The Age-Redshift Relation for Standard Cosmology

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Abstract

We present compact, analytic expressions for the age-redshift relation $\tau(z)$ for standard Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology. The new expressions are given in terms of incomplete Legendre elliptic integrals and evaluate much faster than by direct numerical integration.

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I. INTRODUCTION

Since Type Ia supernova observations [19, 14] have favored a Universe with $\Lambda \neq 0$, interest in FLRW cosmologies has grown. For this family of models most observational relations, e.g., the Hubble curve, are given by integral expressions; however, Kantowski et al. [9] recently succeeded in giving useful analytic distance-redshift relations for them. In this paper we extend these analytic results to include the age-redshift relation $\tau(z)$. These new expressions are useful for any computation that requires a transformation $\tau(z)$ from the observed redshift variable z to the age τ of the Universe at that z. Feige [7] provides related light travel times in terms of Legendre elliptic integrals; however, his expressions are not easy to make use of. A presentation closer to what we give appears in [6]. There, light travel time as a function of redshift was given for the $\Omega_0 = 1$ universe, see (18). For all other cases, Edwards [6] was only able to give τ and z parametrically as Jacobi elliptic functions of conformal time $\omega \equiv \int dt/R(t)$.[1]

We have concentrated on giving useful and valid expressions for $\tau(z)$ appropriate for <u>all</u> big bang models in the first quadrant of the Ω_m–Ω_Λ plane. Because the incomplete Legendre elliptic integrals have branch points, more than one expression is necessary to completely cover this domain, e.g., see (8) and (10). These new expressions should be quite useful for everything from gravitational lensing to high z evolution studies. As an example [13, 18, 17] and [5] all estimate event rates of supernovae at very high ($z > 1$) redshifts, given an observed star formation rate. Computing such event rates for any choice of (Ω_m,Ω_Λ) requires the transformation $\tau(z)$. We have made similar estimates of event rates and find that our computations are reduced from hours down to minutes when our analytic $\tau(z)$ is used.[2] In §2 we present our results and in §3 some conclusions.

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[1] The error detected when numerical checks were performed by Campusano et al. [3] was evidently caused by an error in equation 361.54 of Byrd & Friedman [2], see footnote 3 of [9]

[2] FORTRAN 90 and *Mathematica* implementations of the results presented here are available at http://www.nhn.ou.edu/~thomas/z2t.html.

II. AGE OF THE UNIVERSE IN TERMS OF LEGENDRE ELLIPTIC INTEGRALS

The expression for the age of the Universe at the time a source at redshift z emits light is

$$\tau(\Omega_m, \Omega_\Lambda; z) = \frac{1}{H_0} \int_z^\infty \frac{dz}{(1+z)\sqrt{(1+z)^2(1+\Omega_m z) - z(2+z)\Omega_\Lambda}}\,, \tag{1}$$

and can easily be derived. For a Friedmann-Robertson-Walker (FRW) universe, i.e., $\Omega_\Lambda = 0$, (1) can be integrated in terms of elementary functions,

$$\tau(\Omega_m, \Omega_\Lambda = 0; z) = \frac{1}{H_0} \left[\frac{\sqrt{1+\Omega_m z}}{(1-\Omega_m)(1+z)} - \frac{\Omega_m}{(1-\Omega_m)^{3/2}} \sinh^{-1} \sqrt{\frac{\Omega_m^{-1} - 1}{1+z}} \right]. \tag{2}$$

This expression is also valid for the Einstein-de Sitter universe, i.e., limit $\Omega_m \to 1$, as well as when $\Omega_m > 1$. For massless big bang models, $\Omega_m = 0$ and $0 < \Omega_\Lambda < 1$, the integral is, see [11, 16]:

$$\tau(\Omega_m = 0, \Omega_\Lambda; z) = \frac{1}{H_0\sqrt{\Omega_\Lambda}} \sinh^{-1} \left[\frac{1}{(1+z)\sqrt{\Omega_\Lambda^{-1} - 1}} \right]. \tag{3}$$

When $\Omega_\Lambda \neq 0$ and $\Omega_m \neq 0$, (1) becomes an incomplete elliptic integral and can at best be expressed as a combination of the three independent Legendre elliptic integrals $F(\phi, \mathrm{k})$, $E(\phi, \mathrm{k})$, and $\Pi(\phi, \alpha^2, \mathrm{k})$.[3] The form of the resulting expression depends on what portion of the Ω_m–Ω_Λ plane is being investigated. Because the cubic under the radical in (1) is the same as that contained in integrals for the luminosity distance as given by [9] and [10] a similar analysis is required. Below we outline results, hoping to make our expressions easy to use.

