DIRAC'S FIELD OPERATOR Ψ

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Abstract. In this tribute dedicated to Albert Crumeyrolle's contributions to Clifford algebras and spinors, we wish to complete some work on the Dirac theory that we started earlier but never finished. This somewhat sober contribution requires close comparisons with related equations appearing in a previous work by the authors. The purpose is to interpret the quantum field operator $\Psi(x)$ appearing in the Dirac theory as a tensor product valued function on spacetime which is invariant under inhomogeneous group actions.

1. Deriving the Dirac Quantum Field

In an earlier work (Cho *et al.* (1990) [1]), we have shown how a sequential construction of two Clifford algebras, starting from Minkowski space $\mathbb{R}^{1,3}$ and its associated Lorentz metric $(+,-,-,-)$, can yield the free particle spin-1/2 quantum field theory of Dirac. Here we complete the construction by relating the Dirac quantum field operator $\Psi(x)$ as used by the physics community, to an invariant cross section of a field on $\mathbb{R}^{1,3}$ which takes its values in the tensor product of the two constructed Clifford algebras. The reader must necessarily consult Cho *et al.* (1990) [1] to follow the construction presented here. We refer the reader to the references required in this construction to the above citation.

The massive spin-1/2 quantum fields $\Psi(x)$ and $\Psi^\dagger(x)$, as ordinarily understood in the Dirac theory, stand for 4 spacetime-dependent operators (see Bjorken & Drell 1965 [2] or Itzykson & Zuber 1980 [3]). The operators are complex linear combinations of annihilation and creation operators $\mathbf{b}^A_{\vec{p}}, \mathbf{d}^{A\dagger}_{\vec{p}}, \mathbf{b}^{A\dagger}_{\vec{p}}$, and $\mathbf{d}^A_{\vec{p}}$, one operator for each component 'Ω' $(=1,2,3,4)$,

$$\begin{aligned}
\Psi^\Omega(x) &= \int dp\{u^\Omega_{A,\vec{p}}\mathbf{b}^A_{\vec{p}}e^{-ip_\mu x^\mu} + v^\Omega_{A,\vec{p}}\mathbf{d}^{A\dagger}_{\vec{p}}e^{+ip_\mu x^\mu}\},\\
\Psi^{\dagger\Omega}(x) &= \int dp\{\overset{*}{u}{}^\Omega_{A,\vec{p}}\mathbf{b}^{A\dagger}_{\vec{p}}e^{+ip_\mu x^\mu} + \overset{*}{v}{}^\Omega_{A,\vec{p}}\mathbf{d}^A_{\vec{p}}e^{-ip_\mu x^\mu}\},
\end{aligned} \tag{1}$$

where dp is the Lorentz invariant momentum measure given by

$$dp \equiv \frac{1}{(2\pi)^3}\frac{mc}{p^0}d^3p, \tag{2}$$

and repeated indices $A(=1,2)$ and $\mu(=0,1,2,3)$ are summed. The four component positive and negative energy spinors $u^{\Omega}_{A,\vec{p}}$ and $v^{\Omega}_{A,\vec{p}}$ are the familiar momentum, $\vec{p}$, and spin up and down eigenstates ($A=1,2$) of the Dirac wave theory. The imaginary 'i' as is ordinarily understood, provides the complex structure for both the 4-component Dirac spinors and the particle space on which the annihilation and creation operators act ($i^{*}=-i$). To rewrite (1) using the 4-component spinor basis w_{Ω} of ([1].2.15) $\equiv$ equation (2.15) in Cho *et al.* (1990) [1] , we must distinguish between the complex structure ($\gamma\equiv\gamma_0\gamma_1\gamma_2\gamma_3$) of the space of spinors $C\ell_{1,3}\ \frac{1}{2}(1+\gamma_3\gamma_0)$ (called $R_{1,3+}$ in Cho *et al.* (1990) [1]) and 'i' of the ∞-dimensional Clifford algebra of operators $\mathcal{C}$. The correct transcription of (1) is

$$\begin{aligned}\Psi(x) &= \int dp\{u_{A,\vec{p}}e^{-\gamma p_\mu x^\mu}\otimes \mathbf{b}^{A}_{\vec{p}} + v_{A,\vec{p}}e^{+\gamma p_\mu x^\mu}\otimes \mathbf{d}^{A\dagger}_{\vec{p}}\},\\ \Psi^{\dagger}(x) &= \int dp\{u_{A,\vec{p}}e^{-\gamma p_\mu x^\mu}\otimes \mathbf{b}^{A\dagger}_{\vec{p}} + v_{A,\vec{p}}e^{+\gamma p_\mu x^\mu}\otimes \mathbf{d}^{A}_{\vec{p}}\},\end{aligned} \tag{3}$$

