Ch 9. Hartle #15

$$\Delta\phi = 2l \int_{r_1}^{r_2} \frac{dr}{r^2}\Big[\underbrace{e^2(e^2-1) + \frac{2GM}{r}}\ \underbrace{-\frac{l^2}{r^2} + \frac{2GMl^2}{c^2r^3}}\Big]^{-\frac{1}{2}}$$

(9.52)

his l = c*length

$$c^2e^2 - c^2\left(1-\frac{2GM}{c^2r}\right) - \frac{l^2}{r^2}\left(1-\frac{2GM}{c^2r}\right)$$

$$\Delta\phi = 2l\int_{r_1}^{r_2}\frac{dr}{r^2}\left(1-\frac{2GM}{c^2r}\right)^{-\frac{1}{2}}\left[\frac{c^2e^2}{1-\frac{2GM}{c^2r}} - \left(c^2+\frac{l^2}{r^2}\right)\right]^{-\frac{1}{2}}$$

Prob(15) eqn

$$\frac{1}{1-\frac{2GM}{c^2r}} \cong 1 + \frac{2GM}{c^2r} + \left(\frac{2GM}{c^2r}\right)^2 + \mathcal{O}(3)$$

need this order!

$$[\ ] \cong c^2(e^2-1) + c^2e^2\frac{2GM}{c^2r} + \left[c^2e^2\left(\frac{2GM}{c^2}\right)^2 - l^2\right]\frac{1}{r^2}$$

keep 'e'!

$$\cong \left[l^2 - c^2e^2\left(\frac{2GM}{c^2}\right)^2\right]\left[A + Bu - u^2\right]$$

$$\cong l^2\left[1-\left(\frac{2GMe}{cl}\right)^2\right]\left[(\tilde{u}_1-u)(\tilde{u}-u_2)\right]$$

$$\Delta\phi \cong 2\int_{u_2}^{u_1}\frac{du\left(1-\frac{2GM}{c^2}u\right)^{-1/2}}{\left[1-\left(\frac{e\,2GM}{lc}\right)^2\right]^{\frac{1}{2}}\sqrt{(\tilde{u}_1-u)(\tilde{u}-u_2)}}$$

$\tilde{u}_1, \tilde{u}_2$ differ from $u_1 + u_2$

but only by terms of order $\sqrt{\mathcal{O}(3)} \approx (r_s/R)^{3/2}$.

$$\therefore \Delta\phi \cong 2\int_{u_2}^{u_1} \frac{\left(1 + 2\left(\frac{GMe}{\ell c}\right)^2 + \frac{GM}{c^2} u\right) du}{\sqrt{(u_1 - u)(u - u_2)}}$$

eqn in (v)

to integrate substitute $u = (u_1 - u_2)\sin^2\phi + u_2$

$$\cong 2\pi\left[1 + 2\left(\frac{GMe}{\ell c}\right)^2\right] + \frac{2GM}{c^2} \underbrace{\int_{u_2}^{u_1} \frac{u\, du}{\sqrt{(u_1 - u)(u - u_2)}}}_{\frac{\pi}{2}(u_1 + u_2)}$$

$u_1 + u_2 = B$ (above)

$$= \frac{e^2\, 2GM}{\ell^2 - \left(\frac{2GMe}{c}\right)^2} \cong \frac{2GM}{\ell^2}\left[1 + \mathcal{O}\left(\frac{r_s}{R}\right)\right]$$

'e' = lowest order = 1

$$\therefore \Delta\phi \cong 2\pi\left[1 + 3\left(\frac{GM}{\ell c}\right)^2 + \mathcal{O}(3)\right] \quad (9.55)$$

$\frac{GM}{\ell c} = \frac{1}{2}\frac{r_s/R}{\gamma_R \beta_R}$ (how big is this?) $\qquad \ell c = c^2 R \gamma_R \beta_R$

For circular orbits $\beta_r = \sqrt{\frac{\frac{1}{2} r_s/R}{1 - r_s/R}} \quad \therefore \quad \frac{GM}{\ell c} \cong \frac{r_s/R}{\beta_r} \cong \sqrt{\frac{r_s/R}{2}}$

$$e \cong \gamma_R \sqrt{(1 - r_s/R)} \cong 1 - \tfrac{1}{2} r_s/R - \tfrac{1}{2}\beta_R^2 \cong 1 - \tfrac{3}{4} r_s/R + \mathcal{O}(r_s/R)^2$$