A. $\Omega_0 = \Omega_m + \Omega_\Lambda \neq 1$

As seen below, $\tau(z)$ depends on (Ω_m, Ω_Λ) primarily through a single parameter b

$$b \equiv -(27/2)\frac{\Omega_m^2 \Omega_\Lambda}{(1-\Omega_0)^3}. \tag{4}$$

[3] Only two of the three are needed to give $\tau(z)$ and they are defined by: $F(\phi, \mathrm{k}) \equiv \int_0^\phi 1/\sqrt{1-\mathrm{k}^2 \sin^2\phi}\, d\phi$ and $\Pi(\phi, \alpha^2, \mathrm{k}) \equiv \int_0^\phi 1/\left[(1-\alpha^2\sin^2\phi)\sqrt{1-\mathrm{k}^2\sin^2\phi}\,\right]\, d\phi$. The particular integrals needed can be found in [2].

This parameter divides the Ω_m–Ω_Λ plane (see Fig. 1) into four domains where the results of integrating (1) differ. We will ignore one of the four domains and its $b = 2$ boundary where big bangs don't occur. In the following we use the familiar parameter $\kappa \equiv (\Omega_0 - 1)/|\Omega_0 - 1|$, which is determined by the sign of the 3-curvature, to distinguish between open and closed models. When $b < 0$, $\kappa = -1$ and when $b > 0$, $\kappa = +1$. Results for special boundaries other than $\Omega_\Lambda = 0$, i.e., (2) and $\Omega_m = 0$, i.e., (3), are given in subsection **B** below. The three special boundaries needed are: $b = \pm\infty$ i.e., $\Omega_0 \equiv \Omega_m + \Omega_\Lambda = 1$; $b = 2$; and $b = 27(2+\sqrt{2})/8$.

1. Results for the two large domains, $b < 0$ and $2 < b$, can be combined by defining intermediate constants v_κ, y_1 and A:

$$v_\kappa \equiv \left(\kappa(b-1) + \sqrt{b(b-2)}\right)^{1/3}, \tag{5}$$

$$y_1 \equiv \frac{-1 + \kappa(v_\kappa + v_\kappa^{-1})}{3}, \tag{6}$$

$$A \equiv \sqrt{y_1(3y_1 + 2)}. \tag{7}$$

These constants depend on b alone and are only used for convenience of presentation. If the reader desires, the source of these parameters can be found in [10]. For this case we give two expressions for the integral (1). Both are valid except for special combinations of Ω_m, Ω_Λ, and z. If one fails the other can be used. These expressions fail when $\Pi(\phi, \alpha^2, \mathrm{k})$ and the logarithm have canceling infinities. Both can simultaneously fail only when Ω_m, Ω_Λ values are on the $b = 27(2+\sqrt{2})/8$ curve and then only for a specific value of z (see Figs. 2 and 3). This special b case is given in (22) and is good for any z. The first expression is:

$$\tau(\Omega_m, \Omega_\Lambda; z) = \frac{\Omega_m}{H_0|1-\Omega_0|^{3/2}}\Bigg[\frac{1}{2\kappa y_1\sqrt{A}}\, F(\phi_z, \mathrm{k}) + \frac{A-\kappa}{2y_1(1+y_1)\sqrt{A}}\, \Pi\Big(\phi_z, \frac{y_1(1+y_1)}{(A-\kappa y_1)^2}, \mathrm{k}\Big) + \frac{1}{2\kappa y_1\sqrt{\kappa(y_1+1)}}\, \ln(h_z^-/h_z^+)\Bigg], \tag{8}$$

where

$$h_z^\mp \equiv \mp\, 2\kappa y_1\sqrt{(1+y_1)\{y_1^2(1+y_1) - [(1+z)\Omega_m/(1-\Omega_0)]^2[1+(1+z)\Omega_m/(1-\Omega_0)]\}} + [(1+z)\Omega_m/(1-\Omega_0)]^2(A-\kappa y_1) - 2\kappa y_1^2(1+y_1). \tag{9}$$

