SOME CLIFFORD ALGEBRA HISTORY

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Abstract. In this tribute dedicated to the memory of the French mathematician Albert Crumeyrolle for his contribution to Clifford algebras and spinors, we offer a glimps of the history of the subject, as we have found it. Albert Crumeyrolle's treatment of spinor structures remains today a rich source of inspiration for both mathematicians and Physicists. We are deeply indebted to him for his illuminations. We found the historical development interesting as well as inspiring and hope the reader does also.

1. Introduction

This paper offers a quick panoramic survey of the evolution of algebraic and geometrical ideas from the ninth century with the origin of the word *algebra* about 815 A.D. up to the applications of Clifford algebras in physics in the twentieth century, see Gillispie (1970-76), van der Waerden (1985), and Crowe (1967) [1, 2, 3]. We also include some of the applications of Lie groups to quantum mechanics by Wigner (1939,64) [38, 43] which are of particular interest to us. From an abundance of episodes, we have selected the most relevant ones to our subject, and as this suggests we have overlooked many others of equal value. For this we sincerely apologize.

2. The Origin of Algebra

The word *Algebra* originates from the Arabic word *al-jabr* which literally means *completion*. Under the reign of Caliph Al-Ma'mun (813–833 A.D.) at Baghdad, the Arabian mathematician, astronomer and geographer Muhammad Ibn Musa Al-Khwarizmi composed a treatise on the solutions of linear and quadratic equations and their applications to practical mensuration, legacies and trade. The title of the work is: The Compendious Book on calculation by aljabr and al-muqabala. The author defines *aljabr [al-muqabala]* as the operation of adding [subtracting] equal terms to [from] both sides of an equation in order to reduce its form to a standard one whose general solution is called *jidhr* (root). Al-Khwarizmi's Algebra was the chief influence on Medieval European algebra through its translations into Latin by the scholars Robert of Chester (1140) and Ghirardo of Cremona (1150).

3. Complex Numbers

The italian mathematician Geralamo Cardano (1501-1576) was the first to introduce complex numbers as solutions to quadratic and cubic equations in his book

Ars Magna, sive de regulis algebraics (1545). The revolutionary idea that complex numbers can be represented geometrically by points in a plane was first visualized by the British mathematician John Wallis (1616-1703) in his Algebra (1673) and was further developed by Abraham De Moivre in 1722 and Leonhard Euler in 1748. However, the conception of complex numbers as directed line segments in a plane was realized by the Norwegian surveyor Caspar Wessel in 1799 followed by Jean Robert Argand in 1806. In 1831 Carl Friedrich Gauss showed that complex numbers can be regarded as ordered pairs of real numbers for which the operations of addition and multiplication are defined in terms of the component parts of the pairs. The algebra of complex numbers proved to be useful in describing rotations in a plane and found numerous physical applications, particularly in the theory of alternating currents and in the formulation of quantum mechanics. During the following century, the theory of functions of a complex variable was mainly developed by Augustin-Louis Cauchy (1825) and Bernhard Riemann (1854), so that it now constitutes one of the most beautiful branches of mathematics.

4. Quaternions

On the 13th of November 1843, Hamilton (1805-1865) presented to the Royal Irish Academy his famous paper: On a New Species of Imaginary Quantities Connected with the Theory of Quaternions . A quaternion algebra is an associative, non-commutative division ring with four basic elements: a real identity $\mathbf{1}$ and three different kinds of imaginary units $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ called *pure quaternion basis* which obey the multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}\,, \quad \mathbf{i}\,\mathbf{j} = -\mathbf{j}\,\mathbf{i} = \mathbf{k}\ etc. \tag{1}$$

Therefore, the quaternion idea is a logical extension of complex numbers. The geometrical significance of a quaternion became clear when Arthur Cayley (1845) showed that a quaternion operator written in the conventional form

$$\mathbf{q} = q_0\,\mathbf{1} + q_1\,\mathbf{i} + q_2\,\mathbf{j} + q_3\,\mathbf{k}\,. \tag{2}$$

can represent a three-dimensional rotation where the coefficients are the corresponding *Cayley-Klein* parameters. Hamilton was convinced that in quaternions he had found a natural algebra of three-dimensional space suitable for physical applications. He represented a vector by a pure quaternion and wrote the outcome of the product of two pure quaternions as the sum of a scalar part and a vector part:

$$\mathbf{q}\,\mathbf{q}' = -\sum_{i=1}^{3} q_i\,q_i' + (\,q_2\,q_3' - q_3\,q_2'\,)\,\mathbf{i} + (\,q_3\,q_1' - q_1\,q_3'\,)\,\mathbf{j} + (\,q_1\,q_2' - q_2\,q_1'\,)\,\mathbf{k}\,. \tag{3}$$

These parts are equivalent to the negative of the modern scalar (dot) product and to the modern vector (cross) product respectively. Each part of the (whole) product appeared to be a fundamental operation, but the union of the two parts (generally separated afterwards) did not seem a valuable idea to physicists.