where the $\otimes$ indicates that $\Psi(x)$ is a function on spacetime, taking its values in a tensor product space,

$$\Psi:\ \mathbb{R}^{1,3}\longrightarrow C\ell_{1,3}\ \frac{1}{2}(1+\gamma_3\gamma_0)\otimes\mathcal{C}^1, \tag{4}$$

see ([1].4.9) $\equiv$ equation (4.9) Cho *et al.* (1990) [1], for a definition of the 1-particle operator subspace $\mathcal{C}^1$. The $^{\dagger}\ :\ \Psi(x)\longrightarrow\Psi^{\dagger}(x)$ mapping of (3) is the anti-automorphism † of ([1].4.10) and ([1].4.11) and is only applied to the $\mathcal{C}^1$ part of $C\ell_{1,3}\ \frac{1}{2}(1+\gamma_3\gamma_0)\otimes\ \mathcal{C}^1$. The $u_{A,\vec{p}}$ and $v_{A,\vec{p}}$ are the usual spin 'up' and spin 'down', positive and negative energy Dirac spinors defined by

$$\begin{aligned}u_{A,\vec{p}} &= w_{\Omega}u^{\Omega}_{A,\vec{p}}\equiv e^{\frac{\zeta^i}{2}\alpha_i}z_A,\quad A=\{1,2\},\\ v_{A,\vec{p}} &= w_{\Omega}v^{\Omega}_{A,\vec{p}}\equiv e^{\frac{\zeta^i}{2}\alpha_i}z_{A+2}.\end{aligned} \tag{5}$$

The rapidity ζ^i is related to momentum by ([1].3.4), and z_{Ω} are basis vectors from ([1].2.15). The complex structure of the tensor product space (4) is given by γ. Because the complex structure of $C\ell_{1,3}\ \frac{1}{2}(1+\gamma_3\gamma_0)$ (γ on the right) induces an action of γ on $\mathcal{C}^1$ as given in ([1].4.15), the $e^{-\gamma p_\mu x^\mu}$ and $e^{+\gamma p_\mu x^\mu}$ of (3) could appear on the right of $\otimes$, as is, or as $e^{-ip_\mu x^\mu}$ and $e^{+ip_\mu x^\mu}$ multiplying $\mathbf{b}^{A}_{\vec{p}}$ and $\mathbf{d}^{A\dagger}_{\vec{p}}$ respectively or as $e^{+ip_\mu x^\mu}$ and $e^{-ip_\mu x^\mu}$ multiplying $\mathbf{b}^{A\dagger}_{\vec{p}}$ and $\mathbf{d}^{A}_{\vec{p}}$ respectively.

Four-component Dirac wave functions are cross sections of a $C\ell_{1,3}\ \frac{1}{2}(1+\gamma_3\gamma_0)$ valued vector bundle associated with HB (see [1].2.28). In terms of a fixed frame such as ([1].2.15) a wave function would be written as

$$\begin{aligned}\psi:\ \mathbb{R}^{1,3} &\longrightarrow C\ell_{1,3}\ \frac{1}{2}(1+\gamma_3\gamma_0),\\ \psi(x) &= w_{\Omega}\psi^{\Omega}(x).\end{aligned} \tag{6}$$

For the <u>free</u> electron/positron theory, HB possesses an integrable connection whose horizontal slices correspond to the trivial sections of ([1].2.28). Consequently, covariant derivatives are taken without vector potentials or Christoffel symbols,

$$D\psi = w_\Omega \otimes dx^\mu \; \partial_\mu \psi^\Omega(x). \tag{7}$$

The position that we are taking is that the basis of the Clifford algebra $C\ell_{1,3}$ ($R_{1,3}$ in Cho *et al.* (1990) [1]), and hence the 4-component spinor basis w_Ω of $C\ell_{1,3}\ \frac{1}{2}(1+\gamma_3\gamma_0)$, are translationally invariant because the tangent basis vectors $e_\mu \leftrightarrow \gamma_\mu$ (below [1].2.1) are chosen that way. The Dirac wave equation in the presence of an external field $A_\mu(x)$ is written in a basis independent form as

$$\gamma^\mu \left(\hbar \partial_\mu \psi(x)\gamma - \frac{e}{c} A_\mu(x)\psi(x)\right) - mc\psi(x) = 0, \tag{8}$$

from which the discrete transformations ([1].2.22) are found by requiring the set of solutions to be invariant under,

$$\psi_{\mathrm{P}}|_x \equiv \mathbf{P}(\psi(x^0, -\vec{x})),\ \psi_{\mathrm{T}}|_x \equiv \mathbf{T}(\psi(-x^0, \vec{x})),\ \psi_{\mathrm{C}}|_x \equiv \mathbf{C}(\psi(x^0, \vec{x})). \tag{9}$$

