{\left\{z+\left[1+\left(\Omega_m^{-1}-1\right)^{1/3}\right]\left[1+\sqrt{1+\Omega_m z(3+3z+z^2)}\right]\right\}}{\left\{2+3\,z\,\Omega_m+z^2\,\Omega_m\left[1+\left(1+\sqrt{3}\right)\left(\Omega_m^{-1}-1\right)^{1/3}\right]+2\sqrt{1+\Omega_m z(3+3z+z^2)}\right\}}. \qquad (56)
\end{aligned}
$$

2. $b = 2$

See the description for the $\nu = 0$ case in section I.B.2 including (44) for this "critical" value of b:

$$
\begin{aligned}
&D_\ell(\Omega_m, \Omega_\Lambda(\Omega_m), \nu = 2; z) \\
&= \frac{c}{H_0}\frac{9\,\Omega_m\,(1+z)^2}{2|1-\Omega_0|^{3/2}}\left\{\frac{1}{(1+z)}\sqrt{\frac{1}{3}-\frac{(1+z)\Omega_m}{1-\Omega_0}}-\sqrt{\frac{1}{3}-\frac{\Omega_m}{1-\Omega_0}}\right. \\
&\left.+\frac{\Omega_m}{1-\Omega_0}\log\left[\frac{1+\sqrt{1/3-(1+z)\Omega_m/(1-\Omega_0)}}{1+\sqrt{1/3-\Omega_m/(1-\Omega_0)}}\sqrt{\frac{2/3+\Omega_m/(1-\Omega_0)}{2/3+(1+z)\Omega_m/(1-\Omega_0)}}\right]\right\}. \qquad (57)
\end{aligned}
$$

3. $\Omega_\Lambda = 0$

This result was first given by Dyer & Roeder (1972),

$$
\begin{aligned}
&D_\ell(\Omega_m, \Omega_\Lambda = 0, \nu = 2; z) \\
&\quad = \frac{c}{H_0}\frac{\Omega_m(1+z)^2}{4(1-\Omega_m)^{3/2}}\left[\frac{3\Omega_m}{2(1-\Omega_m)}\ln\left\{\left(\frac{1+\sqrt{1-\Omega_m}}{1-\sqrt{1-\Omega_m}}\right)\left(\frac{\sqrt{1+\Omega_m z}-\sqrt{1-\Omega_m}}{\sqrt{1+\Omega_m z}+\sqrt{1-\Omega_m}}\right)\right\}\right. \\
&\quad \left.+\frac{3}{\sqrt{1-\Omega_m}}\left(\frac{\sqrt{1+\Omega_m z}}{1+z}-1\right)+\frac{2\sqrt{1-\Omega_m}}{\Omega_m}\left(1-\frac{\sqrt{1+\Omega_m z}}{(1+z)^2}\right)\right], \qquad (58)
\end{aligned}
$$

and can be rewritten using the identity

$$\sinh^{-1}\sqrt{\frac{1-\Omega_m}{\Omega_m(1+z)}} = \frac{1}{2}\ln\left(\frac{\sqrt{1+\Omega_m z}+\sqrt{1-\Omega_m}}{\sqrt{1+\Omega_m z}-\sqrt{1-\Omega_m}}\right). \qquad (59)$$

When $\Omega_m > 1$ equation (58) is analytically continued using $\sqrt{1-\Omega_m} \longrightarrow \pm i\sqrt{\Omega_m - 1}$, which simplifies by using, $\sinh^{-1}(ix) = i\sin^{-1}(x)$ to give a form containing only real variables. The $\Omega_m = 1$ result for all ν was given by Dashevskii & Slysh (1966):

$$D_\ell(\Omega_m = 1, \Omega_\Lambda = 0, \nu; z) = \frac{c}{H_0}\frac{1}{(\nu+\frac{1}{2})}\left[(1+z)^{(\frac{\nu}{2}+1)} - (1+z)^{(-\frac{\nu}{2}+\frac{1}{2})}\right]. \qquad (60)$$

4. $\Omega_m = 0$

This result is exactly the same as the $\nu = 0$ and $\nu = 1$ result (29). If there is no mass in the universe then removing 100% of no mass from the beam removes nothing.