In 1873, James Clerk Maxwell (1831-1879), Cavendish professor of physics at Cambridge University, published his theoretical treatise on Electricity and Magnetism, in which he used in some measure the language of quaternions but little of its operational methods, perhaps, because he was not sure enough that quaternions were appropriate for the exposition of his theory.

5. Grassmann's Theory of Extensions

In 1844, Hermann Grassmann (1809-1877), professor of mathematics at the Stettin Gymnasium in Germany (now Szczecin, Poland), published his celebrated treatise entitled: Die Lineale Ausdehnungslehre (The Theory of Linear Extension), in which he expounded a general theory of analysis of manifolds of arbitrary dimensions far beyond the reach of his contemporary fellow mathematicians who were still thinking in terms of a three-dimensional space. This remarkable work was neglected for more than twenty years due to the very abstract and philosophical language in which the subject was written. The second edition of Grassmann's theory (1862) was more readable than the first and consequently led to the development of Clifford Algebras, vector analysis, linear algebra and differential forms, though the last subject was not immediately noticed by the mathematical public. The algebraic entities of Grassmann's theory of Extension are called *extensive quantities* among which is a special class of fundamental elements with geometric properties that pertain to vectors of any finite dimension. The author originated the *inner product* for vectors to define a quadratic form on a vector space over the field of real numbers. Moreover, he introduced a second kind of multiplication for vectors called *exterior product* to form multivectors (bivectors, trivectors, etc.) or in general antisymmetric tensors of arbitrary rank. These geometric quantities obey the rules $\mathbf{u}\wedge\mathbf{v} = -\mathbf{v}\wedge\mathbf{u}$, $\mathbf{u}\wedge\mathbf{u} = 0$ and serve to conveniently represent the projective geometry of n-dimensions.

The need for geometric objects that would fit physics better than quaternions led Gibbs in 1881 and 1884 [4] to develop a three-dimensional vector analysis based on Grassmann's and Hamilton's ideas. For example, in the three-dimensional case, Gibbs substituted the cross product for the exterior product and axial vectors for bivectors. This could be done because an axial vector is geometrically equivalent to a bivector; its magnitude and direction represent the same oriented area formed by the bivector.

The third multiplication of vectors, the *indeterminate product*, introduced by Grassmann, which today is called *tensor product*, inspired Gibbs (1884) [4] to create what he called *dyadics*. For instance, in three-dimensions, a sum of the indeterminate products (dyads) of the second rank constitutes a dyadic or a second rank tensor of which the scalar coefficients represent the nine components of a 3×3 matrix ($\mathbf{T} = \sum_{i,j=1}^{3} a_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$). Thus, the birth of linear matrix algebra, often associated with the publication of Cayley's classic Memoir on the Theory of Matrices in 1858, may be said to have originated in 1844. However, there is no doubt that the association of dyadics with linear vector functions, their algebraic operations as well as their application as linear transformations on a vector space, constitute the most original part of Gibbs's contributions to vector analysis. The operational method of dyadics continued to be used by theoretical physicists until it was replaced by a more general theory of analysis introduced by the Italian geometer Gregorio Ricci-Curbastro and his pupil Tulio Levi-Civita in their memoir (1901) Methode de Calcul differentiel absolu et leurs applications which played a fundamental role in Einstein's general theory of relativity. Ricci's theory was mainly based on the intrinsic metrical geometry of n-dimensional manifolds and the invariant differential quadratic forms developed by Riemann (1854), Christoffel (1869) and Lipschitz (1869).

Through the work of the French mathematician Elie Cartan (1901), Grassmann's theory has since shown its utility in the study of Exterior Calculus and differential forms applied to non-metrical differential geometry which later in the second half of the twentieth century provided an elegant modern setting for the description of physical tensor fields such as the electromagnetic field and gravity.

