October 29, 2008

PHYS 6333 (General Relativity)

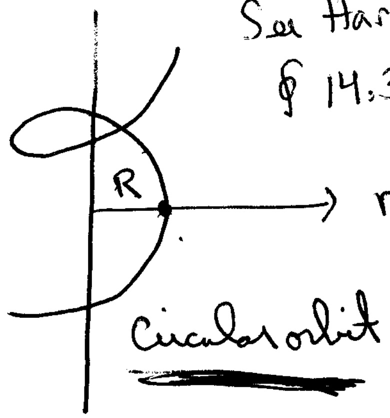
Assignment #9, Due pm Wednesday November 5, 2008

1. A gyroscope circles the earth at a constant altitude h above it's surface. If the spinning gyroscope initially points vertically, in what direction does it point after one revolution of the earth? Assume the gyroscope's orbit is a circular geodesic of the Schwarzschild metric and it's pointing direction is parallelly transported. Assume the orbit is in the $\theta = \pi/2$ plane and in one revolution around the earth ϕ changes by 2π.

2. Compute the scalar $R_{abcd}R^{abcd}$ for the Schwarzschild metric.

#1 Schiff effect!

γ + β measured by stationary observer ($r\theta\phi$) = con



See Hartle § 14.3

$$\frac{dt}{d\tau} = \frac{e}{1 - r_s/R} \Rightarrow e = \gamma_R \sqrt{\left(1 - \frac{r_s}{R}\right)}$$

$$\frac{d\phi}{c\,d\tau} = \frac{l}{r^2} \Rightarrow l = \gamma_R \beta_R R$$

$$\boxed{R \cong R_E + h} \quad R_E = 6.38 \times 10^3 \text{ km}$$

$$\boxed{\frac{d\phi}{dt} \equiv \Omega = c\sqrt{\frac{r_s}{2R^3}}}$$

angular rotation rate as measured by observer at $r = $ "∞".

time-like geodesic

$$-(\ )\left(\frac{ce}{1 - r_s/R}\right)^2 + \frac{\left(\frac{dr}{d\tau}\right)^2}{(\ )} + r^2\left(\frac{cl}{r^2}\right)^2 = -c^2$$

$$\left(\frac{dr}{c\,d\tau}\right)^2 + \left(\frac{l^2}{r^2} + 1\right)\left(1 - r_s/r\right) = e^2$$

or

$$\left(\frac{dr}{c\,d\tau}\right)^2 + \left\{-\frac{r_s}{r} + \frac{l^2}{r^2} - \frac{l^2}{r^2}\frac{r_s}{r}\right\} = e^2 - 1$$

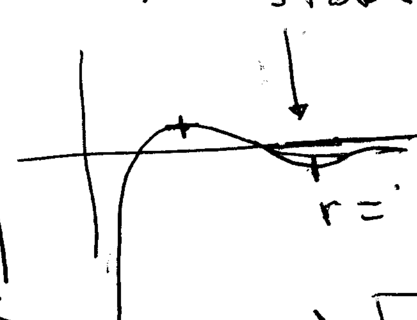
$$l = \sqrt{\frac{r_s R}{2\left(1 - \frac{3}{2} r_s/R\right)}}$$

$$e = \frac{1 - r_s/R}{\sqrt{1 - \frac{3}{2} r_s/R}}$$

$$\gamma_R^2 = \frac{2(1 - r_s/R)}{2 - 3r_s/R}$$

$$\beta_R^2 = \frac{r_s/R}{2(1 - r_s/R)}$$

stable circular orbit



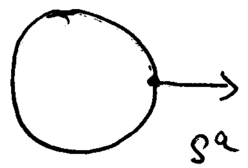
$$r = l^2/r_s + \sqrt{\left(\frac{l^2}{r_s}\right)^2 - 3l^2}$$

$$\Rightarrow \boxed{\frac{R}{r_s} = \left(1 + \frac{1}{2}\beta_R^2\right)}$$

stable circular orbit cond ↑.