The second expression is obtained from the first by using a "special addition formula" analytically extended from a corrected version of equation 17.03 of [2]. This transformation changes the α^2 value of $\Pi(\phi, \alpha^2, \mathrm{k})$ and hence moves the associated branch point. The resulting second expression is:

$$\tau(\Omega_m, \Omega_\Lambda; z) = \frac{\Omega_m}{H_0|1-\Omega_0|^{3/2}} \frac{1}{\sqrt{A}} \Big[-\frac{F(\phi_z, \mathrm{k})}{A+\kappa y_1} - \frac{A-\kappa y_1}{2\kappa y_1(A+\kappa y_1)} \Pi\Big(\phi_z, \frac{(A+\kappa y_1)^2}{4A\kappa y_1}, \mathrm{k}\Big) - \frac{\sqrt{A}}{2\kappa y_1\sqrt{\kappa(y_1+1)}} \ln\Big(\frac{1-h_z}{1+h_z}\Big)\Big], \tag{10}$$

where

$$h_z \equiv \sqrt{\frac{(1+y_1)[y_1-(1+z)\Omega_m/(1-\Omega_0)]}{y_1^2+[1+(1+z)\Omega_m/(1-\Omega_0)][y_1+(1+z)\Omega_m/(1-\Omega_0)]}}. \tag{11}$$

In both cases k and ϕ_z, respectively the modulus and argument of the elliptic integrals, are defined by:

$$\mathrm{k} \equiv \sqrt{\frac{2A+\kappa(1+3y_1)}{4A}}, \tag{12}$$

$$\phi_z \equiv \cos^{-1}\Big(\frac{\kappa y_1 + (1+z)\Omega_m/|1-\Omega_0| - A}{\kappa y_1 + (1+z)\Omega_m/|1-\Omega_0| + A}\Big). \tag{13}$$

In (8) and (10) the z dependence of τ is contained in ϕ_z, $h_z^{\mp}$, and h_z. All other terms depend on Ω_m and Ω_Λ, and are easily evaluated using (4)-(7). In Figures 2 and 3 the dotted lines show points where the first expression (8) fails for $z = 1$ and $z = 2$ respectively. Failure of the second expression (10) is shown by the dashed lines. Notice that these curves always intersect somewhere on the $b = 27(2+\sqrt{2})/8$ curve for a common redshift.

2. If $0 < b \leq 2$, we define the three different intermediate parameters y_1, y_2 and y_3

$$\begin{aligned} y_1 &\equiv -\frac{1}{3} + \frac{1}{3}\cos\Big(\frac{\cos^{-1}(1-b)}{3}\Big) + \frac{1}{\sqrt{3}}\sin\Big(\frac{\cos^{-1}(1-b)}{3}\Big), \\ y_2 &\equiv -\frac{1}{3} - \frac{2}{3}\cos\Big(\frac{\cos^{-1}(1-b)}{3}\Big), \\ y_3 &\equiv -\frac{1}{3} + \frac{1}{3}\cos\Big(\frac{\cos^{-1}(1-b)}{3}\Big) - \frac{1}{\sqrt{3}}\sin\Big(\frac{\cos^{-1}(1-b)}{3}\Big). \end{aligned} \tag{14}$$

For this case (1) integrates to give

$$\tau(\Omega_m, \Omega_\Lambda; z) = \frac{\Omega_m}{H_0(\Omega_0-1)^{3/2}} \frac{2}{y_1\sqrt{y_1-y_2}} \Big[\Pi\Big(\phi_z, \frac{y_1}{y_1-y_2}, \mathrm{k}\Big) - F(\phi_z, \mathrm{k})\Big], \tag{15}$$

where k and ϕ_z are defined by

$$k \equiv \sqrt{\frac{y_1 - y_3}{y_1 - y_2}}, \tag{16}$$

$$\phi_z \equiv \sin^{-1} \sqrt{\frac{y_1 - y_2}{y_1 - (1+z)\Omega_m/(1-\Omega_0)}}\,. \tag{17}$$

In (15) the z dependence of τ is contained in ϕ_z. All other terms depend on Ω_m and Ω_Λ, and are easily evaluated using (4) and (14). There are two domains in the Ω_m–Ω_Λ plane where $0 < b \le 2$; however, the result for this case (15) applies only to those models which have big bangs.