$\theta = \pi/2$

ⓐ $(1 - r_s/r)\,\dot{t} = e$

$$\boxed{\cdot = \frac{d}{d\tau}}$$

ⓑ $r^2 \dot{\phi} = l$

V_{eff}

ⓒ $$\frac{\dot{r}^2}{2} + \left[\frac{(1 - r_s/r)\, l^2}{2r^2} - \frac{c^2}{2}\frac{r_s}{r} \right] = \underbrace{\frac{c^2}{2}(e^2 - 1)}_{\mathcal{E}}$$

ⓒ $$\boxed{-(\ )\,\dot{t}^2 c^2 + \frac{\dot{r}^2}{(\ )} + r^2 \dot{\phi}^2 = -c^2} \rightarrow 0 \text{ for photon}$$

R_c photon ←

$$\boxed{\begin{aligned} \frac{l}{e} &= \frac{R V_R}{\sqrt{1 - r_s/R}} \\ e &= \gamma_R \sqrt{1 - r_s/R} \end{aligned}}$$

V_R = observed vel at SS coord R!

$\Rightarrow l = R V_R \gamma_R$

$$\boxed{\begin{aligned} e &= \gamma_\infty \\ l &= b V_\infty \gamma_\infty \end{aligned}}$$

HW#5 Prob 12

$(R, v_R) \longrightarrow (b, v_\infty) \Rightarrow (R, b)$

can do in this case!

$$e = \gamma_R \sqrt{1 - r_s/R} = \gamma_{\infty}$$

Put in here

$$l = R v_R \gamma_R = b v_\infty \gamma_\infty$$

$$= bc\sqrt{1-\gamma_\infty^{-2}}\,\gamma_\infty$$

$$Rc\sqrt{\gamma_R^2 - 1} = bc\sqrt{\gamma_\infty^2 - 1}$$

$$Rc\sqrt{\gamma_R^2 - 1} = bc\sqrt{\gamma_R^2(1 - r_s/R) - 1}$$

Solve for γ_R as fn of (R, b)

$$\gamma_R^2 = \frac{1 - (R/b)^2}{1 - (R/b)^2 - r_s/R}$$

$$\Rightarrow \quad \beta_R = \sqrt{\frac{r_s/R}{1 - (R/b)^2}}$$

$$\dot r^2 = \Big[\overbrace{(\ )^2 \dot t^2}^{e^2} c^2 - \underbrace{(\ ) c^2}_{\to\, 0} - (\ )\underbrace{r^2 \dot\phi^2}_{\ell^2/r^2} \Big]$$

$$\dot r = \pm\sqrt{e^2 c^2 - (\ )\Big[\underbrace{c^2}_{\to\, 0} + \frac{\ell^2}{r^2}\Big]}$$

$$\frac{dr}{d\phi} = \frac{\dot r}{\dot\phi} = \frac{\dot r}{\ell/r^2} = \pm r^2 \sqrt{\left(\frac{ec}{\ell}\right)^2 - (\ )\left[\frac{c^2}{\ell^2} + \frac{1}{r^2}\right]}$$

$\frac{c^2}{\ell^2} \to 0$ photons

$$\left(\frac{ec}{\ell}\right)^2 \to \frac{1 - r_s/R}{R^2 \beta_R^2} \qquad \frac{c^2}{\ell^2} \to \frac{1}{R^2 \beta_R^2 \gamma_R^2} \to \frac{1}{R^2}\left(\frac{1}{\beta_R^2} - 1\right)$$

$\boxed{\text{Put } \beta_R \to 1 \text{ for photon}}$

$\gamma_R \to \infty$!

cancel for plt

$$\frac{dr}{d\phi} = \pm r^2 \sqrt{\frac{(1 - r_s/R) - (1 - r_s/r)}{R^2 \beta_R^2} + (1 - r_s/r)\left[\frac{1}{R^2} - \frac{1}{r^2}\right]}$$

$$= \pm r^2 \sqrt{\frac{r_s\left(-\frac{1}{R} + \frac{1}{r}\right)}{R^2 \beta_R^2} + (1 - r_s/r)\left[\left(\frac{1}{R} - \frac{1}{r}\right)\left(\frac{1}{R} + \frac{1}{r}\right)\right]}$$

