Mon. Not. R. Astron. Soc. **000**, 1–10 (2002) Printed 12 February 2007 (MN LATEX style file v2.2)

Steps for Solving the Radiative Transfer Equation for Arbitrary Flows in Stationary Spacetimes

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12 February 2007

ABSTRACT
We derive the radiative transfer equation for arbitrary stationary relativistic flows in stationary spacetimes, i.e., for steady-state transfer problems. We show how the standard characteristics method of solution developed by Mihalas and used throughout the radiative transfer community can be adapted to multi-dimensional applications with isotropic sources. Because the characteristics always coincide with geodesics and can always be specified by constants, direct integration of the characteristics derived from the transfer equation as commonly done in 1-D applications is not required. The characteristics are known for a specified metric from the geodesics. We give details in both flat and static spherically symmetric spacetimes. This work has direct application in 3-dimensional simulations of supernovae, gamma-ray bursts, and active galactic nuclei, as well as in modeling neutron star atmospheres.

Key words: radiative transfer — relativity.

1 INTRODUCTION

The solution of the equation of radiative transfer in relativistic flows is of considerable astrophysical interest. Steady-state and dynamical solutions of the transfer equation are particularly important for supernovae (SNe), gamma-ray bursts (GRB), and active galactic nuclei (AGN). The general form of the general relativistic transfer equation was derived by Lindquist (1966), who also derived the equation needed for neutrino transport in spherically symmetric flows such as the core-collapse of massive stars. This work was further extended in Wilson (1971), Bruenn (1985), and Baron et al. (1989). In the stellar community the fully-special relativistic transfer equation was derived and discussed by Mihalas (1980). General relativistic versions of the transfer equation have been derived and discussed by Morita & Kaneko (1984, 1986), Schinder & Bludman (1989), Zane et al. (1996), as well by Castor (2004). Mihalas (1980) proposed solving the transfer equation in the "comoving" frame using the method of characteristics. In this frame the momenta are measured by an observer moving with the flow, and the spatial coordinates are those of an inertial observer. In the steady-state spherically symmetric case the specific intensity I_ν is a function of only three variables, the *inertial frame* radius r, and two comoving momentum coordinates, the energy $\varepsilon = \mathrm{h}\nu$ and μ the cosine of the angle between the direction of the photon's momentum and the radial direction. Comoving frames are particularly useful because they simplify the form of the the collision terms in the transport equation. Mihalas made further use of the spherical symmetry by treating the spatial and momentum angle variation of I_ν separately from the $\partial I_\nu/\partial\nu$ term. Somewhat confusingly he plotted "characteristic lines", which have one variable in real space and another in momentum space (see Fig. 1 of Mihalas (1980)). These plots are curved lines and even though practitioners know that they are working in a mixed frame, it is common parlance to say "the characteristics are curved". Here we re-emphasize that photons move along geodesics (straight lines in flat spacetime) but demonstrate that for isotropic sources only one momentum variable (the energy) must be be co-moving. We show that even in arbitrary 3-dimensional flows one can choose parameters (coordinates) to label geodesics which do not change along phase-space characteristics (except for the affine parameter, or "distance" along the characteristic itself). In addition we show that the change in comoving wavelength along the characteristic can be handled by standard finite difference techniques. This procedure should simplify the development of fully 3-D radiation transfer codes both in flat space (applicable to variable stars, supernova and GRB spectra) and in curved spacetime (applicable to neutron star atmospheres and AGN).

Schinder & Bludman (1989) recognized that the momentum variables can be chosen as constants and the transfer equation simplified in the spherically symmetric case in

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the absence of a fluid flow. Although we developed our formulation independently, our work is an extension of theirs to the general 3-D case incorporating the effects of fluid flow. Their work used the method of variable Eddington factors, whereas our method is a characteristic based method and is specifically applicable to the case of arbitrary fluid flow.

In Sections 2 and 3 we introduce the Boltzmann and radiative transfer equations and the relevant phase-space quantities. In Sections 4 and 5 we look at characteristics in flat and spherically symmetric spacetimes respectively. In Section 6 we discuss the logical steps necessary to solve the steady state transfer equation for stationary spacetimes and in Section 7 we give our concluding remarks.

2 THE BOLTZMANN EQUATION FOR PHOTONS

In this section we do not restrict spacetime, but we neglect polarization effects. The Boltzmann Equation is an integro-differential equation for the invariant photon distribution function $F(x,p)$ on the photon's 7-dimensional phase-space (x,p). This can be thought of as the photon "on-shell" subspace of a full 8 dimensional particle phase-space. The number of photons ΔN found by observer $u(x)$ in a small 6-element $\Delta V_x \Delta P$ of phase-space at (x,p) is measured by the 6-form δN, i.e., $\Delta N = \delta N(\Delta V_x, \Delta P)$ where

$$\delta N \equiv F(x,p)\,\delta V_6, \tag{1}$$

and where (see Lindquist 1966, for details)

$$\delta\, V_6 \equiv -(u(x)\cdot p)\,\delta\, V_x\,\delta P. \tag{2}$$

In the above, $u(x)$ is an arbitrary observer's unit 4-velocity at spacetime point x,

$$-(u(x)\cdot p) = \mathrm{h}/\lambda \tag{3}$$

is the magnitude of the photon's 3-momentum as seen by observer $u(x)$, $\delta\, V_x$ is the observer dependent 3-dimensional volume element at x, and δP is the covariant volume element on the photons' 3-dimensional momentum space at (x,p). Here, h is Planck's constant and λ is the wavelength measured by observer $u(x)$.

The collisionless Boltzmann equation simply states that $F[x(\xi),p(\xi)]$ remains invariant (constant) along the Lagrangian flow of photons in phase-space generated by their geodesic motion in spacetime. Constancy of $F[x(\xi),p(\xi)]$ is a natural consequence of Liouville's theorem, i.e., δV_6 is invariant under this flow, and the constancy of ΔN due to the absence of non-gravitational interactions of the photons. Any lack of constancy of ΔN in a finite volume ΔV_6 is accounted for by a collision term (Oxenius (1986) not withstanding). To exhibit covariance, the Boltzmann equation with collisions, is often written as a differential equation

$$\frac{dF}{d\xi} = \frac{dx^\alpha}{d\xi}\frac{\partial F}{\partial x^\alpha} + \frac{dp^\alpha}{d\xi}\frac{\partial F}{\partial p^\alpha} = \left(\frac{dF}{d\xi}\right)_{coll}, \tag{4}$$

with $F(x,p)$ explicitly given as a function of 8 variables (all 4 components of momentum are included but constrained by $p{\cdot}p = 0$). The collision term on the R.H.S. is a measure of the rate of change of the number of photons ΔN in a ΔV_6 transported along the would-be paths of non-interacting photons in phase-space.