6. Clifford Algebras and Spinors

In 1876, the philosopher and geometer William Kingdon Clifford (1845-1879), professor of applied mathematics at the University College of London, wrote a paper entitled: On The Classification of Geometric Algebras [5]. The paper was found amongst Clifford's notes, of which the forewords are the abstract that he communicated to the London Mathematic Society on the 10th of March 1876. This work remained unfinished due to the deterioration of his health by tuberculosis from which he died on March 3, 1879 at the age of thirty-five.

Let us pause the chronological development of the subject for the moment and discuss the important discoveries in Clifford's paper. In his geometric algebras, Clifford applies a general definition of a product given by Grassmann: A product of two extensive quantities $\mathbf{u}$ and $\mathbf{v}$ is either defined as extensive quantity or a scalar. First, he considers n-fundamental units $\mathbf{e}_1$, $\mathbf{e}_2, ..., \mathbf{e}_n$ such that all $(\mathbf{e}_j)^2 = \mathbf{1}$ (or $-\mathbf{1}$) and $\mathbf{e}_j\,\mathbf{e}_k = -\mathbf{e}_k\,\mathbf{e}_j$ $(j \neq k)$, where $\mathbf{1}$ denotes the identity. All products of linear factors must therefore contain either basic terms of odd order ($\mathbf{e}_j$, $\mathbf{e}_j\,\mathbf{e}_k\,\mathbf{e}_l, ...$such that $j \neq k \neq l$), or basic terms of even order ($\mathbf{1}$, $\mathbf{e}_j\,\mathbf{e}_k, ...$such that $j \neq k$). There can be no such term of higher order than the n^{th}, and since we have altogether one term of order zero, n of order 1, $\frac{1}{2}(n-1)$ of order 2,..., one of order n; the total number of basic terms is 2^n units. He uses the selective symbols V_0, $V_1, ..., V_n$ to denote the subspaces containing expressions of order $0, 1, ..., n$. Second, he writes ω for the product of all n-units, $\omega \equiv \mathbf{e}_1\,\mathbf{e}_2 ... \mathbf{e}_n$ and enquires into the value of its square:

$$\omega^2 \equiv \mathbf{e}_1\,\mathbf{e}_2 ... \mathbf{e}_n\,\mathbf{e}_1\,\mathbf{e}_2 ... \mathbf{e}_n \,, \tag{4}$$

which after r-interchanges becomes:

$$\omega^2 = (-1)^r\,{\mathbf{e}_1}^2\,{\mathbf{e}_2}^2 ... {\mathbf{e}_n}^2 = \begin{cases} (-\mathbf{1})^r, & \text{if } {\mathbf{e}_j}^2 = +\mathbf{1}; \\ (-\mathbf{1})^{r+n}, & \text{if } {\mathbf{e}_j}^2 = -\mathbf{1}. \end{cases} \tag{5}$$

To find the value of r, he observes that it takes $(n-1)$ interchanges to bring the second $\mathbf{e}_1$ between the first $\mathbf{e}_1$, and $\mathbf{e}_2$, then $(n-2)$ interchanges and so on. Therefore,

$$r = 1 + 2 + \cdots + (n-1) = \frac{1}{2}n(n-1)\,, \tag{6}$$

$$r + n = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1)\,, \tag{7}$$

hence, if ${\mathbf{e}_j}^2 = +\mathbf{1}$, then $\omega^2 = \mathbf{1}$ or $-\mathbf{1}$ according to whether $\frac{1}{2}n(n-1)$ is even or odd; and if ${\mathbf{e}_j}^2 = -\mathbf{1}$, then $\omega^2 = \mathbf{1}$ or $-\mathbf{1}$ according to whether $\frac{1}{2}n(n+1)$ is even or odd. Next, he considers the nature of the multiplication $\omega\,\mathbf{e}_j$. He observes that:

$$\mathbf{e}_1\,\omega \;=\; \begin{cases} +\mathbf{e}_2\,\mathbf{e}_3\,\cdots\mathbf{e}_n, & \text{if } {\mathbf{e}_j}^2 = +\mathbf{1}; \\ -\mathbf{e}_2\,\mathbf{e}_3,\,\cdots\mathbf{e}_n, & \text{if } {\mathbf{e}_j}^2 = -\mathbf{1}; \end{cases} \tag{8}$$

and

$$\omega\,\mathbf{e}_1 \;=\; \begin{cases} (-\mathbf{1})^{n-1}\,\mathbf{e}_2\,\mathbf{e}_3\,\cdots\mathbf{e}_n, & \text{if } {\mathbf{e}_j}^2 = +\mathbf{1}; \\ (-\mathbf{1})^{n}\,\mathbf{e}_2\,\mathbf{e}_3\,\cdots\mathbf{e}_n, & \text{if } {\mathbf{e}_j}^2 = -\mathbf{1}. \end{cases} \tag{9}$$