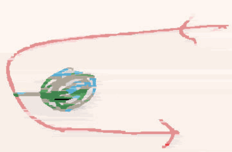
$$\frac{dr}{d\phi} = \pm\, r^2 \sqrt{\frac{1}{R} - \frac{1}{r}}\;\sqrt{\left(1-\frac{r_s}{r}\right)\left(\frac{1}{R}+\frac{1}{r}\right) - \frac{r_s}{R^2\beta_R^2}}$$

$$= r\sqrt{1 - R/r}\,\left(\frac{r}{\sqrt{R}}\right)\sqrt{\left(\frac{1}{R} - \frac{r_s}{R^2\beta_R^2}\right) + \frac{1}{r}\left(1-\frac{r_s}{R}\right) - \frac{r_s}{r^2}}$$

$$\frac{dr}{d\phi} = \pm\, r\sqrt{1-R/r}\;\sqrt{\left(1 - \underbrace{\frac{r_s}{R\beta_R^2}}\right)\frac{r^2}{R^2} + \left(1-\frac{r_s}{R}\right)\frac{r}{R} - \frac{r_s}{R}}$$

put $\beta_R \to 1$ for photon

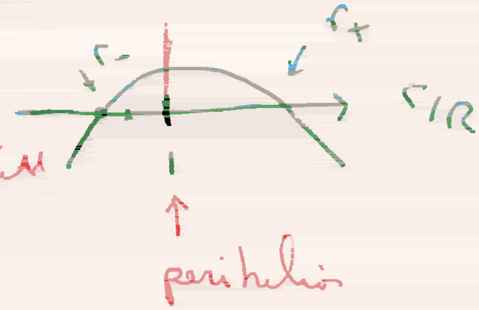
$$\left(\frac{GM/R}{\frac{1}{2}v_R^2}\right) > 1 \quad \text{Bound}$$

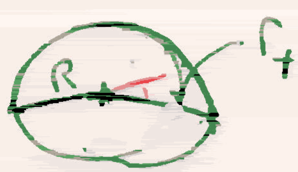


$$< 1 \quad \text{free}$$

$$r_\pm = \frac{\left(1 - r_s/R\right) \pm \sqrt{\left(1 - r_s/R\right)^2 - \frac{4 r_s}{R}\,(\ \ )}}{2\left(\frac{r_s}{R\beta_R^2} - 1\right)}$$

$= r_{max}$





Motion with $\dot{r} = 0$ somewhere ($r > R > r_s$)

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In[2]:= Plot[{1/(2/beta - 1), beta^2}, {beta, 0, 1},
  PlotStyle → {{RGBColor[1, 0, 0]}, {RGBColor[0, 1, 0]}}]
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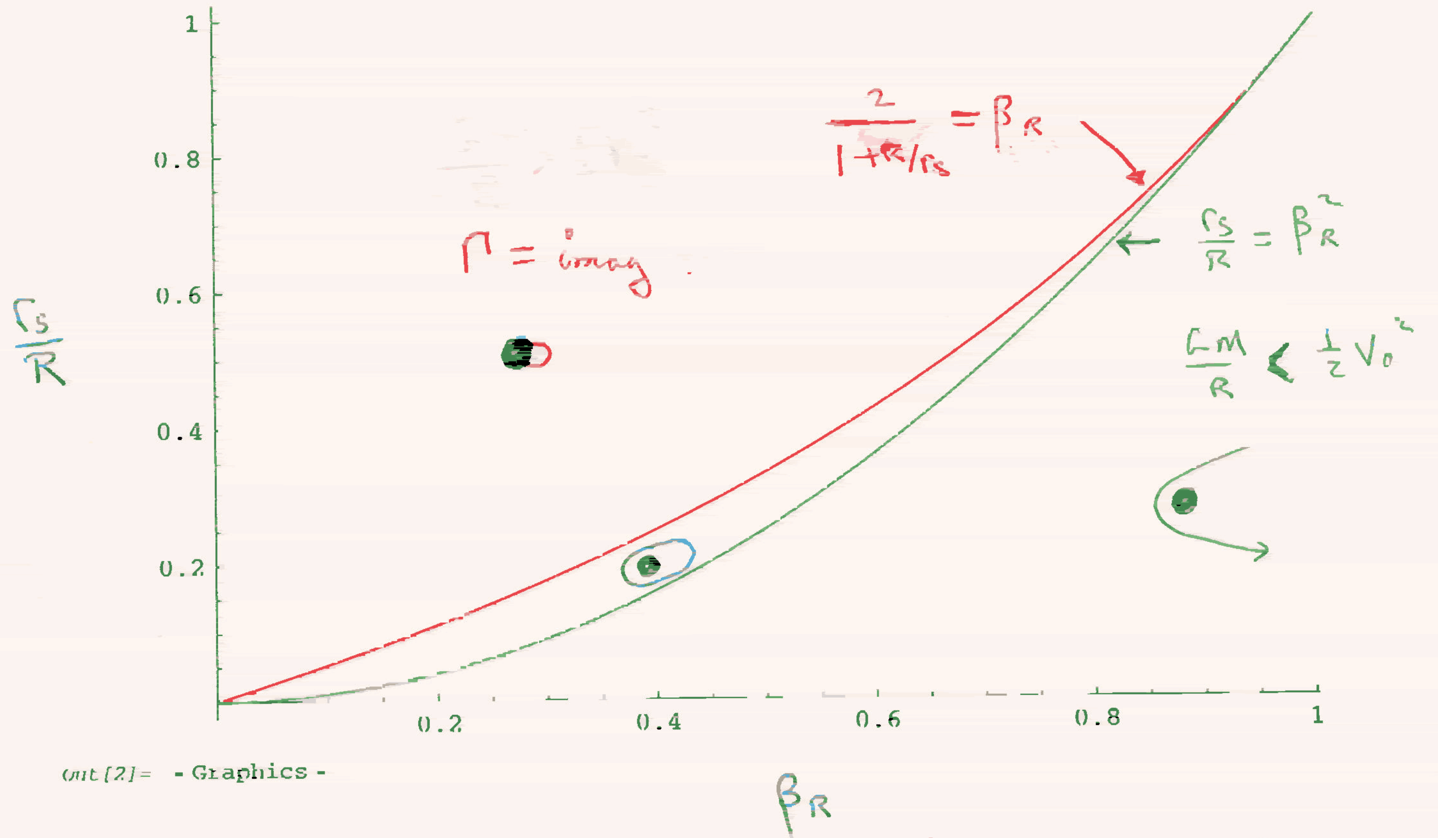