According to the geometrical optics approximation, photons travel on null spacetime geodesics independently of their wavelengths. Affine parameters, ξ, unique to each wavelength, can be chosen which generate the following orbits on phase-space:

$$\frac{dx^\alpha}{d\xi} = p^\alpha, \tag{5}$$

$$\frac{dp^\alpha}{d\xi} = -\Gamma^\alpha_{\beta\gamma}p^\beta p^\gamma, \tag{6}$$

which reduces (4) to

$$p^\alpha\frac{\partial F}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma}p^\beta p^\gamma\frac{\partial F}{\partial p^\alpha} = \left(\frac{dF}{d\xi}\right)_{coll}. \tag{7}$$

The R.H.S. is typically separated into absorption and emission terms

$$\left(\frac{dF}{d\xi}\right)_{coll} = -fF + g, \tag{8}$$

where $f(x,p)$ and $g(x,p)$ are identified respectively with the invariant absorptivity and emissivity (Morita & Kaneko 1986). These quantities implicitly depend on macroscopic properties of the interacting medium such as temperature, pressure, and density. The geodesic equations (5) and (6) are equations for the characteristic curves of the integro-differential PDE (7). These characteristic curves $(x(\xi),p(\xi))$ are simply affinely parameterized spacetime geodesics, lifted to the 7-dimensional photon phase-space, which project back onto the null geodesics of spacetime $x(\xi)$. It is important to understand that changing phase-space coordinates or changing parameters for phase-space curves, doesn't alter these curves at all, only their description, e.g., when mixed coordinates are used as in Mihalas (1980) straight line geodesics naturally appear curved.

Logically, solving the Boltzmann equation is a two step process: first solve the geodesic equations (5) and (6) for the required set of null geodesics and second solve the Boltzmann equation (4) with appropriate boundary conditions. These two steps are combined in what is commonly called the characteristics method where (7) is solved by changing the momentum variables to a comoving frame and the characteristic parameter to a nonphysical distance. These phase-space coordinate changes necessarily involve the fluid flow, and are unique only in highly symmetric cases. To tackle non-symmetric flows/spacetimes, however, the two steps are best kept separate. First solve the geodesic equations by using coordinates that are constant along the geodesics (or by finding the geodesics in any coordinate system and then transforming to constant coordinates) and second proceed to solve equation (4) in the form of the transfer equation as given in the next section.

3 THE RADIATION TRANSPORT EQUATION

The radiation transport equation is an integro-differential equation for the specific intensity $I(x,p)$ which is equivalent to the Boltzmann equation (4) or (7) for $F(x,p)$. Both are functions on the photon's 7-dimensional phase-space (x,p);

however, $I(x,p)$ depends on a choice of an observer at each point of spacetime through [1]

$$I_\lambda(x,p) = -\frac{c^2}{\mathrm{h}}(u(x)\cdot p)^5 F(x,p). \tag{9}$$

We have chosen to follow our group's convention and use λ rather than ν as often appears in the literature; however, it is straightforward to change between I_λ and I_ν using $\lambda I_\lambda = \nu I_\nu$. Once the observers are chosen, $I_\lambda(x,p)$ like $F(x,p)$, is a scalar. Defining a set of observers is equivalent to giving a unit time-like vector field on spacetime, $u(x)$, which appears in Eq. (9). Just as in equation (3), $-u(x)\cdot p$ is equal to the the photon's momentum as seen by observer $u(x)$. If $u(x)$ describes the material fluid with which the photons interact, I_λ is called the comoving specific intensity and λ the comoving wavelength.

The transport equation for $I_\lambda(x,p)$ is obtained from equations (4) and (8) by substituting $I_\lambda(x,p)$ for $F(x,p)$ using Eq. (9)

$$\frac{dI_\lambda}{d\xi} = -(\chi_\lambda\frac{\mathrm{h}}{\lambda} + \frac{5}{\lambda}\frac{d\lambda}{d\xi})I_\lambda + \eta_\lambda\frac{\mathrm{h}}{\lambda}, \tag{10}$$

where the observer dependent absorptivity χ_λ and emissivity η_λ are related to f and g by

$$\begin{aligned} \chi_\lambda &= -\frac{f}{(u(x)\cdot p)}, \\ \eta_\lambda &= \left(\frac{c^2}{\mathrm{h}}\right)(u(x)\cdot p)^4 g. \end{aligned} \tag{11}$$

The other term ($\propto d\lambda/d\xi$) on the right in Eq. (10) is present because the definition of I_λ, Eq. (9), explicitly depends on comoving λ. If as is customary we divide the extinction into two parts: "true absorption" κ_λ and "scattering" σ_λ, then $\chi_\lambda = \kappa_\lambda + \sigma_\lambda$. For a comoving observer we will also assume the emissivity is given by thermal emission (true absorption opacity κ_λ times the Planck function B_λ) and that scattering is elastic and isotropic. This assumption is inherently required for our present formulation. It is beyond the scope of this work to consider anisotropic or inelastic scattering, but it is not entirely clear to us that the method cannot be extended to the more general case. For a comoving observer, χ_λ depends only on the magnitude of the momentum and not its direction (given isotropic sources), and consequently is a function of only x and $u\cdot p$ in an arbitrary coordinate system.

If the energy density in the radiation field is written as $\epsilon_\lambda = 4\pi J_\lambda/c$ (where J_λ is the classic 0^{th} Eddington moment (Mihalas 1978), and ϵ_λ is defined in Eq. (47)), the emissivity becomes

$$\eta_\lambda = \kappa_\lambda B_\lambda + \sigma_\lambda J_\lambda. \tag{12}$$

When choosing coordinates (x,p) on phase-space for the purpose of evaluating I_λ it is obvious that λ itself should be one of the choices because it simplifies the evaluation of χ_λ. This is in fact the raison d'etre for using λ as one of the momentum coordinates. When attempting a numerical solution, any dependence of χ_λ on direction requires the use of a large number of angles in the kinematically favored forward direction. Since as one moves around the interaction region, the forward direction changes, numerically resolving the variation of χ_λ with direction can make the computational requirements enormous.

The reader should note that the right hand side of Eq. (10) differs slightly from the standard form of the non-relativistic static radiation transfer equation because the affine parameter ξ is not a physical distance. As we discuss below it coincides with a distance (up to a constant) in some spaces, e.g., flat spacetime.

4 FLAT SPACE SIMPLIFICATIONS

When solving Eq. (10) for the comoving intensity $I_\lambda(x,p)$ or Eq. (4) for the Boltzmann distribution function $F(x,p)$, the dimension of phase-space can effectively be reduced if there are common symmetries in spacetime, the interacting medium, and the boundary conditions. This is because the transport equation has to be solved on only one of each equivalent set of characteristics on the full phase-space. For example if the spacetime, $u(x)$, and the boundary conditions are stationary (i.e., have a timelike symmetry), a time dimension can be factored out and the required part of phase-space reduces to 6 dimensions (3 space and 3 momentum). If the spacetime, the flow, and the boundary conditions are stationary and spherically symmetric, phase-space reduces to 3 dimensions (1 space and 2 momentum).