Hence, for both cases ${\mathbf{e}_j}^2 = \pm\mathbf{1}$, $\omega\,\mathbf{e}_j = \pm\mathbf{e}_j\,\omega$ according as n is odd or even. In other words, the multiplication product of ω with the units $\mathbf{e}_j$ is commutative or anti-commutative according to whether n is odd or even. For example, in the case where $n = 3$ and ${\mathbf{e}_j}^2 = +\mathbf{1}$; $(j = 1, 2, 3)$, it follows that $\omega^2 = -\mathbf{1}$ and ω commutes with all the elements of the algebra. Therefore, we can put $\omega \equiv i\mathbf{1} = (\sqrt{-1})\,\mathbf{1}$, and reduce the eight-dimensional algebra over the reals to a four-dimensional algebra over the complex field. Third, he considers the factors of units of even order, namely: $\mathbf{1}$, $\mathbf{e}_j\,\mathbf{e}_k$, $\mathbf{e}_j\,\mathbf{e}_k\,\mathbf{e}_l\,\mathbf{e}_m$, ...such that $j \neq k \neq l \neq m$; they form an algebra by themselves which he calls the *even subalgebra*. This is obvious, since the set contains the identity $\mathbf{1}$ and the product of two elements of even order forms an element of an even order. For $n = 3$, ${\mathbf{e}_j}^2 = +\mathbf{1}$, the even subalgebra with basic elements $\{\mathbf{1}; \mathbf{e}_2\mathbf{e}_3; \mathbf{e}_1\mathbf{e}_3; \mathbf{e}_1\mathbf{e}_2\}$ gives the quaternions with $\mathbf{i} \equiv \mathbf{e}_2\mathbf{e}_3$, $\mathbf{j} \equiv \mathbf{e}_1\mathbf{e}_3$, $\mathbf{k} \equiv \mathbf{e}_1\mathbf{e}_2$; therefore, a pure quaternion is actually a bivector in the context of Clifford algebras (a pseudovector in this case for $n = 3$). For the case $n = 4$, ${\mathbf{e}_j}^2 = +\mathbf{1}$, we have $\omega^2 = \mathbf{1}$, and $\omega\,\mathbf{e}_j = -\mathbf{e}_j\,\omega$ $(j = 1, 2, 3, 4)$. The general element of the even subalgebra can be expressed as:

$$\begin{aligned} \mathbf{E} \;&=\; q_0\,\mathbf{1} + \sum_{j\neq k=1}^{4} q_{jk}\,\mathbf{e}_j\,\mathbf{e}_k \;+\; q_0'\,\omega && (10)\\ &=\; \left[q_0\,\mathbf{1} + \sum_{j\neq k=1}^{3} q_{jk}\,\mathbf{e}_j\,\mathbf{e}_k\right] \;+\; \omega\;\left[q_0'\,\mathbf{1} + \sum_{j\neq k=1}^{3} {q'}_{jk}\,\mathbf{e}_j\,\mathbf{e}_k\right] && (11)\\ &=\; \mathbf{q} + \mathbf{q}'\,, && (12) \end{aligned}$$

where $\mathbf{q}$ and $\mathbf{q}'$ are quaternions. The above expression represents what Clifford called a *biquaternion*.