4. Conclusions

We have given useful forms for the luminosity distance in three currently relevant cosmologies. They are all dynamically FLRW cosmologies in the large but differ in how gravitating matter effects optical observations. The models are labeled by an additional parameter ν ($\nu = 0$, 1, and 2) beyond the familiar H_0, Ω_m, and Λ. The $\nu = 0$ model is standard FLRW where all matter is homogeneous and transparent on the scale of the observing beam widths. This model is called the 'filled-beam' model. The $\nu = 2$ model assumes the opposite; all matter is inhomogeneous and excluded from the observing beams. This extreme case is called the 'empty-beam' model. The $\nu = 1$ model assumes that 1/3 of the mass density of the universe is excluded from observing beams and hence it is the 'two-thirds filled-beam' model. These three cases were singled out because their distance-redshift relations can be given in terms of incomplete elliptic integrals; functions which are universally available in computer libraries and very efficiently evaluated.[5] For the $\nu = 1$ and 2 cases, somewhat simpler expressions than what we have given exist, but only for complex arguments of the elliptic integrals. We chose to give expressions whose arguments are real and which can be rapidly evaluated. Results are available for all $0 \leq \nu \leq 2$ but only in terms of the less familiar and unavailable Heun functions, Kantowski (1998a). We have extended the flat space, $\Omega_0 = 1$, results given here to arbitrary filling parameter ν. These new results will be available shortly. Related results have been independantly found by Damianski et al. (2000). A calculation similar to the $\nu = 0$ case given here is that of the age of the Universe as a function of redshift and can be found in Thomas & Kantowski (2000).

R. Kantowski wishes to thank VP for Research, E. Smith, for funds to support J.K. Kao's visit to OU during the summer of 1998 when the first elliptic integral results were obtained. R. C. Thomas thanks P. Helbig for discussions of his code, see Kayser et al. (1997), and E. Baron for benchmarking discussions.

A. Appendix

One expected practical use of the results given in this paper is to speedup distance evaluations for the $\nu =$ 0,1,2 partially filled beam FLRW models. We have implemented and made publicly avail-

[5]The results appearing in Section 3 have been coded and are posted at http://www.nhn.ou.edu~thomas/z2dl.html. This code is discussed in the Appendix and compared to the numerical integration times of Kayser et al. (1997).

able a FORTRAN 90 version of this work called Z2DL (see http://www.nhn.ou.edu∼thomas/z2dl.html for Z2DL with documentation and extensive CPU-time benchmark results). Z2DL uses Carlson elliptic integrals (see Press et al. (1994) and references therein) and results in a fast distance calculator. We have benchmarked Z2DL by comparing it with the commonly used and fast numerical integration routine ANGSIZ (see Kayser et al. (1997)). For a given $(\Omega_m, \Omega_\Lambda)$, the total CPU-time required to convert 5×10^5 redshifts (equally spaced between z=0 and z=5) to luminosity distance using Z2DL and ANGSIZ separately were recorded. By calculating the ratio of ANGSIZ CPU-time to Z2DL CPU-time on a grid of points in $(\Omega_m, \Omega_\Lambda)$ we have generated three speedup surfaces, one for each value of ν = 0,1,2 (see Fig. 3 for the $\nu = 0$ surface). The results for all three comparisons are given as contour plots at the web site. Using an IBM AIX 375 MHz Power III approximately 7 hours was required to generate each $(\Omega_m, \Omega_\Lambda)$ grid of 30 x 30 points (minus models without a big bang).

For the purpose of a clearer presentation, we omitted speedup points along the $\Omega_m = 0$ and $\Omega_\Lambda = 0$ lines. Along these boundaries speedup factors are greater than 100. The large open domains of the Ω_m-Ω_Λ plane, i.e., subsection 'A' cases, constitute the majority of models in the grid and also those with the least impressive speedup. However, even for these cases, the improvement is substantial: typically 17-20 for $\nu = 0$ (standard filled beam FLRW), 6-8 for $\nu = 1$ (66% filled beam FLRW), and 11-13 for $\nu = 2$ (empty beam FLRW).