∥ transport eqns for a vector s^a

$\theta = \pi/2$ plane



$$\frac{ds^a}{d\tau} + \left\{ {a \atop bc} \right\} s^b u^c = 0$$

$$u^c \downarrow \left(\frac{dt}{d\tau}, 0, 0, \frac{d\phi}{d\tau} \right)$$

$$\frac{ds^a}{d\tau} + \left\{ {a \atop bt} \right\} s^b \left(\frac{e}{1 - r_s/R} \right) + \left\{ {a \atop b\phi} \right\} s^b \frac{c\ell}{R^2} = 0$$

$$\left\{ {t \atop rt} \right\} = \frac{r_s}{2R^2(1 - r_s/R)}$$

$$\left\{ {r \atop tt} \right\} = \frac{r_s c^2}{2R^2}(1 - r_s/R)$$

$$\left\{ {r \atop \phi\phi} \right\} = -R(1 - r_s/R) \underbrace{\sin^2\theta}_{=1}$$

$$\left\{ {\phi \atop \phi r} \right\} = \frac{1}{R}$$

$$\left\{ {\phi \atop \theta\phi} \right\} = \cot\theta = 0$$

$$\left\{ {\theta \atop \phi\phi} \right\} = -\sin\theta\cos\theta = 0$$

$\therefore \quad S^a(0) = \left(0, \sqrt{1 - r_s/R}, 0, 0\right)$ unit radial

① $\dfrac{dS^t}{d\gamma} + \dfrac{r_s}{2R^2(\;)} S^r \left(\dfrac{e}{(\;)}\right) = 0$

② $\dfrac{dS^r}{d\gamma} + \dfrac{r_s c^2}{2R^2} \cancel{\left(1 - \dfrac{r_s}{R}\right)} S^t \left(\dfrac{e}{\cancel{(\;)}}\right) + (-R)(\;) S^\phi \dfrac{c\ell}{R^2} = 0$

$\dfrac{dS^\theta}{d\gamma} = 0$

③ $\dfrac{dS^\phi}{d\gamma} + \dfrac{1}{R} S^r \dfrac{c\ell}{R^2} = 0$

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differentiate ② and use ①+③ to obtain

$\text{"}\cdot\text{"} = \dfrac{d}{d\gamma}$

$$\ddot{S}^r + \frac{r_s c^2 e}{2R^2} \underset{\downarrow}{\dot{S}^t} - \frac{(1 - r_s/R)\, c\ell}{R} \underset{\downarrow}{\dot{S}^\phi} = 0$$

$$\left(\frac{-r_s e S^r}{2R^2(\;)^2}\right) \qquad\qquad -\frac{c\ell}{R^3} S^r$$

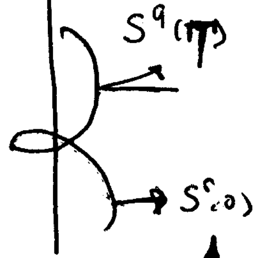
$$\therefore \quad \ddot{S}^r + \underbrace{\left[-\frac{c^2 e^2 r_s^2}{4R^4(\;)^2} + \frac{c^2\ell^2}{R^4}(\;)\right]}_{\eta^2 \,=\, \omega^2 \,=\, c^2 \frac{r_s}{2R^3}} S^r = 0$$

$\leftarrow$ use ℓ + e from page 0 to get this!

$$\therefore \quad \boxed{S^r(\tau) = S^r(0)\cos(\omega\tau)}$$

where $S^a(0) = [\,0, S^r(0), 0, 0\,] =$ unit radial

$$S^r(0) \downarrow \sqrt{1 - r_s/R}$$



From ①
$$\dot{S}^t = -\frac{r_s e}{2R^2(\;)^2} S^r(0)\cos(\omega\tau)$$

$$\Rightarrow \quad S^t = -\frac{r_s e}{2R^2(\;)^2}\frac{S^r(0)}{\omega}\sin(\omega\tau)$$