It is worth noticing that if we assume that the square of an element of rank one is equal to the sum of its squared coefficients (quadratic form), that is

$$\left(\sum_{j=1}^{n} q_j\,\mathbf{e}_j\right)^2 \;=\; \textstyle\sum_{j=1}^{n} {q_j}^2\,, \tag{13}$$

$$\sum_{j,k=1}^{n} q_j\,q_k\,\frac{1}{2}(\mathbf{e}_j\,\mathbf{e}_k \;+\; \mathbf{e}_k\,\mathbf{e}_j) \;=\; \textstyle\sum_{j,k=1}^{n} q_j\,q_k\,\delta_{jk}\,, \tag{14}$$

we get Clifford's rule:

$$\mathbf{e}_j\,\mathbf{e}_k\,+\,\mathbf{e}_k\,\mathbf{e}_j\,=\,2\,\delta_{j\,k}\,\mathbf{1},\quad (j,k=1,...,n). \tag{15}$$

The same idea in a different form enabled the eminent physicist P.A.M. Dirac in 1928 [6] to linearize the Hamiltonian of the Klein-Gordon equation (a second-order Quantum relativistic wave equation) which may be written for a free massive particle as:

$$P^\mu\,P_\mu\,\Psi\,=\,m^2\,c^2\,\Psi\,,\qquad (\mu=0,1,2,3) \tag{16}$$

where the contravariant and covariant energy-momentum operators are given by

$$P^\mu\,=\,i\,\hbar\,g^{\mu\nu}\,\partial_\nu\,,\qquad P_\mu\,=\,i\,\hbar\,\partial_\mu\,, \tag{17}$$

where $g^{\mu\nu}\,=\,diag(1,-1,-1,-1)$ is the Minkowski metric. Now, the linear matrix wave equation,

$$\gamma_\mu\,P^\mu\,\Psi\,=\,mc\,\Psi\,, \tag{18}$$

satisfies (14) if and only if,

$$\begin{aligned}(\gamma_\mu\,P^\mu)^2\quad &=\,P^\mu\,P_\mu\,I\,,\\ P^\mu\,P^\nu\,\frac{1}{2}(\gamma_\mu\,\gamma_\nu\,+\,\gamma_\nu\,\gamma_\mu)\,&=\,P^\mu\,P^\nu\,g_{\mu\nu}\,I\,,\end{aligned} \tag{19}$$

or equivalently if and only if Clifford's rules are satisfied,

$$\gamma_\mu\,\gamma_\nu\,+\,\gamma_\nu\,\gamma_\mu\,=\,2\,g_{\mu\nu}\,I\,. \tag{20}$$

In (19) and (20) 'I' is the identity 4×4 matrix. Therefore, the Dirac γ-matrices which brought in the spin of the electron turn out to be representations of elements of a Clifford algebra associated with Minkowski space. Next, Dirac shows that his wave equation (18) is form-invariant under an arbitrary Lorentz transformation provided that the following fundamental relation is satisfied:

$$S\,\gamma_\mu\,S^{-1}\,=\,\gamma_\nu\,\Lambda^\nu{}_\mu\,=\,\gamma'_\mu\,, \tag{21}$$

where $\Lambda^\nu{}_\mu$ are the components of a 4×4 Lorentz matrix, and S is a non-singular complex matrix. The group of all matrices S (spinor representation), form two-valued matrix representations of the Lorentz group. Finally, the solutions of the free-particle wave equation are the four-components of a geometric object called *spinor* (represented by a column matrix) multiplied by the Lorentz invariant quantity $e^{\mp i p_\mu x^\mu}$.

Fifteen years earlier, spinor representations defined in a pure geometrical manner, were discovered by Elie Cartan (1913) [7] in his investigations on the linear representations of simple Lie groups. Among these, he found two valued matrix representations of the group of rotations in a space with any finite number of dimensions. For instance, the real orthogonal group of rotations in three dimensions is double covered by the special unitary group $SU(2)$. In this geometric scheme, a spinor appears as an element of the representation space of the covering group of the orthogonal group.

In their paper entitled: Spinors in n dimensions, Richard Brauer and Hermann Weyl (1935) [8] succeeded in constructing the global two valued representations of orthogonal groups by means of Clifford algebras. A rigorous formulation of Clifford algebras as well as a new and complete treatment of Clifford modules were provided by M. Atiyah, R. Bott and A. Shapiro in 1964 [9].

The first explicit suggestion that a spinor can be looked upon as an algebraic entity, was made by Marcel Riesz (1947, 1953) [10, 11]. The author defines spinors as elements of minimal left ideals contained in Clifford algebras. General construction of these ideals and of scalar products of algebraic spinors were developed by I. Porteous in 1969 [12] and later by Pertti Lounesto in 1980 and 1981 [13, 14]. In his book The Algebraic Theory of Spinors Claude Chevalley (1954) [15] generalized the theory of finite Clifford algebras by considering an arbitrary quadratic form defined on an n-dimensional vector space over an arbitrary basic field.