The quantum fields $\Psi(x)$ and $\Psi^\dagger(x)$ of (3) are easily seen to satisfy the same equation satisfied by the Dirac wave theory, *i.e.* (8) with the vector potential $A_\mu = 0$. Rather than postulating this equation for the quantum field we show that it can be defined simply as an invariant function of the form (4). In group representation language (4) is a cross section of a complex vector bundle associated with the principal bundle HB of the homogeneous group H (see [1].2.28). The representation of H is a tensor product of the 4-dimensional Dirac spinor representation given in Sec. II Cho *et al.* (1990) [1] and the ∞-dimensional representation of $H \subset P = R^4 \circledS H$ given in ([1].4.17-19). The $C\ell_{1,3}\ \frac{1}{2}(1+\gamma_3\gamma_0)$ parts of the function (4) are the same as in (6-9) and the reader should consult ([1].2.29-31) on how to compare the values of functions at different points $x \in \mathbb{R}^{1,3}$. The $\mathcal{C}^1$ part of the function requires some additional discussion. Recall from section IV of Cho *et al.* (1990) [1] that $\mathcal{C}^1$ is a complexified real orthogonal space $\mathcal{C}^1_R \cong \mathcal{H}_R$ which acts as a representation space for P acting as orthogonal transformations $\mathbf{O}_\gamma \cong \mathbf{U}_{\mathrm{H}}$. Consequently the value of a function $\mathbb{R}^{1,3} \longrightarrow \mathcal{C}^1$ at x is acted on by $H \cong H_x \subset P$ see ([1].2.31). This unorthodox action of the bundle's group occurs because members of $\mathcal{C}^1$ are not translationally invariant *e.g.,* the basis functions are phase rotated as in ([1].4.19) when transported from x to $x + r$. To carry out the imposed invariance calculation it is not necessary to unravel the H actions except at $x = 0$.

The inhomogeneous group P acts on the associated vector bundle by virtue of its action on the two parts of the tensor product $C\ell_{1,3}\ \frac{1}{2}(1+\gamma_3\gamma_0)$ and $\mathcal{C}^1$. If we ask for all invariant cross sections (4), we find combinations of fields like $\Psi(x)$ and its Hermitian conjugate $\Psi^\dagger(x)$ of (3). To make the derivation as simple as possible we choose a fixed basis for the tensor product space $C\ell_{1,3}\ \frac{1}{2}(1+\gamma_3\gamma_0) \otimes \mathcal{C}^\infty$, $\left\{ z_\Omega \otimes \mathbf{b}^A_{\vec{p}},\ z_\Omega \otimes \mathbf{d}^{A\dagger}_{\vec{p}},\ z_\Omega \otimes \mathbf{b}^{A\dagger}_{\vec{p}},\ z_\Omega \otimes \mathbf{d}^A_{\vec{p}} \right\}$ and write the value of an arbitrary cross section at 'x' as

$$\mathbf{\Psi}(x) = \int dp \;\; \{ \;\; z_\Omega B^\Omega_{A,\vec{p}}(x) \otimes \mathbf{b}^A_{\vec{p}} + z_\Omega \overline{D}^\Omega_{A,\vec{p}}(x) \otimes \mathbf{d}^{A\,\dagger}_{\vec{p}} + \;\; z_\Omega \overline{B}^\Omega_{A,\vec{p}}(x) \otimes \mathbf{b}^{A\,\dagger}_{\vec{p}} + z_\Omega D^\Omega_{A,\vec{p}}(x) \otimes \mathbf{d}^A_{\vec{p}} \}, \tag{10}$$

where $B^\Omega_{A,\vec{p}}(x), \overline{B}^\Omega_{A,\vec{p}}(x), \overline{D}^\Omega_{A,\vec{p}}(x)$,and $D^\Omega_{A,\vec{p}}(x)$ are arbitrary $\gamma-$complex components. These constants can be shifted, without consequence, to the right of $\otimes$ and changed to i-complex by using ([1].4.15). Invariance of $\mathbf{\Psi}$ under the action of the inhomogeneous group requires

$$\begin{aligned}
\big[(a,h)\mathbf{\Psi}\big] &= \mathbf{\Psi}, \\
\leftrightarrow \big[(a,h)\mathbf{\Psi}\big](\mathrm{x}) &= \mathbf{\Psi}(\mathrm{x}), \\
\text{where} \quad \big[(\mathrm{a},\mathrm{h})\mathbf{\Psi}\big](\mathrm{x}) &\equiv (a,h)\big\{\mathbf{\Psi}(\mathrm{h}^{-1}(\mathrm{x}-\mathrm{a}))\big\},
\end{aligned} \tag{11}$$