4.1 Arbitrary Stationary Flows

For flat space with a stationary flow and static boundary conditions, 6 dimensions are required (3 space and 3 momentum). For space coordinates we choose the 3 Euclidean values $\mathbf{r}$ of the flat spacetime inertial system in which the flow is stationary

$$u(x) = u(\mathbf{r}) = \gamma(\mathbf{r})(1, \boldsymbol{\beta}(\mathbf{r})). \tag{13}$$

For a boundary we choose a sphere of fixed radius R surrounding the origin. We assume the emission and absorption coefficients vanish on and beyond $r = R$. On $r = R$ we assume $I_\lambda(x,p) = 0$ for photons with incoming directions, i.e., with $\mathbf{n}\cdot\mathbf{r} < 0$ [see Eq. (14) below], and what we look for by integrating the transfer equation is the outgoing intensity on $r = R$, i.e., we seek $I_\lambda(x,p)$ for photons with $\mathbf{n}\cdot\mathbf{r} > 0$. To make Eq. (10) as easy to solve as possible, we choose two of the three momentum variables from the direction cosines of the photon's direction in the inertial system

$$\frac{dx}{d\xi} = \left(\frac{dt}{d\xi}, \frac{d\mathbf{r}}{d\xi}\right) = \frac{\mathrm{h}}{\lambda_\infty}(1,\mathbf{n}) = \frac{\mathrm{h}}{\lambda_\infty}(1, n^x, n^y, n^z), \tag{14}$$

e.g., we choose (n^x, n^y). The subscript "∞" refers to wavelength as seen by inertial observers. If we were to use λ_∞ as a 3^{rd} coordinate, all momentum coordinates would be constant, see Eq. (14), and the characteristic equations (5) and (6) would be trivial. However, to accommodate the procedure used to solve Eq. (10) we must use the comoving wavelength λ as the third momentum coordinate:

[1] $I_\lambda(x,p)d\lambda dA d\Omega$ is the rate observer $u(x)$ detects energy crossing normal to his area dA in the direction p, within his solid angle $d\Omega$, and in his waveband $d\lambda$. Any locally-flat comoving reference frame at x associated with the comoving observer $u(x)$ can be used to evaluate dA and $d\Omega$. These frames are arbitrary up to a rotation at each point x, however, actually defining one at every point x isn't necessary.

$$\lambda = -\frac{\mathrm{h}}{(u(x)\cdot p)} = \frac{\lambda_\infty}{\gamma(\mathbf{r})(1-\mathbf{n}\cdot\boldsymbol{\beta}(\mathbf{r}))}. \tag{15}$$

The comoving specific intensity $I_\lambda(x,p) = I_\lambda(\mathbf{r},\mathbf{n})$ then depends on the spatial position $\mathbf{r}$, two direction angles in $\mathbf{n}$, and the comoving wavelength λ; however, the comoving absorption χ_λ and emission η_λ coefficients are independent of the direction angles. The obvious reason for choosing the first 5 of these 6 phase-space variables is that their dependence on the affine parameter ξ along any characteristic is easy to determine. The Euclidean position $\mathbf{r}(\xi)$ is linear in ξ and given by

$$\mathbf{r}(\xi) = \mathbf{b} + s\ \mathbf{n}, \tag{16}$$

where

$$s \equiv \left(\frac{\mathrm{h}}{\lambda_\infty}\xi\right). \tag{17}$$

The Euclidean distance s is measured in the fixed inertial frame and $\mathbf{b}$ the impact vector defined by $\mathbf{b}\cdot\mathbf{n} = 0$ (see Fig. 1). If λ_∞ is written as a combination of λ and $\mathbf{r}$ using Eq. (15), the combination in parenthesis in Eq. (16) still remains constant and the position coordinate part of a characteristic curve is still a straight line. The photon's direction $\mathbf{n}$ remains constant and although $\lambda(\xi)$ is a complicated function of ξ, it is given explicitly by substituting Eq. (16) into Eq. (15).

To logically connect I_λ with distant observations we can (somewhat arbitrarily) smoothly distort the comoving fluid $u(x)$ to coincide with the rest observers at some point beyond $r = R$ (see Fig. 1). Because χ_λ and η_λ vanish in this domain, the fluid's effect on observations is sterile and λ conveniently coincides with λ_∞. In most applications, the transfer from comoving to stationary observers is done abruptly at the boundary and is implemented by a Lorentz boost. The reason for using the comoving wavelength as the 6^{th} coordinate will become clearer when we alter the transport equation in such a way as to have characteristic curves which keep λ constant. Rewriting Eq. (10) by explicitly separating out the λ dependence we obtain

$$\left.\frac{\partial I_\lambda}{\partial \xi}\right|_\lambda + \left(\frac{d\lambda}{d\xi}\right)\frac{\partial I_\lambda}{\partial\lambda} = -\left(\chi_\lambda\frac{\mathrm{h}}{\lambda} + \frac{5}{\lambda}\frac{d\lambda}{d\xi}\right) I_\lambda + \eta_\lambda\frac{\mathrm{h}}{\lambda}. \tag{18}$$

If Euclidean coordinates on flat space are used as in Eq. (16) the first term in Eq. (18) is related to the $\mathbf{r}$ dependence of I_λ by

$$\left.\frac{\partial I_\lambda}{\partial\xi}\right|_\lambda = \frac{d\mathbf{r}}{d\xi}\cdot\nabla I_\lambda. \tag{19}$$

The $d\lambda/d\xi$ part of the tangent to the characteristic is computed from Eq. (15). If we would have chosen any 5 variables on phase-space, in addition to the comoving wavelength λ, the transport equation would have been exactly of the form Eq. (18). As long as the new coordinates for phase-space are given as functions of the original six, the characteristics in the new coordinates are found by direct substitution. The transfer equation in the form of Eq. (18) says that I_λ changes along a geodesic as usual because of emission-absorption and its explicit dependence on λ due to its definition Eq. (9); however, it also changes, when written as a function of λ, because of its implicit dependence on a comoving wavelength that changes ($\dot\lambda \neq 0$) along the geodesic. Another way to say the same thing is that when geodesics on spacetime, are lifted to phase-space they are not constrained to $\lambda =$ constant hypersurfaces (see Fig. 1) and hence if λ is used as one of the coordinates, I_λ changes because λ changes.

4.2 Radial Stationary Flows

In this section we try to clarify how our approach differs from the classic approach of Mihalas. To make contact with what is commonly done in the stationary spherically symmetric case where the flow is radial $\boldsymbol{\beta} = \beta(r)\hat{\mathbf{r}}$ and hence where phase-space reduces to 2 independent dimensions beyond λ, we introduce 1 inertial coordinate r and 1 comoving momentum coordinate μ. The wavelength and the radial coordinate can be evaluated from Eqs. (15), (16), and (17)

$$\begin{aligned} r(\xi) &= \sqrt{b^2+s^2}, \\ \lambda(\xi) &= \lambda_\infty\frac{\sqrt{1-\beta^2(r)}}{1-\beta(r)s/r}. \end{aligned} \tag{20}$$

The coordinate μ is the cosine of the angle between the radial direction and the direction of the photon in a frame instantaneously moving with the radial flow at $\mathbf{r}$ and is found by a radial Lorentz boost,

$$\mu(\xi) = \frac{\mathbf{n}\cdot\hat{\mathbf{r}} - \beta(r)}{1-\beta(r)\mathbf{n}\cdot\hat{\mathbf{r}}} = \frac{s/r-\beta(r)}{1-\beta(r)s/r}. \tag{21}$$