Since the beginning of the sixties, Clifford algebras and spinors have been studied extensively in pure mathematics, mathematical physics and physics. In particular, the French mathematician Albert Crumeyrolle (1969, 1971, 1974) [16, 17, 18] presented the fullest systematic exposition of Clifford algebras, spinor structures and spinor groups. In his approach of constructing complex spinor spaces by the use of de Witt decomposition, he succeeded in formalizing a generalized hermitian structure over these spaces as well as in representing the antilinear charge conjugation operation encountered in Dirac's theory. His remarquable works on symplectic Clifford algebras and symplectic spinors (1974, 1977, 1987) [18, 19, 20] led to new connections between Mathematics and Physics and culminated into a new mathematical foundation of quantized physical fields. He further extended the algebra of spinors over curved spaces and established a method of quantization of relativistic free fields over these spaces (1975, 1978) [21, 22]. He defined a quantized field by means of a section of a trivial vector fiber bundle over the curved space-time. Each fiber represents a Hilbert space constructed by means of either orthogonal or symplectic spinor fibrations corresponding to either fermions or bosons fields. Meanwhile, David Hestenes (1966) [23] rediscovered the importance of Clifford algebras in expressing the geometrical ideas in physics. He further developed the mathematical foundations of non-relativistic and relativistic classical and quantum mechanics through his unified language of geometric algebra/calculus (1968, 1971, 1984, 1986) [24, 25, 26, 27] and thereby he set a new pace for the applications of algebra, functional analysis, and group representations to quantum field theory (1990) [28]. Moreover, Clifford algebras constructed from infinite-dimensional Hilbert spaces and spinor representations of infinite orthogonal groups have been discussed by David Shale and W. Forrest Stinespring (1964-65) [29, 30], and by P. de la Harpe (1972) [31]. Finally, the mathematical quantization technique introduced by I.E. Segal (1947, 1960, 1962) [32, 33, 34] has been further developed and applied to physics in a Clifford algebra context by David Shale and W. Forrest Stinespring (1964-65) [29, 30], P.J.M. Bongaarts (1970, 1972) [35, 36], R.J. Plymen (1976) [37], and by the authors [28] of this paper (1990).

7. Groups of Transformations

As early as 1758, Euler proved that every displacement of a rigid body can be expressed as a product of an axial rotation and a translation. He described rotations by the use of three parameters known as *Euler angles* which are still used by physicists today. Following Euler, Olinde Rodrigues in 1840 parameterized a rotation with four real variables, the first determines the magnitude of the angle and the last three determine the direction of the axis of rotation. If these parameters are replaced by the components of a quaternion, then the formula for the multiplication of rotations is identical to the one developed by Cayley in 1845. It is an historical fact that Rodrigues recognized the non-commutivity of the product of two rotations three years before Hamilton's discovery of quaternions.

The notion of an abstract group is due to Cayley (1854). In 1868-1869 Camille Jordan published a two-part treatise entitled: Memoire sur les groupes de mouvements (Memoirs on groups of movements), in which he revealed a complete enumeration of all possible groups of displacements of rigid bodies in a three-dimensional Euclidean space. The displacements studied by Jordan are helicoidal motions of which the rotations and translations are two particular cases.

The Norwegian mathematician Marius Sophus Lie (1842-1899), professor of mathematics at Christiania University in Oslo, presented in 1888-1893 a clear and extensive exposition of his theory of continuous groups in a monumental three-volume work entitled: Theorie der Transformationgruppen (Theory of group transformations) which culminated the description of the invariance laws of nature. The structure theory of Lie algebras is mainly due to Wilhelm Killing (1888-1890) and Elie Cartan (1894) who published his famous thesis: Sur la structure des groupes de transformations finis et continus (On the structure of finite continuous groups of transformations), in which he corrected the defective proofs in Killing's demonstrations.