and (a,h) on the right of (11) is the tensor product action of $(a,h) \in \;\; R^4 \circledS H$ on the vector $\mathbf{\Psi}(h^{-1}(x-a)) \in C\ell_{1,3} \; \frac{1}{2}(1+\gamma_3\gamma_0)\otimes \mathcal{C}^1$. We start with translational invariance using ([1].4.19),

$$(a,I)\big\{\mathbf{\Psi}(\mathrm{x}-\mathrm{a})\big\} = \mathbf{\Psi}(\mathrm{x}), \quad \Longrightarrow \tag{12}$$

$$\begin{array}{lcl}
\int dp\big\{z_\Omega B^\Omega_{A,\vec{p}}(x-a)\otimes e^{-ip_\mu a^\mu}\,\mathbf{b}^A_{\vec{p}} & & \int dp\big\{z_\Omega B^\Omega_{A,\vec{p}}(x)\otimes \mathbf{b}^A_{\vec{p}} \\
\quad +z_\Omega \overline{D}^\Omega_{A,\vec{p}}(x-a)\otimes e^{+ip_\mu a^\mu}\,\mathbf{d}^{A\,\dagger}_{\vec{p}} & = & \quad +z_\Omega \overline{D}^\Omega_{A,\vec{p}}(x)\otimes \mathbf{d}^{A\,\dagger}_{\vec{p}} \\
\quad +z_\Omega \overline{B}^\Omega_{A,\vec{p}}(x-a)\otimes e^{+ip_\mu a^\mu}\,\mathbf{b}^{A\,\dagger}_{\vec{p}} & & \quad +z_\Omega \overline{B}^\Omega_{A,\vec{p}}(x)\otimes \mathbf{b}^{A\,\dagger}_{\vec{p}} \\
\quad +z_\Omega D^\Omega_{A,\vec{p}}(x-a)\otimes e^{-ip_\mu a^\mu}\,\mathbf{d}^A_{\vec{p}}\big\} & & \quad +z_\Omega D^\Omega_{A,\vec{p}}(x)\otimes \mathbf{d}^A_{\vec{p}}\big\}.
\end{array} \tag{13}$$

By moving all complex coefficients to the left of $\otimes$ using ([1].4.15) and equating coefficients of basis vectors $z_\Omega \otimes \mathbf{b}^A_{\vec{p}}$ *etc.* we have

$$\begin{array}{l}
B^\Omega_{A,\vec{p}}(x) = B^\Omega_{A,\vec{p}}(0)e^{-\gamma p_\mu x^\mu}, \;\; D^\Omega_{A,\vec{p}}(x) = D^\Omega_{A,\vec{p}}(0)e^{+\gamma p_\mu x^\mu}, \\
\overline{D}^\Omega_{A,\vec{p}}(x) = \overline{D}^\Omega_{A,\vec{p}}(0)e^{+\gamma p_\mu x^\mu}, \;\; \overline{B}^\Omega_{A,\vec{p}}(x) = \overline{B}^\Omega_{A,\vec{p}}(0)e^{-\gamma p_\mu x^\mu}.
\end{array} \tag{14}$$

Using the results of (14) in (13) and leaving out '(0)' from the components we next require $s \in SL(2,\mathbb{C})$ invariance. From (11) and ([1].4.16)

$$\begin{aligned}
(0,s)\big\{\mathbf{\Psi}(\Lambda^{-1}(\mathrm{x}))\big\} = \int \mathrm{dp} \;\; \{ \;\; & S(z_\Omega)B^\Omega_{A,\vec{p}} \otimes e^{-ip_\mu \Lambda^{-1}(x)^\mu} D^{\frac{1}{2}}[-\vec{\omega}(s,\vec{\Lambda p})]^A{}_B \;\, \mathbf{b}^B_{\vec{\Lambda p}} \\
+ \;\; & S(z_\Omega)\overline{D}^\Omega_{A,\vec{p}} \otimes e^{+ip_\mu \Lambda^{-1}(x)^\mu} \sigma_1{}^A{}_B D^{\frac{1}{2}}[-\vec{\omega}(s,\vec{\Lambda p})]^B{}_C \sigma_1{}^C{}_D \;\, \mathbf{d}^{D\,\dagger}_{\vec{\Lambda p}} \\
+ \;\; & S(z_\Omega)\overline{B}^\Omega_{A,\vec{p}} \otimes e^{+ip_\mu \Lambda^{-1}(x)^\mu} \overset{*}{D}{}^{\frac{1}{2}}[-\vec{\omega}(s,\vec{\Lambda p})]^A{}_B \;\, \mathbf{b}^{B\,\dagger}_{\vec{\Lambda p}} \\
+ \;\; & S(z_\Omega)D^\Omega_{A,\vec{p}} \otimes e^{-ip_\mu \Lambda^{-1}(x)^\mu} \sigma_1{}^A{}_B \overset{*}{D}{}^{\frac{1}{2}}[-\vec{\omega}(s,\vec{\Lambda p})]^B{}_C \sigma_1{}^C{}_D \;\, \mathbf{d}^D_{\vec{\Lambda p}}\} \\
= \int dp \;\; \{ \;\; & z_\Omega S^\Omega_\Sigma \; B^\Sigma_{B,\vec{\Lambda^{-1}p}} D^{\frac{1}{2}}[-\vec{\omega}(s,\vec{p})]^B{}_A \;\; \otimes e^{-ip_\mu x^\mu}\mathbf{b}^A_{\vec{p}}
\end{aligned}$$