From ③
$$\dot{S}^\phi = -\frac{c\ell}{R^3} S^r(0)\cos\omega\tau$$

$$\Rightarrow \quad S^\phi = -\frac{c\ell}{R^3}\frac{S^r(0)}{\omega}\sin(\omega\tau)$$

To find a proper time period integrate

$$\frac{d\phi}{c\,d\tau} = \frac{\ell}{R^2} \Rightarrow \boxed{T} = \frac{2\pi R^2}{\ell c} = \boxed{\frac{2\pi}{c}\sqrt{\frac{(1-\frac{3}{2}r_s/R)}{\frac{r_s}{2R^3}}}}$$

$\Omega \cong 1.2\times10^{-3}\,\text{sec}^{-1}$
$\Rightarrow 6.0\times10^{3}$ rev/yr
1 rad = 206,265 arc-sec

$$\therefore \boxed{\omega T = 2\pi\sqrt{(1 - 3/2\, r_s/R)}} \cong 2\pi - \delta$$

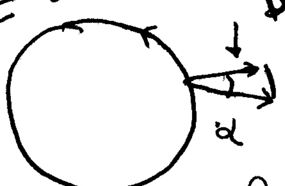
$$\boxed{\delta \equiv \frac{3\pi}{2}\frac{r_s}{R}} << \frac{\pi}{2}$$

$\delta \cong \frac{\pi}{2}\,\frac{0.88\text{ cm}}{6{,}400\text{ km}}$

$= 6.5\times10^{-9}$ rad

$\delta \sim 1.3\times10^{-3}$ arc sec

$\cong 8$ arcsec/yr



$$\alpha \cong \tan\alpha = \frac{RS^\phi}{S^r/\sqrt{\;}} = +\frac{\sqrt{(\;)}\,Rc\ell}{R^3\omega}\,\frac{\sin\omega T}{\cos\omega T}$$

Angular lag per Revolution $= \alpha = \boxed{\frac{(1-r_s/R)}{(1-\frac{3}{2}r_s/R)}}\tan\omega T \cong \boxed{-\delta}$

#2

$$R_{abcd}R^{abcd} = 4\Big(R_{trtr}R^{trtr} + R_{t\theta t\theta}R^{t\theta t\theta}$$

$$+ R_{t\phi t\phi}R^{t\phi t\phi} + R_{r\theta r\theta}R^{r\theta r\theta} + R_{r\phi r\phi}R^{r\phi r\phi} + R_{\theta\phi\theta\phi}R^{\theta\phi\theta\phi}\Big)$$

$$= 4\Big((R^t{}_{rtr})^2 (g^{rr})^2 + (R^t{}_{\theta t\theta})^2 (g^{\theta\theta})^2 + (R^t{}_{\phi t\phi})^2 (g^{\phi\phi})^2$$

$$+ (R^r{}_{\theta r\theta})^2 (g^{\theta\theta})^2 + (R^r{}_{\phi r\phi})^2 (g^{\phi\phi})^2 + (R^\theta{}_{\phi\theta\phi})^2 (g^{\phi\phi})^2 \Big)$$

$$= 4\Big(\frac{r_s^2 (\ )^2}{r^4(r-r_s)^2} + \left(-\frac{r_s}{2r}\right)^2 \frac{1}{r^4} + \left(-\frac{r_s}{2r}\sin^2\theta\right)^2 \frac{1}{\sin^4\theta\, r^4}$$

$$+ \left(-\frac{r_s}{2r}\right)^2 \frac{1}{r^4} + \left(-\frac{r_s}{2r}\sin^2\theta\right)^2 \frac{1}{\sin^4\theta\, r^4} + \left(\frac{r_s}{r}\sin^2\theta\right)^2 \frac{1}{\sin^4\theta\, r^4} \Big)$$

$$= 4\frac{r_s^2}{r^6}\left[1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + 1 \right]$$

$$= \boxed{\frac{12\, r_s^2}{r^6}}$$