B. Special Cases

1. $\Omega_0 = \Omega_m + \Omega_\Lambda = 1$

This is the spatially flat model ($b \to \pm\infty$) and for it the age-redshift integral takes on a simpler form. This result is easily obtained using elementary integration methods. This result is well known:

$$\begin{aligned}\tau(\Omega_m, \Omega_\Lambda = 1 - \Omega_m; z) &= \frac{1}{H_0} \int_z^\infty \frac{dz}{(1+z)\sqrt{1 + \Omega_m z(3 + 3z + z^2)}} \\ &= \frac{2}{3H_0\sqrt{1-\Omega_m}}\ \sinh^{-1}\left(\sqrt{\frac{\Omega_m^{-1} - 1}{(1+z)^3}}\right).\end{aligned} \tag{18}$$

2. $b = 2$

This value of b can be identified with "critical" values of the cosmic parameters [8]. The following result is equivalent to the $b = 2$ value given in (15); however, it is a much simpler expression:

$$\begin{aligned}\tau(\Omega_m, \Omega_\Lambda(\Omega_m); z) = \frac{1}{H_0\sqrt{\Omega_\Lambda}} \ln\ &\left[\left(\frac{\sqrt{1/3 - (1+z)\Omega_m/(1-\Omega_0)} + 1}{\sqrt{1/3 - (1+z)\Omega_m/(1-\Omega_0)} - 1}\right)^{1/\sqrt{3}}\right. \\ &\left.\times \left(\frac{\sqrt{1 - 3(1+z)\Omega_m/(1-\Omega_0)} - 1}{\sqrt{1 - 3(1+z)\Omega_m/(1-\Omega_0)} + 1}\right)\right].\end{aligned} \tag{19}$$

This $\tau(z)$ doesn't apply to the Einstein-Lemaître universe ($b = 2$ where $\Omega_\Lambda > \Omega_m/2$) [12], which starts expanding from the finite static Einstein radius at $t = -\infty$. However, it does apply to the $\Omega_\Lambda < \Omega_m/2$ models which start with a big bang and expand to the Einstein radius at $t = +\infty$. In the Ω_m–Ω_Λ plane the static Einstein universe itself is a point at ∞ on the two $b = 2$ curves where $\Omega_m/\Omega_\Lambda \to 2$. If wanted, the $b = \text{constant} \geq 2$ curves can be drawn using the following expressions. Because $\Omega_\Lambda(\Omega_m)$ is double valued, two expressions must be given. For the upper part of the curve:

$$\Omega_0 - 1 = 3\sqrt{2/b}\;\Omega_m\;\cosh\left[\frac{\cosh^{-1}\left[\sqrt{b/2}\;(\Omega_m^{-1} - 1)\right]}{3}\right], \tag{20}$$

where $0 \leq \Omega_m \leq 1/(1 - \sqrt{2/b})$. In this expression hyperbolic cosine analytically becomes cosine for $\Omega_m \geq 1/(1 + \sqrt{2/b})$.

For the lower part of the curve:

$$\Omega_0 - 1 = 3\sqrt{2/b}\;\Omega_m\;\cos\left[\frac{\cos^{-1}\left[\sqrt{b/2}\;(1 - \Omega_m^{-1})\right] + \pi}{3}\right], \tag{21}$$

where $1 \leq \Omega_m \leq 1/(1-\sqrt{2/b})$. For $b = 2$ (see Fig. 1) the max value of Ω_m is '∞' (the static Einstein universe); however, for the next case (see Fig. 3) the upper and lower parts of the curve meet at finite $\Omega_m \approx 1.7$.