Equations (20) and (21) are the integrated characteristic equation for the resulting transport equation for $I_\lambda(r,\mu)$,

$$\left(\frac{dr}{d\xi}\frac{\partial I_\lambda}{\partial r} + \frac{d\mu}{d\xi}\frac{\partial I_\lambda}{\partial\mu}\right) + \frac{d\lambda}{d\xi}\frac{\partial I_\lambda}{\partial\lambda} = -\left(\chi_\lambda\frac{\mathrm{h}}{\lambda} + \frac{5}{\lambda}\frac{d\lambda}{d\xi}\right)I_\lambda + \eta_\lambda\frac{\mathrm{h}}{\lambda}, \tag{22}$$

which is equivalent to equation (3.1) of Mihalas (1980) for I_ν. Mihalas' parameter s_M is related to the standard affine parameter ξ by $ds_M = -(u\cdot p)d\xi = (\mathrm{h}/\lambda)d\xi = (\lambda_\infty/\lambda)ds$, and equals the differential spatial distance traveled by the photon as measured in the instantaneous local rest frame of the comoving observer (and shouldn't be confused with distance in any global frame). The "comoving" distance parameter s_M is the same for all photons with identical paths $\mathbf{r}(\xi)$ and depends on the fluid velocity $u(x)$ they intercept, but not on their individual wavelengths. When s_M is used Eq. (22) becomes

$$\left(\frac{dr}{ds_M}\frac{\partial I_\lambda}{\partial r} + \frac{d\mu}{ds_M}\frac{\partial I_\lambda}{\partial\mu}\right) + a(s_M,\lambda)\frac{\partial I_\lambda}{\partial\lambda} = -\left(\chi_\lambda + \frac{5}{\mathrm{h}}\frac{d\lambda}{d\xi}\right)I_\lambda + \eta_\lambda, \tag{23}$$

where $a(s_M,\lambda)$ is defined as

$$a(s_M,\lambda) \equiv \frac{\lambda}{\mathrm{h}}\frac{d\lambda}{d\xi}, \tag{24}$$

and is related to Mihalas's $a_M(s_M,\nu)$ by

$$a(s_M,\lambda) = \frac{\lambda^2}{\mathrm{h}}a_M(s_M,\nu) \tag{25}$$

By using Eqs. (17), (20), and (21), $a(s_M,\lambda)$ is easily evaluated,

$$a(s_M,\lambda) = \gamma\left[(1-\mu^2)\frac{\beta}{r} + \gamma^2\mu(\mu+\beta)\frac{d\beta}{dr}\right]\lambda, \tag{26}$$

as are the characteristic equations,

$$\frac{dr}{ds_M} = \gamma(\mu+\beta), \tag{27}$$

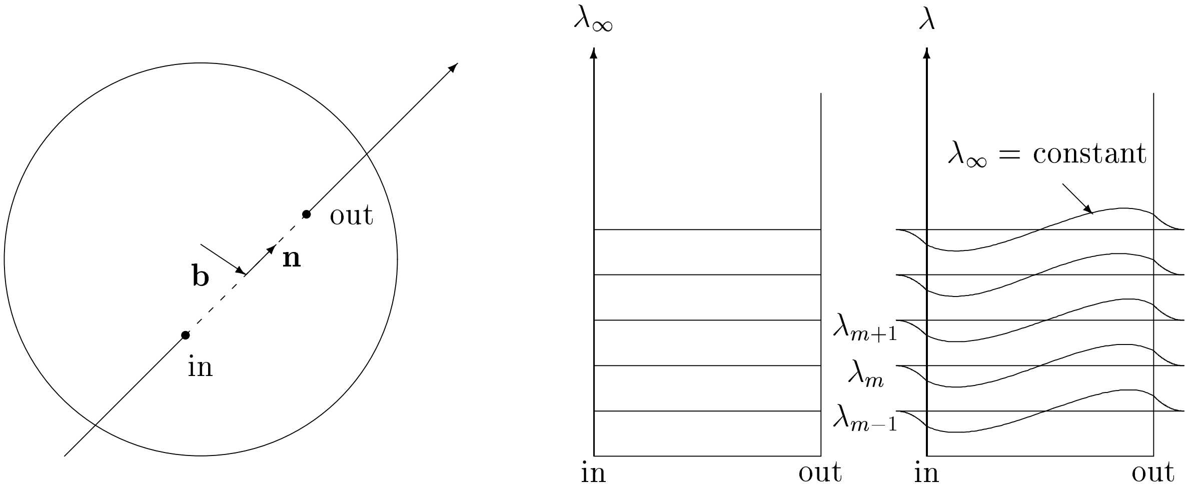

Figure 1. On the left a single flat-space geodesic, Eq. (16), described by the impact vector **b** and tangent **n**, enters and exits the boundary of the interaction region $r = R$. On the right, the related characteristic curves in phase-space corresponding to a discrete set of wavelengths are also shown. If λ_∞ is used as the momentum coordinate the characteristics of Eq. (18) are all straight lines; however, if the comoving λ is used the characteristics deviate from being straight but return to their λ_∞ values beyond the boundary where the comoving fluid coincides with the rest observers. The characteristics of the differenced equation (43) are defined only when λ_m is constant.

$$\frac{d\mu}{ds_M} = \gamma(1-\mu^2)\left[\frac{1+\mu\beta}{r} - \gamma^2(\mu+\beta)\frac{d\beta}{dr}\right]. \qquad (28)$$

These are to be compared with Eqs. (3.4a), (3.4b) and (3.9) of Mihalas (1980). We have arrived at the original transport equation obtained by Mihalas (1980), characteristics and all, but do not propose solving Eq. (23) However, if we were to now follow Mihalas, we would solve the single equation, $ds_M = (\mathrm{h}/\lambda)d\xi$, and obtain the integrated characteristics from (16) and (17). By first finding the geodesics and secondly deciding on what variables to use on phase-space, we obtain the characteristics by substitution. For phase-space coordinates used in Eq. (23), the characteristic curves had three non-vanishing tangent vectors Eqs. (26), (27), and (28). The procedure we use is to choose as many coordinates on phase-space as possible that remain constant along geodesics (lifted to phase-space). For radial flows one constant coordinate is chosen from **b** and **n** e.g., b, and the other two are ξ itself and λ. The transport equation is of the form (18) with characteristics having only two non-vanishing tangents, i.e., only two coordinates change along any characteristic, $\xi(\xi) = \xi$ and $\lambda(\xi)$ [see Eq. (20)]. Only one coordinate would change if we chose λ_∞ rather than λ. However, the differencing procedure described in §6 requires the comoving energy (i.e., λ for us) to be used as one of the phase-space coordinates.

It is clear from the above discussion that if an affine parameter such as ξ is used, the characteristic curves are completely determined and don't have to be constructed as integral curves of their tangents. If a non-affine parameter is used (e.g., s_M) all that must be done is to relate it to ξ (e.g., by inverting $s_M = -\int(u(x)\cdot p)d\xi$) along the geodesic and substitute. For a numerical example of see Baron & Hauschildt (2004). The affine parameter ξ is changed to optical depth τ_λ by a similar substitution. A single reference wavelength such as $\lambda_{\rm std} = 5000$ Å is usually chosen, making $d\tau_{\rm std} = \pm\chi_{\lambda_{\rm std}} ds_M$.[2] This choice greatly facilitates the generation of the spatial numerical grid.

5 STATIC SPHERICALLY SYMMETRICAL SPACETIMES WITH STATIONARY FLOWS

The relevant spacetime is

$$ds^2 = -e^{2\Phi(r)}c^2dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (29)$$

which we assume is asymptotically flat i.e., $\Lambda(\infty) = \Phi(\infty) = 0$. We assume boundary conditions similar to the flat case, i.e., beyond $r = R$ the fluid becomes transparent ($\eta_\lambda = \chi_\lambda = 0$) and that $I(x,p)$ vanishes for incoming photons. With a stationary, but otherwise arbitrary fluid flow, the effective dimension of phase-space reduces to 6 (3 space and 3 momentum). For this case we also distort $u(x)$ to coincide with static observers, i.e., (r,θ,ϕ)=constants, at some finite r value beyond $r = R$.