$$
\begin{aligned}
&+\ z_\Omega S^\Omega_{\ \Sigma}\ \overline{D}^\Sigma_{B,\vec{\Lambda^{-1}p}}\sigma_1{}^B{}_C D^{\frac{1}{2}}[-\vec{\omega}(s,\vec{p})]^C{}_D\sigma_1{}^D{}_A \ \otimes e^{+ip_\mu x^\mu}\,\mathbf{d}^{A\dagger}_{\vec{p}} \\
&+\ z_\Omega S^\Omega_{\ \Sigma}\ \overline{B}^\Sigma_{B,\vec{\Lambda^{-1}p}} D^{\frac{1}{2}}[-\vec{\omega}(s,\vec{p})]^B{}_A \ \otimes e^{+ip_\mu x^\mu}\,\mathbf{b}^{A\dagger}_{\vec{p}} \\
&+\ z_\Omega S^\Omega_{\ \Sigma}\ D^\Sigma_{B,\vec{\Lambda^{-1}p}}\sigma_1{}^B{}_C D^{\frac{1}{2}}[-\vec{\omega}(s,\vec{p})]^C{}_D\sigma_1{}^D{}_A \ \otimes e^{-ip_\mu x^\mu}\,\mathbf{d}^{A}_{\vec{p}}\,\} \\
= \mathbf{\Psi}(\mathrm{x}) = \int \mathrm{dp}\ \ &\{\ z_\Omega B^\Omega_{A,\vec{p}} \otimes e^{-ip_\mu x^\mu}\,\mathbf{b}^A_{\vec{p}} + z_\Omega \overline{B}^\Omega_{A,\vec{p}} \otimes e^{+ip_\mu x^\mu}\,\mathbf{b}^{A\dagger}_{\vec{p}} \\
&+\ z_\Omega \overline{D}^\Omega_{A,\vec{p}} \otimes e^{+ip_\mu x^\mu}\,\mathbf{d}^{A\dagger}_{\vec{p}} + z_\Omega D^\Omega_{A,\vec{p}} \otimes e^{-ip_\mu x^\mu}\,\mathbf{d}^{A}_{\vec{p}}\,\}, \qquad (15)
\end{aligned}
$$

where a momentum integral change $p \to \Lambda^{-1}p$, has been made and S, the spin representative of s, has been written as a matrix using the z_Ω basis, $S(z_\Sigma) = z_\Omega S^\Omega_{\ \Sigma}$. By simply equating coefficients of the tensor product basis *e.g.* $z_\Omega \otimes \mathbf{b}^A_{\vec{p}}$, we obtain,

$$
\begin{aligned}
B^\Omega_{A,\vec{p}} &= [e^{\frac{\varsigma^i}{2}\alpha_i} W(s,\vec{p}) e^{-\frac{\acute{\varsigma}^i}{2}\alpha_i}]^\Omega_{\ \Sigma}\ B^\Sigma_{B,\vec{\Lambda^{-1}p}} D^{\frac{1}{2}}[-\vec{\omega}(s,\vec{p})]^B{}_A, \\
\overline{D}^\Omega_{A,\vec{p}} &= [e^{\frac{\varsigma^i}{2}\alpha_i} W(s,\vec{p}) e^{-\frac{\acute{\varsigma}^i}{2}\alpha_i}]^\Omega_{\ \Sigma}\ \overline{D}^\Sigma_{B,\vec{\Lambda^{-1}p}}\sigma_1{}^B{}_C D^{\frac{1}{2}}[-\vec{\omega}(s,\vec{p})]^C{}_D\sigma_1{}^D{}_A, \\
\overline{B}^\Omega_{A,\vec{p}} &= [e^{\frac{\varsigma^i}{2}\alpha_i} W(s,\vec{p}) e^{-\frac{\acute{\varsigma}^i}{2}\alpha_i}]^\Omega_{\ \Sigma}\ \overline{B}^\Sigma_{B,\vec{\Lambda^{-1}p}} D^{\frac{1}{2}}[-\vec{\omega}(s,\vec{p})]^B{}_A, \\
D^\Omega_{A,\vec{p}} &= [e^{\frac{\varsigma^i}{2}\alpha_i} W(s,\vec{p}) e^{-\frac{\acute{\varsigma}^i}{2}\alpha_i}]^\Omega_{\ \Sigma}\ D^\Sigma_{B,\vec{\Lambda^{-1}p}}\sigma_1{}^B{}_C D^{\frac{1}{2}}[-\vec{\omega}(s,\vec{p})]^C{}_D\sigma_1{}^D{}_A, \qquad (16)
\end{aligned}
$$