3. $b = 27(2 + \sqrt{2})/8$

This result is equivalent to the values given by (8) and (10) except for certain redshifts where canceling infinities appear in $\Pi(\phi, \alpha^2, \mathrm{k})$ and the respective logarithms. It is a simpler expression and is valid for all z values, for this particular b,

$$\tau(\Omega_m, \Omega_\Lambda(\Omega_m); z) = \frac{1}{H_0\sqrt{\Omega_\Lambda}}\Big[\frac{\sqrt{2}-1}{4}\;F\left(\phi_z, \frac{\sqrt{1+2\sqrt{2}}}{2}\right) + \frac{1}{4}\ln(h_z^+/h_z^-)\Big], \tag{22}$$

where

$$h_z^\pm \equiv [1 - (1+z)\Omega_m/(1-\Omega_0)]\sqrt{2[(1+z)\Omega_m/(1-\Omega_0)]^2 + [\sqrt{2}(1+z)\Omega_m/(1-\Omega_0) + 1](\sqrt{2}+1)}$$
$$\pm(\sqrt{2}-1)[(1+z)\Omega_m/(1-\Omega_0) + \sqrt{2} + 1]\sqrt{(\sqrt{2}+1)[1 - \sqrt{2}(1+z)\Omega_m/(1-\Omega_0)]}, \tag{23}$$

and

$$\phi_z \equiv \cos^{-1}\Big(\frac{-1 - (1+z)\Omega_m/(1-\Omega_0)}{\sqrt{2} + 1 - (1+z)\Omega_m/(1-\Omega_0)}\Big). \tag{24}$$

We found it necessary to compute $\tau(z)$ for this particular b value to overcome the occasional simultaneous failures of (8) and (10), see Figures 2 and 3.

III. CONCLUSIONS

We have given valid analytic expressions for $\tau(z)$ in FLRW, the age of the Universe as a function of redshift, which are relatively simple and are quite useful when a fast computer implementation is needed.[4] These expressions completely cover the big bang models of the first quadrant of the $(\Omega_m, \Omega_\Lambda)$ plane. If lookback times are wanted they can additionally be obtained from results given here by simply evaluating $\tau(0) - \tau(z)$. Readers that are interested in adding radiation pressure as a source of gravity should see [1] and [4] and cited references.

Even though we give several expressions for $\tau(z)$, most of the $(\Omega_m, \Omega_\Lambda)$ plane, which includes currently favored values, is covered by case A1, i.e., result (8) or (10). If $\tau(z)$ for the flat model, $\Omega_0 = 1$, is wanted, the simpler result (18) should be used. Results for $\Omega_0 \neq 1$, (8), (10), and (15), appear complicated because of the presence of extra constants, e.g., A and y_1 that have been retained to compactify formulas. The reader should keep in mind that these are simply constants that depend on $(\Omega_m, \Omega_\Lambda)$ through the single combination b of (4). We could have eliminated these auxiliary constants and given $\tau(z)$ directly in terms of the two parameters Ω_m and Ω_Λ; however, such expressions would take up more than a page.

Expressions (8) and (10) for $\tau(z)$ remain real but as presented can contain imaginary terms because of branch points. The threshold is defined by $1-\alpha^2 \sin^2 \phi_z = 0$ in $\Pi(\phi_z, \alpha^2, \mathrm{k})$. If $1-\alpha^2 \sin^2 \phi_z < 0$ canceling imaginary terms appear in $\Pi(\phi_z, \alpha^2, \mathrm{k})$ and the logarithm. For expressions that avoid this imaginary complication the reader simply replaces the argument

[4] FORTRAN 90 implementation of the results presented here and available at http://www.nhn.ou.edu/~thomas/z2t.html are 20-40 times faster than a traditional Bulirsch-Stoer integrator [15].

of the logarithm with its magnitude and $\Pi(\phi_z, \alpha^2, \mathrm{k})$ with its principal part. At threshold points where $1-\alpha^2\sin^2\phi_z = 0$, canceling infinities appear in $\Pi(\phi_z, \alpha^2, \mathrm{k})$ and the logarithm. The infinity problem is avoided by switching between (8) and (10). If both have infinities then (22) gives the correct result.