The metric (29) has 4 Killing vectors, 1 time translation and 3 rotations, which aid in finding photon orbits. In Fig. 2 we show the 1-parameter family of orbits confined to the $\theta = \pi/2$ plane and oriented symmetrically about $\phi = 0$. They are labeled by the single impact parameter b and the wavelength λ_∞ seen by a rest observer at $r = \infty$. The 3 non-vanishing momentum components are

$$cp^t = \frac{dct}{d\xi} = \frac{\mathrm{h}}{\lambda_\infty}e^{-2\Phi(r)}, \qquad (30)$$

$$p^\phi = \frac{d\phi_{\pi/2}}{d\xi} = \frac{\mathrm{h}}{\lambda_\infty}\frac{b\,e^{-\Phi(b)}}{r^2}, \qquad (31)$$

$$p^r = \frac{dr}{d\xi} \qquad (32)$$

[2] In classical plane-parallel and spherically symmetric radiative transfer τ is measured from the outside inward and the minus sign is appropriate.

$$= \pm \frac{\mathrm{h}}{\lambda_\infty} \frac{e^{-\Phi(b)-\Lambda(r)}}{r} \sqrt{r^2 e^{-2[\Phi(r)-\Phi(b)]} - b^2},$$

from which the spacetime orbits are

$$ct(r) = \pm \int_b^r \frac{r e^{\Lambda(r)+\Phi(b)-2\Phi(r)}\, dr}{\sqrt{r^2 e^{-2[\Phi(r)-\Phi(b)]} - b^2}}, \tag{33}$$

$$\phi_{\pi/2}(r) = \pm \int_b^r \frac{b\, e^{\Lambda(r)}\, dr}{r\sqrt{r^2 e^{-2[\Phi(r)-\Phi(b)]} - b^2}}. \tag{34}$$

All other photon orbits are obtained from these by rotations. According to Euler, any active rotation may be described using three rotations $R_z(\phi_p+\pi/2)R_x(\theta_p)R_z(\psi_b)$. For our purposes, (ϕ_p, θ_p) specifies the direction of the normal to the photon's orbital plane $\mathbf{n}_{plane}$, and ψ_b specifies the direction of the impact vector $\mathbf{b}$ within the orbit plane (see Fig. 2). These directions are space-like with no t components and are constrained by $\mathbf{n}_{plane} \cdot \mathbf{b} = 0$. Because the rotated orbit is orthogonal to $\mathbf{n}_{plane}$

$$\cos[\phi(r) - \phi_p] = -\cot\theta_p \cot\theta(r). \tag{35}$$

The actual orbit in parametric form $(r, \theta(r), \phi(r))$ is given by

$$\cos\theta(r) = \sin\theta_p \sin[\phi_{\pi/2}(r) + \psi_b], \tag{36}$$

$$\tan\phi(r) = \frac{\cos\theta_p \tan[\phi_{\pi/2}(r) + \psi_b] - \cot\phi_p}{1 + \cos\theta_p \cot\phi_p \tan[\phi_{\pi/2}(r) + \psi_b]}, \tag{37}$$

where $\phi_{\pi/2}(r)$ is given by Eq. (34). Equation (35) can be used in place of Eq. (37) if desired. For an arbitrary stationary flow, the only net symmetry is a time translation. The specific intensity depends on six variables e.g., $I_\lambda = I_\lambda(\mathbf{r}, \mathbf{p})$. Instead of using the spherical polar angles and spherical polar momentum coordinates, we use the radius r, the comoving wavelength λ, and four constants. It is convenient to choose the 4 constants from the set of 5 given by the impact vector $\mathbf{b}$ and the orbit plane normal $\mathbf{n}_{plane}$, e.g., the impact parameter b and the 3 angles $(\phi_p, \theta_p, \psi_b)$ described above. We now have $I_\lambda = I_\lambda(r, \mathbf{n}_{plane}, \mathbf{b})$, where the λ dependence is implied. We could have eliminated λ in favor of the constant λ_∞, however, as with the flat-space case, resolution of the atomic lines and accurately calculating the angular integral [see Eq. (47)] dictates that we use the comoving wavelength. The transfer equation is still equation (18), but now

$$\left.\frac{\partial I_\lambda}{\partial \xi}\right|_\lambda = \frac{dr}{d\xi}\frac{\partial I_\lambda}{\partial r}, \tag{38}$$

and the single coordinate part of the characteristic to solve is Eq. (33). The $\lambda(\xi)$ part is found by substituting into

$$\begin{aligned}\frac{\mathrm{h}}{\lambda} &= -u\cdot p = \frac{\mathrm{h}}{\lambda_\infty}\gamma e^{-\Phi(b)}\Bigg\{ e^{\Phi(b)-\Phi(r)} \mp \\ &\quad \frac{\sqrt{r^2 e^{-2[\Phi(r)-\Phi(b)]} - b^2}}{r}\beta^R \\ &+ \frac{b}{r}\sin\theta_p \sin[\phi(r) - \phi_p]\,\beta^\Theta - \frac{b}{r}\frac{\cos\theta_p}{\sin\theta(r)}\beta^\Phi\Bigg\},\end{aligned} \tag{39}$$

where the unit comoving 4-velocity $u(x)$ has been written in terms of an orthonormal tetrad adapted to the static spherical polar coordinates of Eq. (29),

$$\begin{aligned}u(x) &= \gamma\Bigg[\left(\frac{e^{-\Phi(r)}}{c}\frac{\partial}{\partial t}\right) + \\ &\quad \beta^R\left(e^{-\Lambda(r)}\frac{\partial}{\partial r}\right) + \beta^\Theta\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right) \\ &+ \beta^\Phi\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\Bigg].\end{aligned} \tag{40}$$

We assume that $u(x)$, and hence the $\beta^i(r, \theta, \phi)$ are given by prior hydrodynamical calculations. Again as was illustrated Fig. 1, if λ is being used as a coordinate, it varies along a characteristic.

Because of the paucity of hydro calculations in GR one can use β's that come from Newtonian gravity calculations as a reasonable approximation as long as the gravity remains weak. The β^i used in Eq. (40) should be Newtonian components relative to a static spherical polar coordinate system. The GR metric associated with the Newtonian calculation is given by

$$\begin{aligned}e^{2\Lambda} &= 1/\left(1 - \frac{2Gm(r)}{c^2 r} - \frac{\Lambda_0}{3}r^2\right) \\ \Phi(r) &= \int_0^r e^{2\Lambda}\left(\frac{4\pi G P_r}{c^4} + \frac{GM(r)}{c^2 r^3} - \frac{\Lambda_0}{3}\right) dr,\end{aligned} \tag{41}$$

here Λ_0 is the cosmological constant, $M(r)$ is approximated as the Newtonian mass contained in radius r, and $P_r(r)$ is the radial component of the pressure as seen by a static observer in the Newtonian hydro calculation. For a numerical application of this section to the static Schwarzschild case see Knop et al. (2007).