To gauge the level of agreement between distances computed by ANGSIZ and Z2DL, a finer grid of $(\Omega_m, \Omega_\Lambda)$ with 3000 x 3000 points (between 0 and 3 in both directions, also excluding models without a big bang) was used. For each $(\Omega_m, \Omega_\Lambda)$, both routines were used to compute luminosity distance for z=1. Most often the results agree to within one part in 10^6. Cases where disagreements greater than one part in 10^3 occur are near the upper b=2 line (see Fig. 1). We found that ANGSIZ was giving less accurate distances near this boundary of non-big bang models as ANGSIZ documentation explains.

REFERENCES

Bertotti, B. 1966, Proc. Roy. Soc. London, A, 294, 195

Byrd, P. F. & Friedman, M. D. 1971, Handbook of Elliptic Integrals for Engineers & Scientists (New York : Springer-Verlag)

Damianski, M., de Ritis, R., Marino, A. A., & Piedipalumbo, E., astro-ph/0004376

Dashevskii, V. M., & Slysh, V. I. 1966, Soviet Ast.–AJ, 9, 671

Dyer, C. C., & Roeder, R. C. 1972, ApJ, 174, L115

Dyer, C. C., & Roeder, R. C. 1973, ApJ, 180, L31

Edwards, D. 1972, MNRAS, 159, 51

Feige, B. 1992, Astron. Nachr., 313, 139

Felton, J. E. & Isaacman, R. 1986, Rev. Mod. Phys., 58, 689

Holz, D. E., & Wald, R. M. 1998, Phys. Rev. D, 58, 063501

Holz, D. E. 1998, ApJ, 506, L1

Kantowski, R. 1969, ApJ, 155, 89

Kantowski, R. 1998a, ApJ, 507, 483

Kantowski, R. 1998b, in Sources and Detection of Dark Matter in the Universe, edited by D.B. Cline (Amsterdam: Elsevier Press).

Kaufman, S. E. & Schucking, E. L. 1971, AJ, 76, 583

Kaufman, S. E. 1971, AJ, 76, 751

Kayser, R., Helbig, P., & Schramm, T. 1997, A&A, 318, 680

Lemaître, A. G. 1931, MNRAS, 91, 483

Mattig, W. 1958, Astro. Nach. 284, 109

McVittie, G. C. 1965, General Relativity and Cosmology, (Urbana: The University of Illinois Press)

Perlmutter, S. et al. 1999, ApJ, 517, 565

Press, W., Teukolsky, S., Vetterling, W., & Flannery, B. 1994, Numerical Recipes (Cambridge: University Press)

Robertson, H. P. 1933, Rev. Mod. Phys. 5, 62

Schmidt, B. P. et al. 1998, ApJ, 507, 46

Thomas, R. C. & Kantowski, R. 2000, astro-ph/0002334

Wang, Y. 1999, astro-ph/9907405

Whittaker, E. T., & Watson, G. N. 1927, A Course in Modern Analysis (Cambridge: Cambridge University Press)