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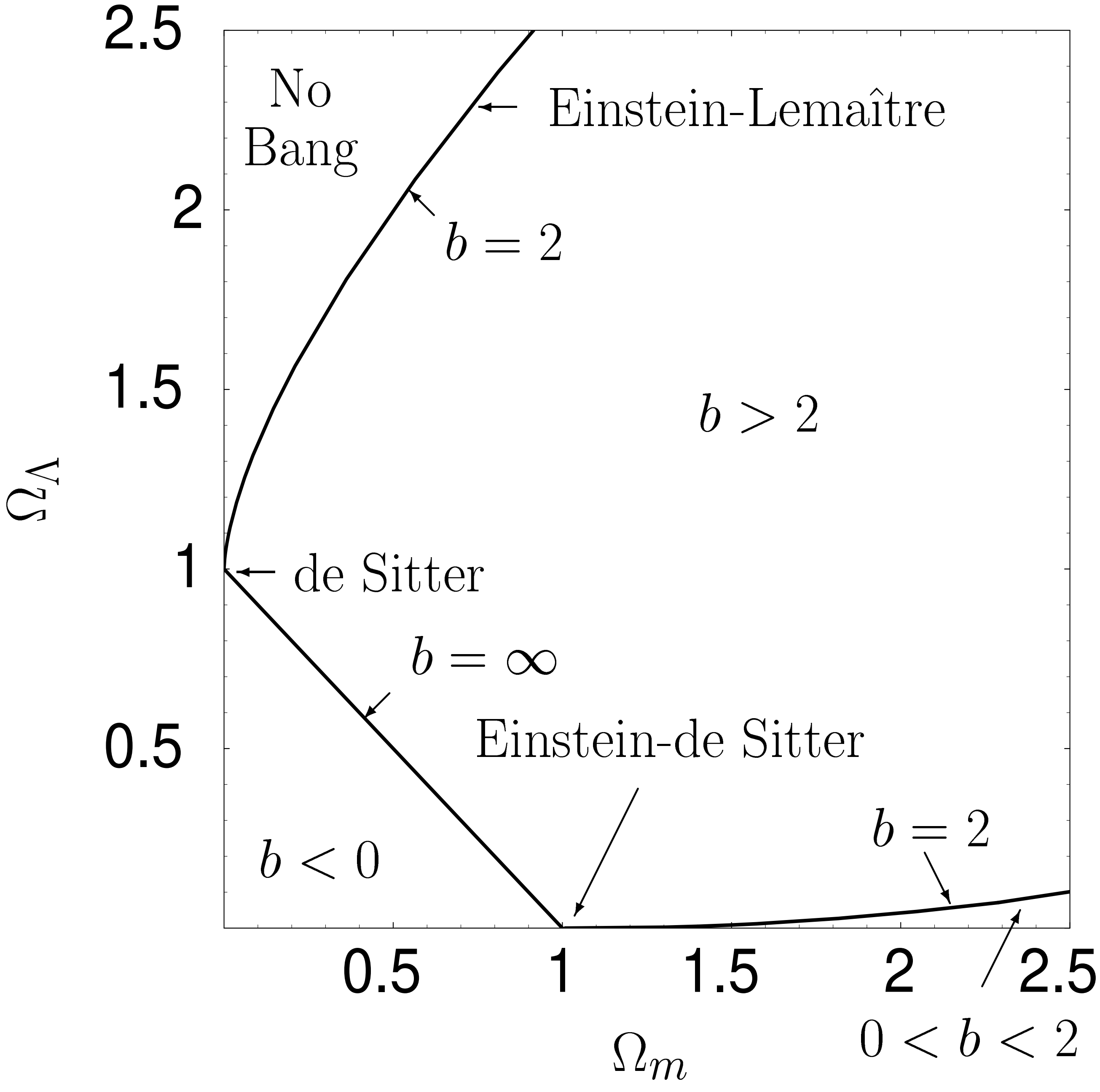

FIG. 1: The Ω_m–Ω_Λ plane showing various b domains that require different expressions for age-redshift $\tau(z)$ for standard FLRW. Expressions (8) and (10) are both appropriate for $b < 0$ and $b > 2$, while (18) is appropriate for $0 < b \leq 2$. Simpler expressions exist for various boundaries: $\Omega_\Lambda = 0$ (2), $\Omega_m = 0$ (3), $b \to \infty \Leftrightarrow \Omega_0 = 1$ (18), and $b = 2$ (19).