6 LOGICAL STEPS FOR NUMERICAL INTEGRATION

The solution of the spherically symmetric transfer equation for radially moving flows has been discussed in detail by Hauschildt and Baron (Hauschildt 1992; Hauschildt & Baron 1999, 2004b,a). In particular Hauschildt & Baron (2004a) showed how to stably difference the $\partial I_\lambda/\partial\lambda$ term. This method will also work in the more general case discussed here. We briefly describe the approach in simple terms, for more details see Hauschildt & Baron (2004a). One first selects a fixed set of comoving wavelengths, λ_m at which to evaluate the specific intensity I_m, and treats $\partial I_\lambda/\partial\lambda$ as a difference, e.g.,

$$\frac{\partial I_\lambda}{\partial\lambda} = \frac{I_m - I_{m-1}}{\lambda_m - \lambda_{m-1}}. \tag{42}$$

This turns the PDE, Eq. (18), into a discrete set of coupled ODEs with a single differential variable ξ for the set of I_m. The choice of the set $\{\lambda_m\}$ is dictated by the variation of χ_λ with λ. Rearranging, Eq. (18) becomes

$$\begin{aligned}\frac{dI_m}{d\xi} &+ \left[\frac{\dot\lambda_m}{\lambda_m - \lambda_{m-1}} + \frac{5\dot\lambda_m}{\lambda_m} + \chi_m\frac{\mathrm{h}}{\lambda_m}\right] I_m \\ &= \eta_m\frac{\mathrm{h}}{\lambda_m} + \frac{\dot\lambda_m}{\lambda_m - \lambda_{m-1}} I_{m-1},\end{aligned} \tag{43}$$

(see Mihalas 1980) where $\dot\lambda_m$ is determined by differentiating equations (15) or (39) with respect to the affine parame-

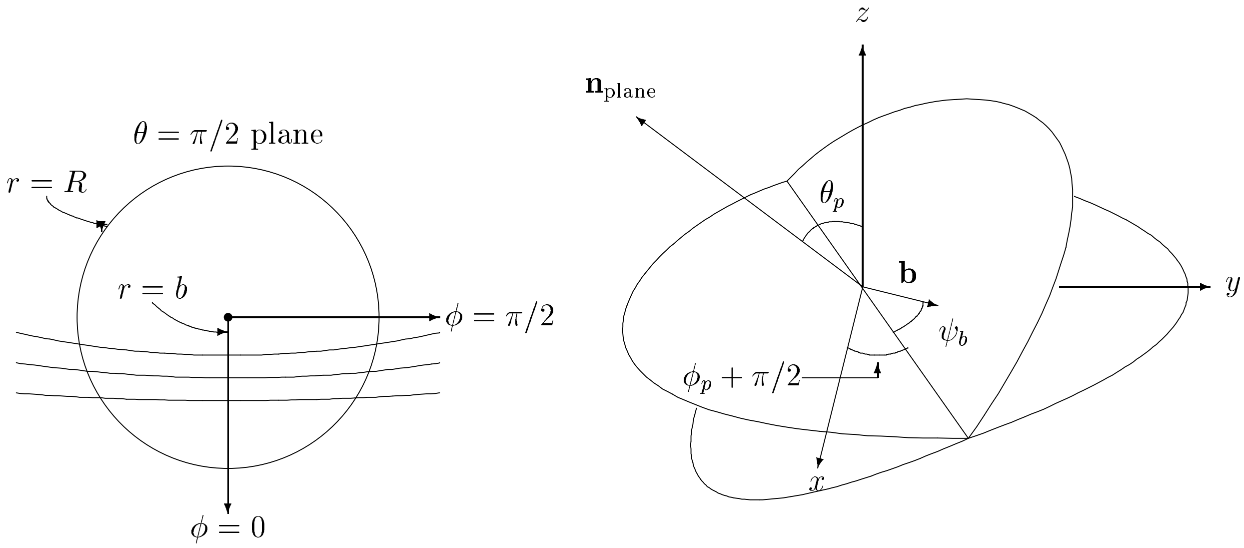

Figure 2. On the left three null geodesics, which lie in the $\theta = \pi/2$ plane and are symmetric about $\phi = 0$, are shown. On the right the active Euler rotations are shown that rotate the $\theta = \pi/2$ plane and its geodesics into the plane whose normal is $\mathbf{n}_{\rm plane}$. The impact vector $b\,\hat{\imath}$ is rotated into the $\mathbf{b}$ direction.

ter ξ. Even though phase-space is still 6-dimensional we are now attempting to find I_λ only on a discrete set of lines corresponding to constant comoving wavelengths λ_m. For both flat and static spherically symmetric spacetimes, the remaining continuous variables are 3 position and 2 momentum coordinates. To simplify solving Eq. (43), coordinates should be adapted to the particular spacetime symmetry and even then there is significant flexibility. For flat space we chose $\mathbf{r}$ and 2 of the $\mathbf{n}$ components making Eq. (16) the characteristics. In Fig. 1 a spacetime geodesic of Eq. (16) is shown as it enters and exits the boundary $r = R$. The related characteristic curves in phase-space corresponding to a discrete set of wavelengths are also shown. If λ_∞ is used as the 3$^{\rm rd}$ momentum coordinate, the characteristics of Eq. (18) are all straight lines; however, if the comoving λ is used the characteristics deviate from being straight but return somewhere outside the boundaries where the comoving fluid coincides with the rest observers. The λ part of the characteristics of the differenced transfer equation (43) has also changed, i.e., comoving λ_m now remains constant. The differencing term in Eq. (42) is an approximation which attempts to account for the change from λ_∞ = constant curves to λ = constant curves along which I_λ is propagated. In practice the differencing procedure used is more complicated (Hauschildt & Baron 2004a). When Eq. (18) is solved for I_λ at an exiting wavelength, e.g., λ_m, its value depends on I_λ values along its prior path for a continuous spectrum of neighboring λ values. However, when Eq. (43) is solved for I_m at an exiting point, its value depends on $I_{m'}$ values for only a discrete set of neighboring wavelength $\lambda_{m'}$. The discrete coupling appears in the RHS of Eq. (43) in η_m and explicitly as I_{m-1}.

The phase-space picture for the static spherically symmetrical spacetime is quite similar. We chose r and 4 constants from $\mathbf{n}_{plane}$ and $\mathbf{b}$ (see Fig. 2) and the only non-constant component of the characteristic curves of Eq. (43) is given by Eq. (33).

Often the source term is the more complicated part of the transfer equation. It contains various moments of the distribution function depending on the assumed physics of the photon-matter interactions. A detailed discussion of the moment formalism for steady state transfer can be found in Thorne (1981). To balance the interaction between the radiation field and the material, i.e., to obtain energy-momentum conservation of the radiation and matter, one must adjust the material's parameters such as temperature. The opacity and Planck function depend on the temperature of the material and thus this temperature has to be consistent with the rate energy is being deposited by the photon gas (which is not in thermal equilibrium with the matter). One obtains this consistency through use of the energy-momentum tensor of the photon gas. Per unit comoving wavelength it is defined as the comoving solid angle integral

$$\begin{aligned} T^{\alpha\beta}_\lambda(x) &\equiv c\int \frac{dP}{d\lambda} p^\alpha p^\beta F(x,p) \\ &= \frac{1}{c}\int I_\lambda(x,p)(n^\alpha_u + u^\alpha)(n^\beta_u + u^\beta)d\Omega, \end{aligned} \tag{44}$$

where we have decomposed the photon 4-momentum into two parts, one along the observer's 4-velocity u, and another perpendicular to it, $u \cdot n_u = 0$, by defining

$$p = -(p\cdot u)(n_u + u). \tag{45}$$

Equation (44) can be decomposed into energy, momentum, and pressure densities per unit wavelength as seen by $u(x)$,