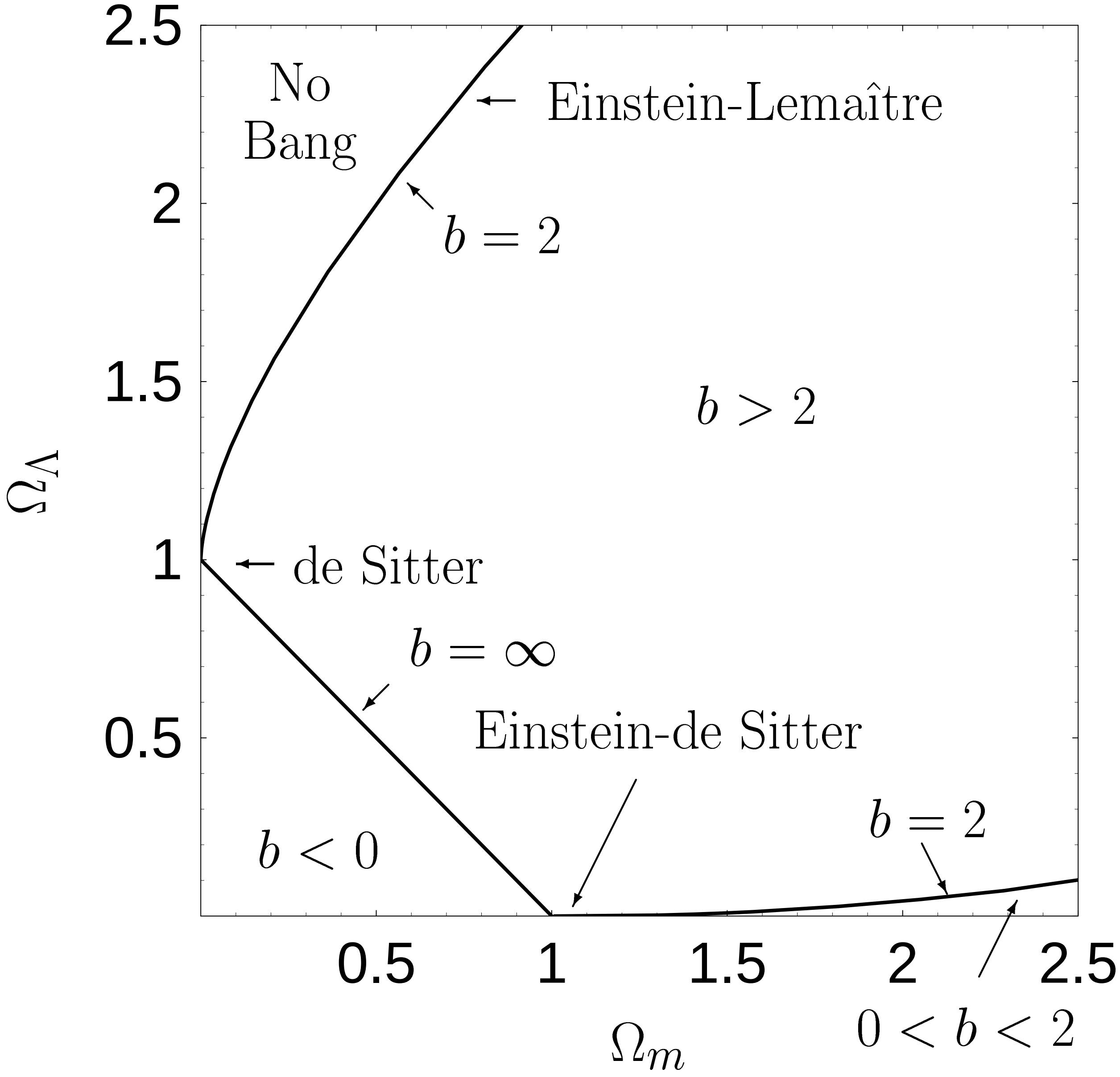
Zel'dovich, Ya. B. 1964, Soviet Ast.–AJ, 8, 13