In 1892, the Dutch physicist Hendrick Antoon Lorentz was the first to introduce the *Pure Lorentz Transformations* as mathematical formulas that describe his contraction hypothesis: *Any moving rigid body approaching the speed of light contracts in the direction of motion.* Lorentz's hypothesis was a direct deduction drawn from the failure of Michelson and Morley's experiment (1887) in determining the motion of earth relative to a hypothetical medium called the *absolute ether.* While Lorentz attempted to explain the contraction phenomenon on the basis of an atomistic theory of material bodies founded on the absolute ether, the brilliant Albert Einstein in 1905 realized that the very foundations of the ether concept were seriously shaken by the results of the above experiment. Einstein associated the phenomenon with the nature of space and time which is founded on his two principles: first, the principle of special relativity which states that *all laws of nature are identical in all inertial systems of reference*; second, the principle of the constancy of the velocity of light in all inertial frames of reference. He derived the pure Lorentz transformations from these principles and proved that the interval between two events in space-time continuum is invariant under these transformations. In other words, he showed that the *boosts* leave invariant the quadratic form defined on Minkowski space. The group formed by products of *spatial rotations* and *boosts* was called the *homogeneous Lorentz group* by the French mathematician Henri Poincaré (1905) who constructed the Inhomogeneous Lorentz or Poincaré group as a semi-direct product of the group of translations with the homogeneous Lorentz group. All these groups are Lie groups.

The irreducible Unitary representations of the Poincaré group were first found by Eugene Wigner in 1939 [38]. Their connection with elementary particles and relativistic wave equations were discussed later by Wigner and Bargmann in 1948 [39]. Moreover, a rigorous mathematical exposition on the induced representations of locally compact groups was supplied by Mackey (1953, 1958) [40, 41]. Finally, the inclusion of *space reflection* and the *time reversal* symmetries was provided by Bargmann (1963) [42], and Wigner (1964) [43]. With these remarks we close our journey into past discoveries and salute again Albert Crumeyrolle's contributions.

References

1. C. C. Gillispie, ed. in chief, *Dictionary of Scientific Biography* (Charles Scribner's Sons, New york, 1970–1976), Vol. I–XV.
2. B. L. van der Waerden, *A History of Algebra* (Springer-Verlag, New York, 1985).
3. M. J. Crowe, *A History of Vector Analysis* (University of Notre Dame Press, Notre Dame, 1967).
4. J. W. Gibbs, "Elements of Vector Analysis," privately printed at New Haven in 1881 and 1884 but never published according to the preface of *Vector Analysis*, vii (1901), reprinted in *The Scientific Papers of J. Willard Gibbs* (Longmans, Green and Co., New York, 1906), Vol. II, pp. 17–90.
5. W. K. Clifford, *Mathematical Papers*, edited by R. Tucker (MacMillan Co., London, 1882).
6. P. A. M. Dirac, "The Quantum Theory of the Electron," *Proc. R. Soc. London* **A117**, 610–624 (1928).
7. E. Cartan, "Les Groups Projectifs qui ne Laissent Invariant Aucune Multiplicité Plane," *Bull. Soc. Math. France* **41**, 53–96 (1913).
8. R. Brauer, and H. Weyl, "Spinors in n dimensions," *Amer. J. Math.* **57**, 425–449 (1935).
9. M. F. Atiyah, R. Bott, and A. Shapiro, "Clifford Modules," *Topology* **3**, suppl. 1, 3–38, 1964.
10. M. Riesz, "Sur Certaines Notions Fondamentales en Théorie Quantiques Relativistes," *Comptes Rendus du Dixième Congrè Math. Scandinaves*, 123–148 (1947).
11. M. Riesz, *Clifford Numbers and Spinors* (University of Maryland, College Park, Maryland, 1958) Facsimile reproduction by Kluwer Press, 1993.
12. I. R. Porteous, *Topological Geometry* (Van Nostrand-Reinhold, London, 1969, also Cambridge University Press, 1981).
13. P. Lounesto, " Sur les Idéaux à Gauche des Algèbres de Clifford et les Produits Scalaires des Spineurs," *Ann. Inst. Henri Poincaré* **A33** , 53–61 (1980).
14. P. Lounesto, "Scalar Products of Spinors and an Extension of Brauer-Wall Groups," *Found. Phys.* **11**, 721–740, 1981.
15. C. C. Chevalley, *The Algebraic Theory of Spinors*, (Columbia University Press, New York, 1954).
16. A. Crumeyrolle, "Structures Spinorielles," *Ann. Inst. Henri Poincaré* **A11**, 19–55 (1969).
17. A. Crumeyrolle, "Groupes de Spinorialité," *Ann. Inst. Henri Poincaré* **A14**, 309–323 (1971).
18. A. Crumeyrolle, *Algèbre de Clifford et Spineurs* (Cours et Seminaire du Departement de Mathematiques de L' Université de Toulouse III, Université Paul Sabatier, Toulouse, 1974).
19. A. Crumeyrolle, "Algèbre de Clifford Symplectique, Revêtement du Groupe Symplectique, Indices de Maslov, et Spineurs Symplectiques," *J. Math. pures et appl.* **56**, 205–230 (1977).
20. A. Crumeyrolle, "Spineurs Symplectiques et Transformation de Fourier," in *Geometrie et Physique*, edited by Y. Choquet-Bruhat, B. Coll, R. Kerner, A. Lichnerowitcz (Hermann, Paris, 1987).
21. A. Crumeyrolle, "Une Théorie d'Einstein-Dirac en Spin Maximum 1," *Ann. Inst. Henri Poincaré* **A22**, 43–61 (1975).
22. A. Crumeyrolle, "Un Programme de Quantification des Champs Libres Relativistes sur les Espaces Courbes," *Ann. Inst. Henri Poincaré* **A29**, 217–231 (1978).
23. D. Hestenes, *Space-Time Algebra* (Gordon and Breach, New York, 1966, 1987, 1992).
24. D. Hestenes, "Multivector Calculus," *J. Math. Anal. Appl.* **24**, 313–325 (1968).