where ([1].3.6) has been used to write $S^\Omega_{\ \Sigma}$ in terms of a Wigner rotation $W(s,\vec{p})$ and two pure boosts, $e^{\frac{\varsigma^i}{2}\alpha_i}$ and $e^{\frac{\acute{\varsigma}^i}{2}\alpha_i}$, of $(mc,0,0,0)$ to p^μ and $\Lambda^{-1\mu}{}_\nu p^\nu$ respectively (see [1].3.4). By letting S be the pure boost $e^{\frac{\varsigma^i}{2}\alpha_i}$, we have $\vec{\omega}(s,\vec{p}) = \vec{\Lambda^{-1}p} = \vec{0}$ and conclude,

$$
\begin{aligned}
B^\Omega_{A,\vec{p}} &= [e^{\frac{\varsigma^i}{2}\alpha_i}]^\Omega_{\ \Sigma} B^\Sigma_{A,\vec{0}}, &\qquad \overline{B}^\Omega_{A,\vec{p}} &= [e^{\frac{\varsigma^i}{2}\alpha_i}]^\Omega_{\ \Sigma} \overline{B}^\Sigma_{B,\vec{0}}, \\
\overline{D}^\Omega_{A,\vec{p}} &= [e^{\frac{\varsigma^i}{2}\alpha_i}]^\Omega_{\ \Sigma} \overline{D}^\Sigma_{A,\vec{0}}, &\qquad D^\Omega_{A,\vec{p}} &= [e^{\frac{\varsigma^i}{2}\alpha_i}]^\Omega_{\ \Sigma} D^\Sigma_{B,\vec{0}}.
\end{aligned} \qquad (17)
$$

Using (17) in (16), but dropping the $\vec{0}$ on the above constants, gives the following constraints on the constants appearing in the tensor coefficients,

$$
\begin{aligned}
B^\Omega_A &= [W(s,\vec{p})]^\Omega_{\ \Sigma}\ B^\Sigma_B\ D^{\frac{1}{2}}[-\vec{\omega}(s,\vec{p})]^B{}_A, \\
\overline{D}^\Omega_A &= [W(s,\vec{p})]^\Omega_{\ \Sigma}\ \overline{D}^\Sigma_B\ \sigma_1{}^B{}_C D^{\frac{1}{2}}[-\vec{\omega}(s,\vec{p})]^C{}_D\sigma_1{}^D{}_A, \\
\overline{B}^\Omega_A &= [W(s,\vec{p})]^\Omega_{\ \Sigma}\ \overline{B}^\Sigma_B\ D^{\frac{1}{2}}[-\vec{\omega}(s,\vec{p})]^B{}_A, \\
D^\Omega_A &= [W(s,\vec{p})]^\Omega_{\ \Sigma}\ D^\Sigma_B\ \sigma_1{}^B{}_C D^{\frac{1}{2}}[-\vec{\omega}(s,\vec{p})]^C{}_D\sigma_1{}^D{}_A,
\end{aligned} \qquad (18)
$$

where the matrix expression for $[W(s,\vec{p})]^\Omega_{\ \Sigma}$ is given in ([1].3.7). The irreducibility of the $j = 1/2$ representation of SU_2 then requires,

$$
\begin{aligned}
B^A_B &\propto \overline{B}^A_B \propto \delta^A_B, &\qquad D^A_B &\propto \overline{D}^A_B \propto \sigma_1{}^A{}_B, \\
B^{A+2}_B &\propto \overline{B}^{A+2}_B \propto \sigma_1{}^A{}_B, &\qquad D^{A+2}_B &\propto \overline{D}^{A+2}_B \propto \delta^A_B.
\end{aligned} \qquad (19)
$$

Consequently, the most general $R^4 \circledS SL(2,\mathbb{C})$ invariant $C\ell_{1,3}\ \frac{1}{2}(1+\gamma_3\gamma_0)\otimes \mathcal{C}^1$ section, see (10), is completely determined up to eight complex constants c^i and