$$\begin{aligned} T^{\alpha\beta}_\lambda(x) &= \epsilon_\lambda(x)u^\alpha(x)u^\beta(x) + \\ &\frac{1}{c}f^\alpha_\lambda(x)u^\beta(x) + \frac{1}{c}f^\beta_\lambda(x)u^\alpha(x) + p^{\alpha\beta}_\lambda(x), \end{aligned} \tag{46}$$

by defining

$$\epsilon_\lambda(x) \equiv \frac{1}{c}\int I_\lambda(x,p)d\Omega, \tag{47}$$

$$f^{\alpha}_{\lambda}(x) \equiv \int I_{\lambda}(x,p) n^{\alpha}_{u} d\Omega, \tag{48}$$

$$p^{\alpha\beta}_{\lambda}(x) \equiv \frac{1}{c}\int I_{\lambda}(x,p) n^{\alpha}_{u} n^{\beta}_{u} d\Omega, \tag{49}$$

where $f^{\alpha}_{\lambda} u_{\alpha} = 0$, $p^{\alpha\beta}_{\lambda} u_{\beta} = 0$ and $g_{\alpha\beta} p^{\alpha\beta}_{\lambda} = \epsilon_{\lambda} = 4\pi J_{\lambda}/c$, see Eq. (12). The rate per unit comoving wavelength per unit volume that 4-momentum (energy/c and momentum) is being transferred to the photons is

$$\begin{aligned} T^{\alpha\beta}_{\lambda\ ;\beta} &= \kappa_{\lambda}\left[-\epsilon_{\lambda} + \frac{4\pi}{c}B_{\lambda}\right]u^{\alpha} - \frac{1}{c}\chi_{\lambda} f^{\alpha}_{\lambda} \\ &= \frac{4\pi}{c}\kappa_{\lambda}\left[B_{\lambda} - J_{\lambda}\right]u^{\alpha} - \frac{1}{c}\chi_{\lambda} f^{\alpha}_{\lambda}. \end{aligned} \tag{50}$$

The familiar statement of radiative equilibrium (total absorptions equals total emissions in steady-state) for a comoving observer is the vanishing of the integral of $u_{\alpha} T^{\alpha\beta}_{\lambda\ ;\beta}$ over all wavelengths. Following Lindquist (1966) it is convenient to define the particle (photon) flux 4-vector

$$\begin{aligned} N^{\alpha}_{\lambda} &= \int \frac{dP}{d\lambda} p^{\alpha} F(x,p), \\ &= \frac{\lambda}{\mathrm{c}^2\mathrm{h}}\int d\Omega\, I_{\lambda}(x,p)(n^{\alpha}_{u} + u^{\alpha}), \\ &= \frac{\lambda}{\mathrm{ch}}\left[\epsilon_{\lambda}(x) u^{\alpha}(x) + \frac{1}{c} f^{\alpha}_{\lambda}(x)\right], \end{aligned} \tag{51}$$

in terms of which the rate per unit comoving wavelength that the photon number density is changing due to absorption and emission by sources can be computed as

$$c\, N^{\alpha}_{\lambda\ ;\alpha} = \frac{4\pi}{c}\kappa_{\lambda}\frac{\lambda}{\mathrm{h}}\left[B_{\lambda} - J_{\lambda}\right]. \tag{52}$$

By inserting Eq. (51) into the left hand side of Eq. (52) we obtain the identity

$$\left[(\epsilon_{\lambda}(x) u^{\alpha}(x))_{;\alpha} + \frac{1}{c} f^{\alpha}_{\lambda}(x)_{;\alpha}\right] = \frac{4\pi}{c}\kappa_{\lambda}\left[B_{\lambda} - J_{\lambda}\right]. \tag{53}$$

Integrating Eq. (53) over λ, together with radiative equilibrium from Eq. (50), leads to the familiar result that the divergence of the flux is zero for a static fluid (i.e., for a fluid where $u(x) \propto$ the timelike Killing field). Equations (50) and (53) are used to enforce the condition of radiative equilibrium (energy conservation, for example see Hauschildt & Baron 1999). At depth, it is necessary to use the vanishing of the LHS of Eq. (53) when integrated over λ, rather than the RHS, because at high optical depth J_{λ} is very close to B_{λ} and both are very large. Numerically, it is difficult to obtain an accurate result from the subtraction of two large numbers.

The above expressions for the decomposition of $T^{\alpha\beta}_{\lambda}$ and N^{α}_{λ} are valid in any coordinate system and only require knowledge of $u(x)$ in that coordinate system. Choosing a comoving frame for an comoving observer in a non-symmetric spacetime essentially introduces an arbitrary rotation at every point in space. Fortunately it is not necessary to pick such a frame. When evaluating the comoving solid angle integrals in Equations (47)–(49) using stationary coordinates one eliminates the comoving element $d\Omega$ in favor of the stationary solid angle $d\Omega_0$ using

$$d\Omega = \left(\frac{u_0 \cdot p}{u \cdot p}\right)^2 d\Omega_0. \tag{54}$$

Here u_0 is a unit vector pointing in the stationary frame's t direction. For the flat space case $u_0 \cdot p = -\mathrm{h}/\lambda_{\infty}$, $u \cdot p$ is given by Eq. (15), and

$$d\Omega = \frac{1 - \beta^2}{(1 - \boldsymbol{\beta}\cdot\mathbf{n})^2} d\Omega_0. \tag{55}$$

For the spherically symmetric gravity field

$$u_0 \cdot p = -\frac{\mathrm{h}}{\lambda_{\infty}} e^{-\Phi(r)}, \tag{56}$$

and $u \cdot p$ is given by Eq. (39). To evaluate $d\Omega$ we might have been tempted to introduce two comoving momentum variables μ, the *cosine* of the comoving polar angle, and ω, the comoving azimuthal angle, as customary. However, because of the arbitrariness of the flow evaluating these variables along a characteristic would have added two more complicated characteristics to solve and done little to aid integrating Eq. (43).

Solving the transfer equation for an arbitrary stationary spacetime is similar to the above flat space and spherically symmetric examples. A stationary spacetime is one that has a timelike Killing vector. If the Killing vector is irrotational or equivalently hypersurface orthogonal, the spacetime is called static as both examples were. Stationary coordinates (t, x^i) can always be adopted to the Killing vector, e.g., $K = \partial/\partial t$, which make the metric components independent of t. If the space is static, all $dt \otimes dx^i$ cross terms of the metric can additionally be made to vanish by appropriately choosing a new t coordinate, $t \to t + f(x^i)$. When stationary coordinates are used, a fixed spatial boundary can be drawn around the source, beyond which photons no longer interact with the comoving fluid. The boundary is 2-dimensional and can be somewhat more complicated than the $r = R$ of the two examples. Incoming photons can be labeled uniquely by 5 constants, 2 from the position they strike the boundary, 2 from the impact orientation with which they strike, and 1 from their wavelengths at ∞ (assuming the space is asymptotically flat). The exact position of any photon within the interaction region is then given by these 5 constants and the affine parameter ξ via the geodesic equations for the given geometry, and the comoving wavelength is given via Eq. (3). Finding coordinates which are constants is also the essence of Hamilton-Jacobi theory of classical particle mechanics. A family of canonical transformation is sought on phase-space which takes the particles in phase-space from their initial positions to their positions at time t. The particles keep their initial values and only time changes. Solving Eq. (43) for the general case then proceeds exactly as in the examples. In the two examples we used the existing symmetry of the spacetime to choose constants to label the photons rather than the boundary impact coordinates and angles. Impact constants could have been used; they can be computed from the symmetry constants we actually used, i.e., from **b** and **n**. However, it is easier to stick with symmetry constants when they exist.