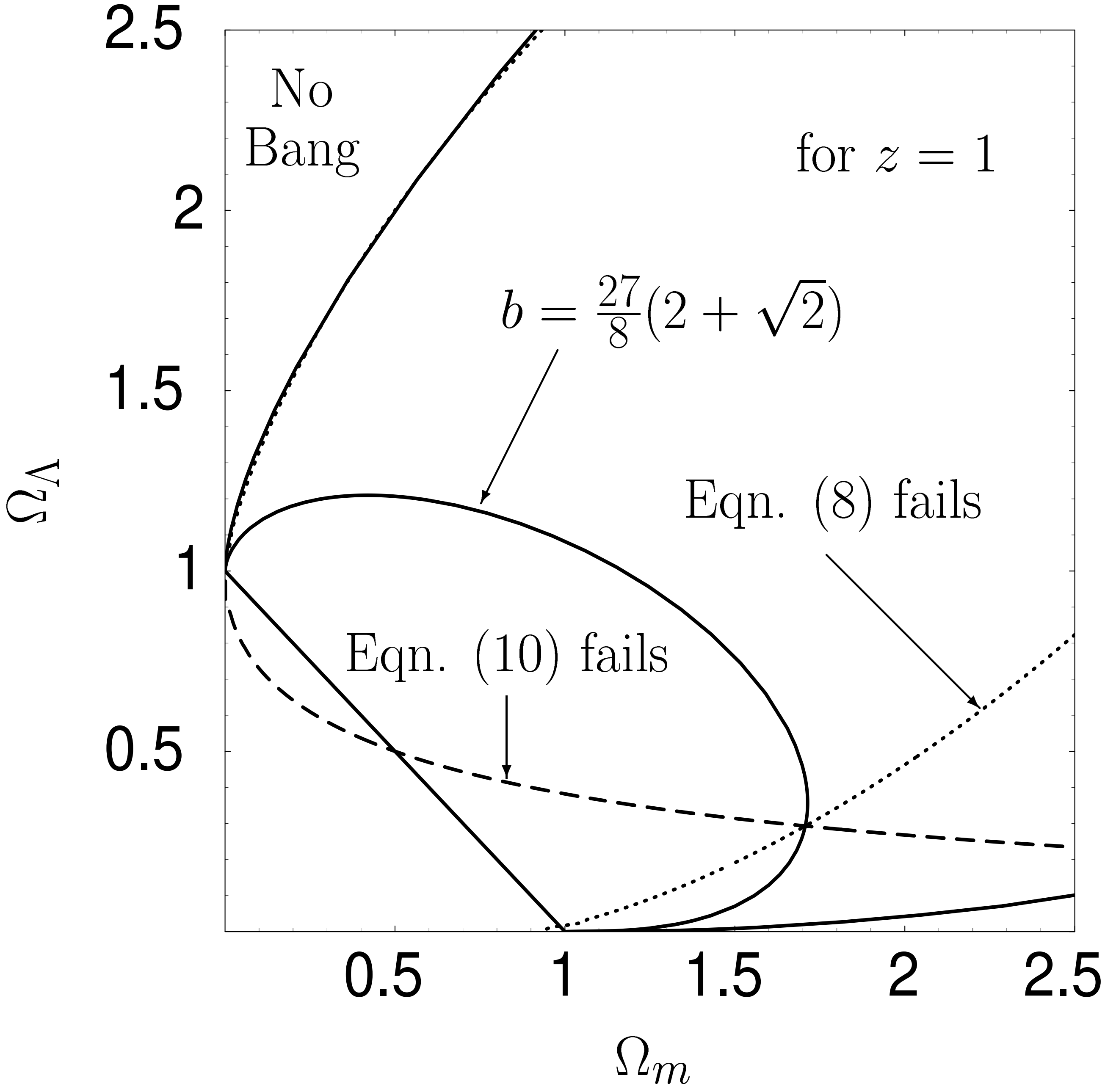

FIG. 2: The Ω_m–Ω_Λ plane showing curves where age-redshift $\tau(z)$ expressions (8) and (10) fail for redshift $z = 1$. Equation (8) fails along the dotted curve and (10) fails along the dashed curve. Both fail where they intersect on the $b = 27(2 + \sqrt{2})/8$ curve; however, (22) gives the $\tau(z)$ value at any point on this curve for all redshifts.