$\acute{c}^i,\ i = \{1,2,3,4\}$,

$$\mathbf{\Psi}(\mathrm{x}) = \int \mathrm{dp} \left\{ \begin{array}{ll} u_{A,\vec{p}} c^1 e^{-\gamma p_\mu x^\mu} \otimes \mathbf{b}^A_{\vec{p}} & +v_{A,\vec{p}} \sigma_1{}^A{}_B \acute{c}^1 e^{-\gamma p_\mu x^\mu} \otimes \mathbf{b}^B_{\vec{p}} \\ +v_{A,\vec{p}} c^2 e^{+\gamma p_\mu x^\mu} \otimes \mathbf{d}^{A\dagger}_{\vec{p}} & +u_{A,\vec{p}} \sigma_1{}^A{}_B \acute{c}^2 e^{+\gamma p_\mu x^\mu} \otimes \mathbf{d}^{B\dagger}_{\vec{p}} \\ +u_{A,\vec{p}} c^3 e^{-\gamma p_\mu x^\mu} \otimes \mathbf{b}^{A\dagger}_{\vec{p}} & +v_{A,\vec{p}} \sigma_1{}^A{}_B \acute{c}^3 e^{-\gamma p_\mu x^\mu} \otimes \mathbf{b}^{B\dagger}_{\vec{p}} \\ +v_{A,\vec{p}} c^4 e^{+\gamma p_\mu x^\mu} \otimes \mathbf{d}^A_{\vec{p}} & +u_{A,\vec{p}} \sigma_1{}^A{}_B \acute{c}^4 e^{+\gamma p_\mu x^\mu} \otimes \mathbf{d}^B_{\vec{p}} \end{array} \right\}. \tag{20}$$

Parity invariance requires

$$(0,\mathsf{P})\left\{\mathbf{\Psi}(\mathrm{x}^0,-\vec{\mathrm{x}})\right\} = \mathbf{\Psi}(\mathrm{x}^0,\vec{\mathrm{x}}), \tag{21}$$

with (0,P) acting on $C\ell_{1,3}\ \frac{1}{2}(1+\gamma_3\gamma_0)$ by ([1].2.22) and on $\mathcal{C}^1$ by ([1].4.18). The needed expressions are

$$\mathbf{P}(u_{A,\vec{p}}) = \pm u_{A,-\vec{p}}, \qquad \mathbf{P}(v_{A,\vec{p}}) = \mp v_{A,-\vec{p}}, \tag{22}$$

which follow from ([1].2.22-23), (5), and ([1].2.15). For example:

$$\begin{aligned} &\mathbf{P}(u_{A,\vec{p}}) = \mathbf{P}(e^{\frac{\zeta^i}{2}\alpha_i} z_A) = \pm\gamma_0 e^{\frac{\zeta^i}{2}\alpha_i} z_A \\ &= e^{-\frac{\zeta^i}{2}\alpha_i}\mathbf{P}(z_A) = e^{-\frac{\zeta^i}{2}\alpha_i}(\pm z_A) = \pm u_{A,-\vec{p}}. \end{aligned} \tag{23}$$

After using (22) and the $O_{(0,\mathsf{P})}$ terms from ([1].4.18), and changing the momentum integration variable $\vec{p} \to -\vec{p}$, we see that the four terms in (20) containing c^i are invariant but the four terms containing $\acute{c}^i$ change signs and hence must vanish. By using a projective representation for the parity component rather than a representation, *i.e.* $\mathbf{P}(c) = \gamma_0 c e^{\phi_P \gamma\alpha_3}$, the c^i terms can be made to vanish and the $\acute{c}^i \neq 0$, (choose $\phi_P = \pm\pi/2$). Since we are aiming for the Dirac theory we choose invariance and eliminate the $\acute{c}^i$ terms. Time reversal invariance requires

$$(0,\mathsf{T})\left\{\mathbf{\Psi}(-\mathrm{x}^0,\vec{\mathrm{x}})\right\} = \mathbf{\Psi}(\mathrm{x}^0,\vec{\mathrm{x}}), \tag{24}$$

with (0,T) acting on $C\ell_{1,3}\ \frac{1}{2}(1+\gamma_3\gamma_0)$ by ([1].2.22) and on $\mathcal{C}^1$ by $A_{(0,\mathsf{T})}$ of ([1].4.18). The needed expressions are

$$\mathbf{T}(u_{A,\vec{p}}) = u_{B,-\vec{p}}\sigma_2{}^B{}_A e^{\phi_T\gamma\alpha_3}, \qquad \mathbf{T}(v_{A,\vec{p}}) = -v_{B,-\vec{p}}\sigma_2{}^B{}_A e^{\phi_T\gamma\alpha_3}, \tag{25}$$