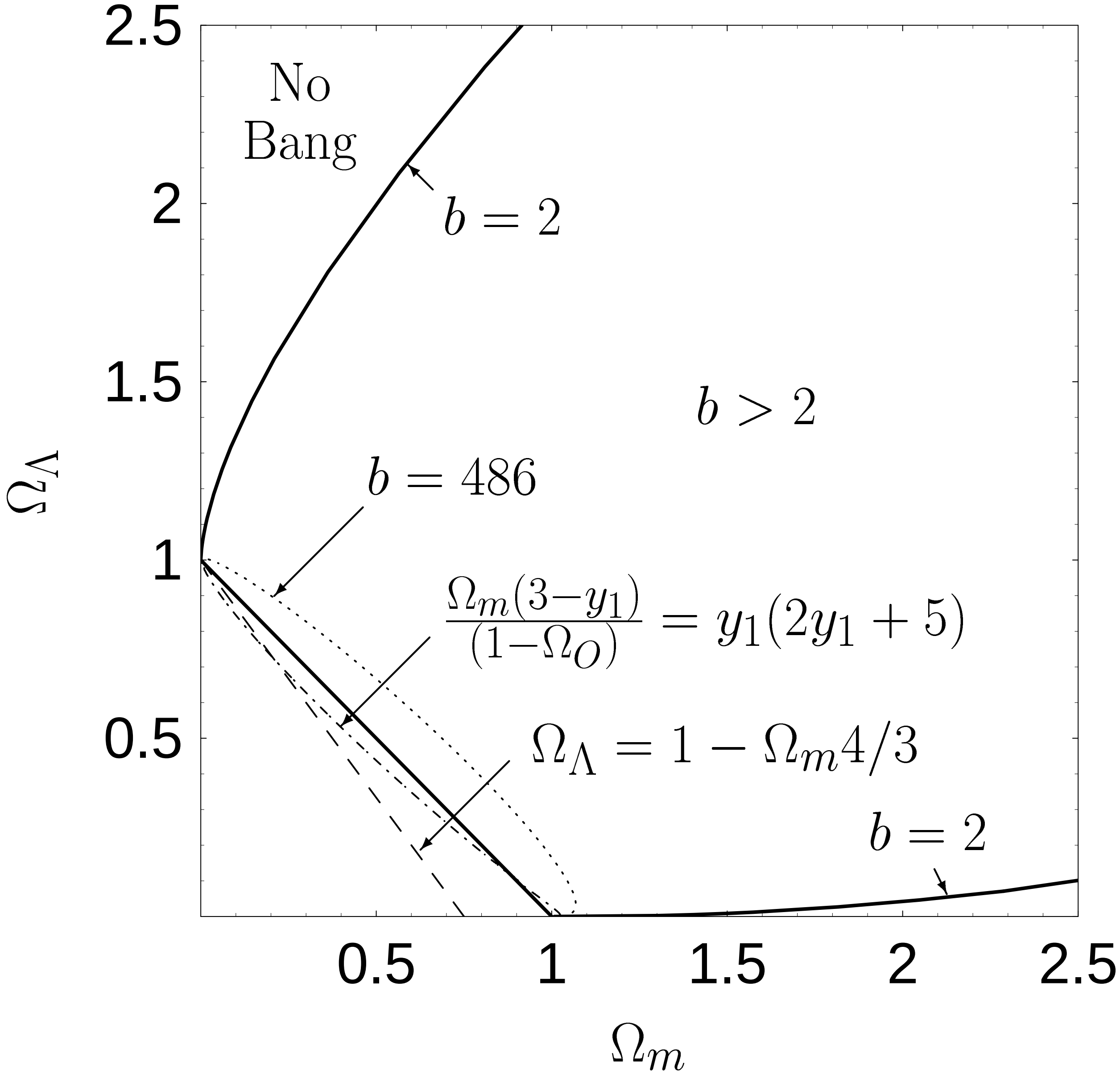
This preprint was prepared with the AAS LaTeX macros v5.0.

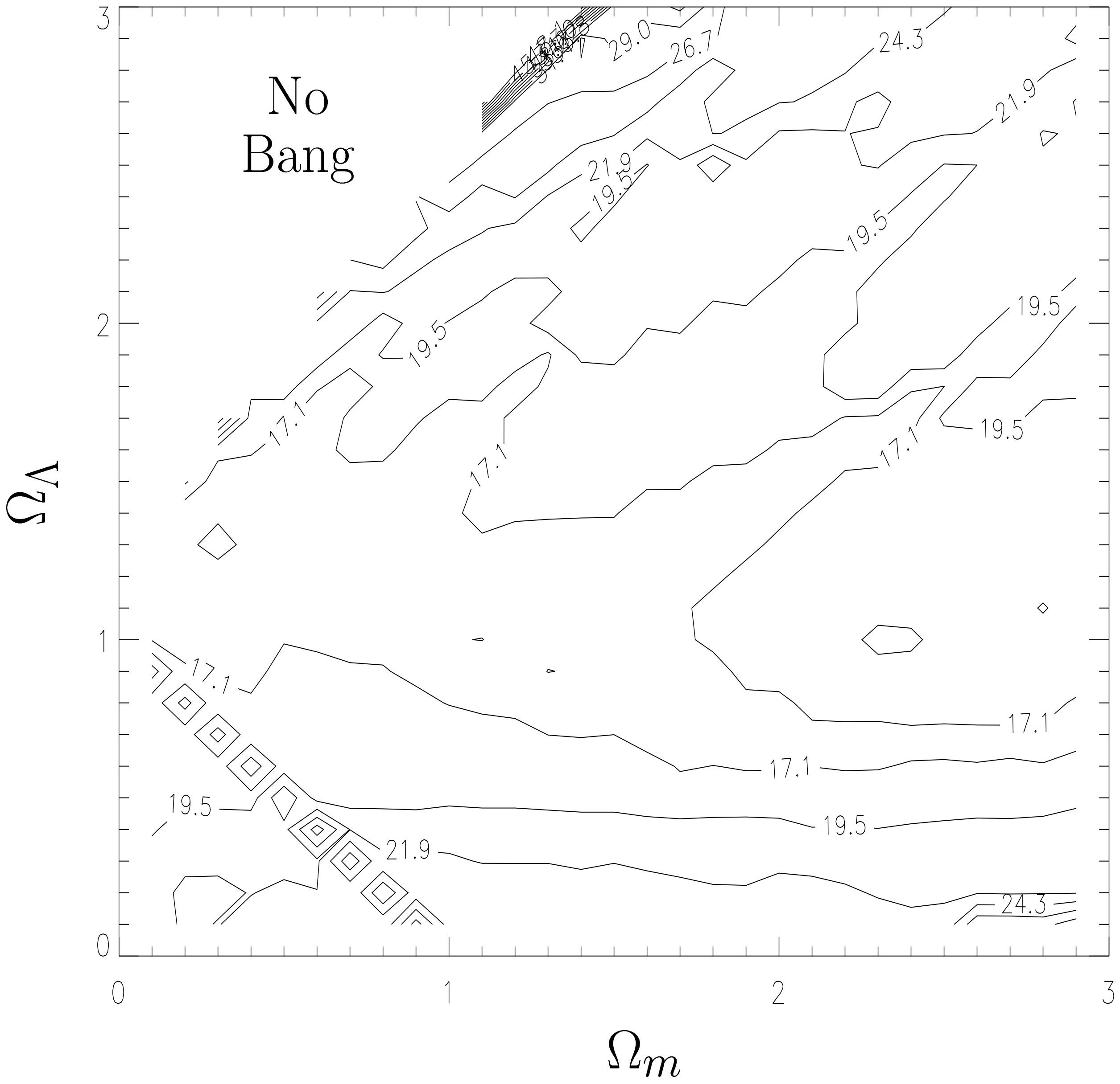
Fig. 1.— The Ω_m-Ω_Λ plane showing various b domains that require different expressions for distance-redshift D_ℓ for all three cases: $\nu = 0, 1, 2$ i.e., filled-beam, 66% filled-beam, and empty-beam.