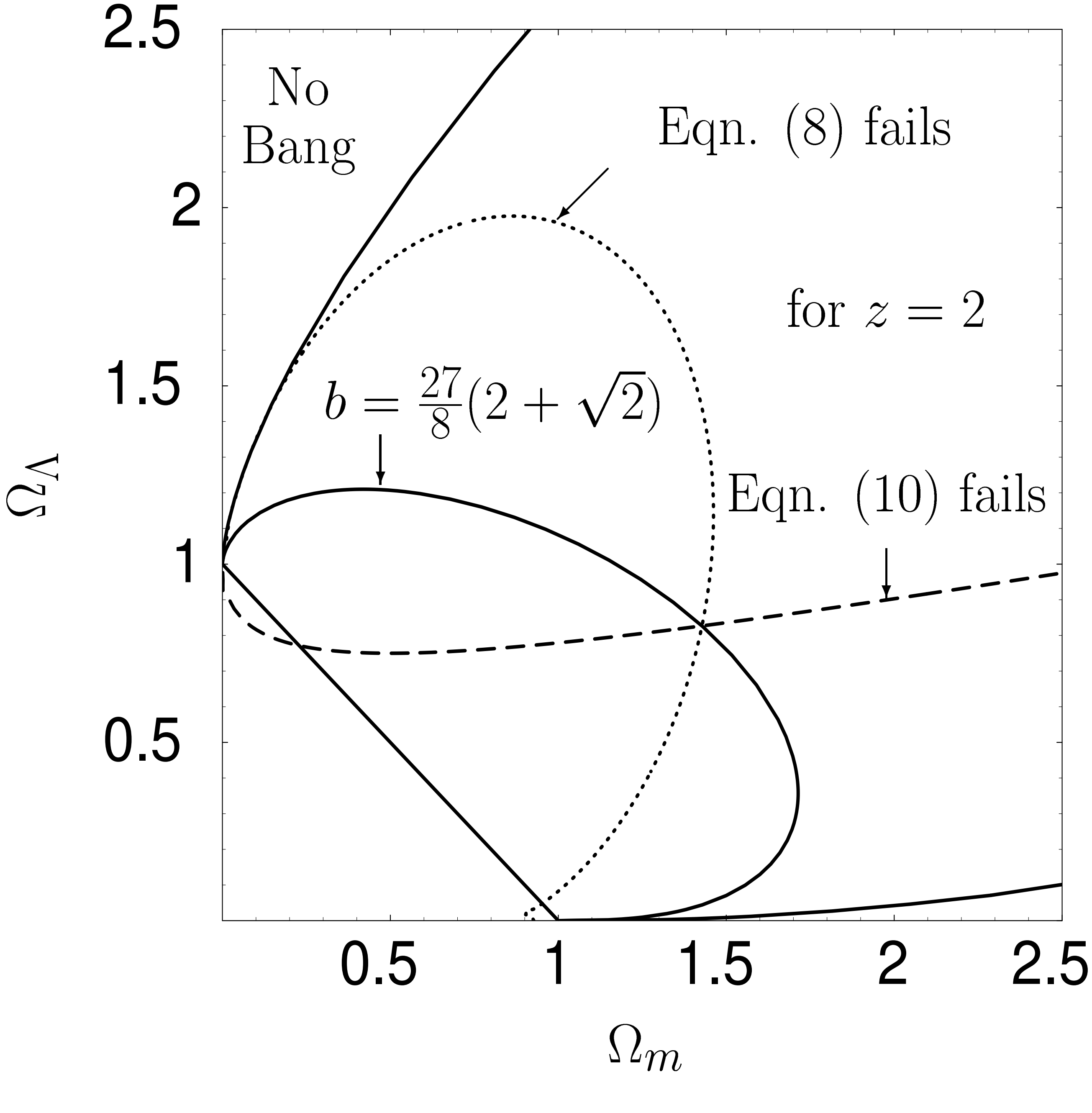

FIG. 3: Same as Fig. 2 except for redshift $z = 2$.