7 DISCUSSION

Our formulation of the radiative transfer equation in terms of comoving wavelengths and stationary coordinates, and the recognition that the momentum directions can be prechosen by constants is the fundamental result of this paper.

Schinder & Bludman (1989) recognized this for the case of purely static (no flow) transfer in spherical symmetry. Since the directions of the geodesics may be chosen for example at the boundary, the solution of the full 3-D radiative transfer problem in the presence of arbitrary hydrodynamic flows is very similar to the purely static case (no flow) described for example by Hauschildt & Baron (2006). In that method space is divided up into a 3-D rectangular grid and long characteristics are followed from the outer boundary through the computational domain. The directions that the characteristics followed were simply chosen by dividing up the angular space at the boundary into equal parts. Since we have shown that the momentum directions may be chosen by constants in both the flat-space and curved-space, the procedure of Hauschildt & Baron (2006) can be used with only the modification that Eqs. (15) and (54) have to be evaluated once at each grid point. Naturally, detailed numerical tests have to be performed to ensure that the material properties such as density and composition are well resolved, and that there are enough "angles" to resolve the momentum-space variation of I_λ. This differs from more classical methods in which the equations of the characteristics are solved in phase-space, with all momentum coordinates co-moving.

If horizons are present in the spacetime, modification of the procedure outlined in this section may be necessary. However, for most astrophysical systems the emission of the radiation and the radiative transfer occur in the accretion disk (or in winds and jets) i.e., outside of any spacetime horizon. The obvious case where one would like to calculate both photon and neutrino transport in the presence of horizons, would be the formation of a collapsar, see for example MacFadyen & Woosley (1999); however, this would involve general relativistic MHD with radiative transfer, a feat that is far beyond current computational capabilities.

Extending 1-D transfer calculations (e.g., spherical symmetry with radial flows) to 3-D applications with arbitrary flows is currently of wide interest because of the desire to include the effects of rotation and Rayleigh-Taylor instabilities into stellar wind models, supernovae, and gamma-ray bursts; as well as the rapid improvement in computing capabilities which makes the extension possible. Some confusion exists in which 1-D structures can be extended to 3-D and which cannot. We have pointed out that the wavelength λ (or equivalently frequency) seen by a comoving observer $u(x)$ is one that not only can be used, it is perhaps essential. Whereas a comoving frame is useful in the 1-D case, it should not be used in 3-D. Not that a comoving frame can't be defined for a comoving observer, it's just that there are too many possibilities and no natural way to make a unique choice between them. Consequently, as many as three additional functions of position (e.g., three angles at each point of spacetime) will be unnecessarily introduced into the transfer equation by introducing a comoving frame. The procedure we advocate for solving non-symmetric transfer problems is (1) start with the given spacetime geodesics, (2) change to appropriate coordinates on phase-space, and (3) solve the transfer using Eq. (34).

We have concentrated on stationary flows in stationary spacetimes (steady-state) applications, emphasizing the fact that, in spite of how coordinates might be chosen, characteristics are really geodesics extended to phase-space as described by Lindquist (1966) and Ehlers (1971). We have also shown that by choosing an appropriate set of parameters (coordinates) to label the geodesics, characteristics can essentially become straight lines. When one of the coordinates is chosen as comoving wavelength λ, only that particular coordinate changes non-linearly. This slight complication is rewarded by a resulting simplification in the absorption and emission terms on the "collision" side of the transfer equation. Additionally the use of λ allows one to convert the transfer equation from an integro-PDE to a system of integro-ODE's by a differencing procedure which restricts comoving wavelengths to a discrete set $\{\lambda_m\}$. These wavelengths remain constant along the characteristics of the now differenced transfer equation and greatly simplifies constructing the formal solution using long or short characteristic methods.

Other authors have recognized the current need for adapting 1-D methods of solution to 3-D problems in radiative transport. Cardall et al. (2005) derive the relativistic equation of transfer in flat spacetime, with similar goals to this paper; however, their approach is somewhat geared to the discrete ordinates matrix exponential method of numerical solution, or to generalized variable Eddington factor (moment) methods.

Broderick (2005) also recognized the advantage of using constant momentum variables, which he attempts to define through the use of Fermi-Walker transported tetrads. Even though his use of these tetrads isn't clear, parallel transport itself results in constant momentum components. If the spacetime is curved, parallel transporting along every geodesic results at an infinite number of tetrads at every point. And, if scattering exists, the relation between these tetrads must be ascertained before the emissivity integral [see Eqs. (12) and (47)] can be evaluated.

In conclusion, we have presented a workable outline of solving the radiative transport equation for many 3-D steady-state problems.

ACKNOWLEDGMENTS

This work was supported in part by NASA grants NAG5-3505 and NAG5-12127, and NSF grants AST-0204771 and AST-0307323, PHH was supported in part by the Pôle Scientifique de Modélisation Numérique at ENS-Lyon.

REFERENCES

Baron E., Hauschildt P. H., 2004, A&A, 427, 987
Baron E., Myra E. S., Cooperstein J., van den Horn L. J., 1989, ApJ, 339, 978
Broderick A., 2005, MNRAS, 366, L10
Bruenn S., 1985, ApJS, 58, 771
Cardall C., Lentz E. J., Mezzacappa A., 2005, Phys. Rev., D72, 043007
Castor J., 2004, Radiation Hydrodynamics. Cambridge Univ. Press, Cambridge
Ehlers J., 1971, in Sachs R. K., ed., General Relativity and Cosmology, Proceedings of the International School of Physics "Enrico Fermi" General Relativity and Kinetic Theory. Academic Press, London, p. 1
Hauschildt P. H., 1992, JQSRT, 47, 433

Hauschildt P. H., Baron E., 1999, J. Comp. Applied Math., 109, 41
Hauschildt P. H., Baron E., 2004a, A&A, 417, 317
Hauschildt P. H., Baron E., 2004b, Mitteilungen der Mathematischen Gesellschaft in Hamburg, 24, 1
Hauschildt P. H., Baron E., 2006, A&A, 451, 273
Knop S., Hauschildt P. H., Baron E., 2007, A&A, 463, 315
Lindquist R., 1966, Annals of Physics, 37, 487
MacFadyen A. I., Woosley S. E., 1999, ApJ, 524, 262
Mihalas D., 1978, Stellar Atmospheres. W. H. Freeman, New York
Mihalas D., 1980, ApJ, 237, 574
Morita K., Kaneko N., 1984, Ap&SS, 107, 333
Morita K., Kaneko N., 1986, Ap&SS, 121, 105
Oxenius J., 1986, Kinetic Theory of Particles and Photons: Theoretical Foundations of Non-Lte Plasma Spectroscopy (Springer Series in Electronics and Photonics). Springer, Berlin
Schinder P. J., Bludman S., 1989, ApJ, 346, 350
Thorne K. S., 1981, MNRAS, 194, 439
Wilson J. R., 1971, ApJ, 163, 209
Zane S., Turolla R., Nobili L., Erna M., 1996, ApJ, 466, 871

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