Fig. 2.— Additional domains in the Ω_m-Ω_Λ plane for $\nu = 1$, i.e., for 66% filled-beam observations, where complications due to divergent terms occur in the analytic results. For Ω_m– Ω_Λ values on the dashed and dot-dashed lines, define respectively by $\Omega_\Lambda = 1 - \Omega_m 4/3$ and $\Omega_m(3-y_1)/(1-\Omega_0) = y_1(2y_1+5)$, expression (31) must be evaluated by taking a numerical limit. For points to the left of the straight dashed line and points between the dot-dashed and $b = 486$ curves, a single value of z exits for which (31) also diverges. These z values are defined respectively by $(z+1) = 3(1-\Omega_0)/\Omega_m$ and $(1+z)\Omega_m(3-y_1)/(1-\Omega_0) = y_1(5+2y_1)$. For Ω_m, Ω_Λ, and z satisfying either equation a limiting process must be used to evaluate D_ℓ via (31), see the Appendix. For points on the divergent $b = 486$ curve an analytic limit was obtained in (45).

Fig. 3.— Contour plot of the Ω_m-Ω_Λ plane showing speedup factors for Z2DL over ANGSIZ when $\nu = 0$ (standard filled beam FLRW cosmology). Speedup factors for the other two cases considered in this paper, $\nu = 1, 2$ i.e., the 66% filled-beam and empty-beam can be found at the web site.


2.5
2
1.5
1
0.5
Ω_Λ
No Bang
Einstein-Lemaître
$b = 2$
$b > 2$
de Sitter
$b = \infty$
Einstein-de Sitter
$b = 2$
$b < 0$
0.5
1
1.5
2
2.5
Ω_m
$0 < b < 2$


2.5
2
1.5
1
0.5
No Bang
$b = 2$
$b > 2$
$b = 486$
$\frac{\Omega_m(3-y_1)}{(1-\Omega_O)} = y_1(2y_1+5)$
$\Omega_\Lambda = 1 - \Omega_m 4/3$
$b = 2$
0.5
1
1.5
2
2.5
Ω_Λ
Ω_m


No
Bang
29.0
26.7
24.3
21.9
19.5
17.1
0
1
2
3
Ω_Λ
Ω_m