25. D. Hestenes, "Vectors, Spinors and Complex Numbers in Classical and Quantum Physics," *Amer. J. Phys.* **39**, 1013–1028 (1971).
26. D. Hestenes, G. Sobczyk, *Clifford Algebra to Geometric Calculus* (Reidel, Dordrecht, 1984, 1987).
27. D. Hestenes, *New Foundations for Classical Mechanics* (Reidel, Dordrecht, 1986, 1987).
28. R. Kantowski, H. T. Cho, and A. Diek, "A Clifford Algebra Quantization of Dirac's Electron Positron Field," *J. Math. Phys.* **31**, 2192–2200 (1990).
29. D. Shale, and W. F. Stinespring, "States of the Clifford Algebra," *Ann. Math.* **80**, 365–381 (1964).
30. D. Shale, and W. F. Stinespring, "Spinor Representations of Infinite Orthogonal Groups," *J. Math. Mech.* **14**, 315–322 (1965).
31. P. de la Harpe, "The Clifford Algebra and the Spinor Group of a Hilbert Space," *Compositio Mathematica* **25**, 245–261 (1972).
32. I. E. Segal, "Postulates for General Quantum Mechanics," *Ann. Math.* **48**, 930–948 (1947).
33. I. E. Segal, "Mathematical Characterization of the Physical Vacuum for a Linear Bose–Einstein Field," *Illinois J. Math.* **6**, 500–523 (1962).
34. I. E. Segal, *Mathematical Problems in Relativistic Physics* (Amer. Math. Soc., Providence, 1963).
35. P. J. M. Bongaarts, "The Electron-Positron Field, Coupled to External Electromagnetic Potentials, as an Elementary $C^\star$ Algebra Theory," *Ann. Phys.* **56**, 108–139 (1970).
36. P. J. M. Bongaarts, *Mathematics of Contemporary Physics*, edited by. R. F. Streater (Academic Press, New York, 1972).
37. R. J. Plymen, "Spinors in Hilbert Space," *Math. Proc. Cambridge. Philos. Soc.* **80**, 337–347 (1976).
38. E. P. Wigner, "On Unitary Representations of the Inhomogeneous Lorentz Group," *Ann. Math.* **40**, 149–204 (1939).
39. V. Bargmann, and E. P. Wigner, "Group Theoretical Discussion of Relativistic Wave Equations" *Proc. Nat. Acad. Sci. U.S.A.* **34**, 211–223 (1948).
40. G. W. Mackey, "Induced Representations of Locally Compact Groups," *Ann. Math.* **55**, 101–139 (1952).
41. G. W. Mackey, "Unitary Representations of Group Extensions," *Acta Math.* **99**, 265–311 (1958).
42. V. Bargmann, *Gruppentheoretische Analyse der Lorentzinvarianz* (Eidgenossische Technische Hochschule, Zurich, 1963).
43. E. P. Wigner, "Unitary Representations of the Inhomogeneous Lorentz Group Including Reflections," in *Group Theoretical Concepts and Methods in Elementary Particle Physics*, edited by F. Gursey (Gordon and Breach, New York, 1964).