which follow from ([1].2.22-23), (5), and ([1].2.15). For example:

$$\begin{aligned} &\mathbf{T}(u_{A,\vec{p}}) = -\gamma_0 e^{\frac{\zeta^i}{2}\alpha_i} z_A \gamma_2 e^{\phi_T\gamma\alpha_3} \\ &= e^{-\frac{\zeta^i}{2}\alpha_i}\mathbf{T}(z_A) = e^{-\frac{\zeta^i}{2}\alpha_i}(z_B\sigma_2{}^B{}_A)e^{\phi_T\gamma\alpha_3} = u_{B,-\vec{p}}\sigma_2{}^B{}_A e^{\phi_T\gamma}. \end{aligned} \tag{26}$$

After using $c^i e^{-\gamma p_\mu x^\mu}\gamma_2 = \gamma_2 c^{\star i} e^{+\gamma p_\mu x^\mu}$, [where $(\gamma^\star = -\gamma)$ from ([1].2.18)], (24) and ([1].4.18), and changing the momentum integration variable $\vec{p} \to -\vec{p}$, the c^i part of (15) is invariant provided

$$c^i = |c^i| e^{(\phi_T + n_i\pi)\gamma}, \quad n_i \in \{0, \pm1, \pm2, \cdots\}. \tag{27}$$

To require invariance up to a global phase change (U_1 group action) would seem more appropriate in view of the projective nature of T and C actions on the individual pieces of the tensor product; however, as we see below, it is possible to require invariance.

Charge conjugation invariance requires

$$(0,\mathsf{C})\left\{\mathbf{\Psi}(\mathrm{x}^0,\vec{\mathrm{x}})\right\} = \mathbf{\Psi}(\mathrm{x}^0,\vec{\mathrm{x}}), \tag{28}$$

with (0,C) acting on $C\ell_{1,3}\ \frac{1}{2}(1+\gamma_3\gamma_0)$ by ([1].2.22) and on $\mathcal{C}^1$ by $O_{(0,\mathsf{C})}$ of ([1].4.18). The needed expressions are

$$\mathbf{C}(u_{A,\vec{p}}) = -v_{B,\vec{p}}\sigma_3{}^B{}_A e^{\phi_C\gamma\alpha_3}, \qquad \mathbf{C}(v_{A,\vec{p}}) = -u_{B,\vec{p}}\sigma_3{}^B{}_A e^{\phi_C\gamma\alpha_3}, \tag{29}$$

which follow from ([1].2.22-23), (5), and (21). For example:

$$\begin{aligned}&\mathbf{C}(u_{A,\vec{p}}) = -\gamma e^{\frac{\varsigma^i}{2}\alpha_i} z_A\gamma_2 e^{\phi_C\gamma\alpha_3} = e^{\frac{\varsigma^i}{2}\alpha_i}\mathbf{C}(z_A)\\ &= e^{\frac{\varsigma^i}{2}\alpha_i}(-z_{B+2}\sigma_3{}^B{}_A)e^{\phi_C\gamma\alpha_3} = -v_{B,\vec{p}}\sigma_3{}^B{}_A e^{\phi_C\gamma}.\end{aligned} \tag{30}$$

After using (28) and ([1].4.18), the c^i part of (20) is invariant provided

$$c^1 = -c^{\star 4}e^{2\phi_C\gamma}, \qquad c^2 = -c^{\star 3}e^{2\phi_C\gamma}, \tag{31}$$

which, when put together with (26), gives

$$\begin{aligned}&|c^1| = |c^4|, \quad |c^2| = |c^3|, \quad \phi_C = \phi_T + \tfrac{(n_1+n_4\pm 1)\pi}{2}\\ &(n_1+n_4)-(n_2+n_3) = 0, \quad \mathrm{mod}\ 2.\end{aligned} \tag{32}$$

The remaining freedom in $\mathbf{\Psi}$ is now relatively trivial, two arbitrary real constants and one phase, and can be removed by requiring self (anti)conjugation (*i.e.* $\mathbf{\Psi}^\dagger = \pm\mathbf{\Psi}$) and appropriate normalization. The result is $\mathbf{\Psi} = \Psi \pm \Psi^\dagger$, where Ψ and $\Psi^\dagger$ are given in (3) with only a relative sign between the u and v spinors being left to convention.

With this construction we feel we have completed the description of Dirac's free quantum field theory of massive spin 1/2 particles in terms of Clifford algebras. The next obvious challenge is to find the appropriate extension of this work to the representation theory for interacting particles [i.e. where translational invariance is broken by the presence of the vector potential A_μ in (8)].

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