Out[2]= - Graphics -

for Γ = real there are 2 more roots $\dot{r} = 0$ but only one is outside R, ie $\frac{r_+}{R} > 1$ but $\frac{r_-}{R} < 1$!

If $\frac{GM}{R} < \frac{1}{2} V_0^2$ both roots are negative, ie non physical!

$$u = R/r$$

$$\frac{dr}{r^2} = -\frac{du}{R} \Rightarrow \frac{du}{d\phi} = -\frac{R}{r^2}\frac{dr}{d\phi}$$

$$\therefore \quad \frac{du}{d\phi} = \mp \sqrt{1-u}\sqrt{\left(1-\frac{r_s}{R}u\right)(1+u) - \frac{r_s}{R\beta_R^2}}$$

$$= -\sqrt{1-u}\sqrt{\frac{r_s}{R}(u-\underbrace{u_+}_{R/r_+})(\underbrace{u_-}_{R/r_-}-u)}$$

$$\int_{u_+}^{1} \frac{du}{\sqrt{1-u}\sqrt{\frac{r_s}{R}(u-u_+)(u_--u)}} = \pi + \frac{1}{2}\Delta\phi$$

$$u_+ < u \leq 1$$

Change variable $u \equiv \frac{r_0}{r} \le 1 \Rightarrow du = -\frac{r_0}{r^2}\,dr$ & $\frac{dr}{r_0} = -\frac{1}{u^2}\,du$

$$\frac{du}{d\phi} = -\sqrt{1-u}\,\sqrt{\left(1-\frac{r_s}{r_0}\frac{c^2}{v_0^2}\right) + u\left(1-\frac{r_s}{r_0}\right) - \frac{r_s}{r_0}u^2}$$

$$= -\sqrt{1-u}\,\sqrt{\frac{r_s}{r_0}(u-u_+)(u_- - u)}$$

$$\int_u^1 \frac{du}{\sqrt{1-u}\,\sqrt{\frac{r_s}{r_0}(u-u_+)(u_- - u)}} = \phi - \phi_0$$

Elliptic Integral

To find the perihelion precession compute

$$\int_{u_+}^{1} \frac{du}{\sqrt{1-u}\,\sqrt{\frac{r_s}{r_0}(u-u_+)(u_- - u)}} = \pi + \frac{1}{2}\Delta\phi$$

$$u_+ \le u \le 1 < u_-$$

This can be transformed into standard complete elliptic integral form.

$$u = 1 - (1-u_+)\cos^2\Phi \qquad \int_{u_+}^{1} = \int_0^{\pi/2} d\Phi$$

$$du = 2(1-u_+)\sin\Phi\cos\Phi\, d\Phi$$

$\Phi \neq \text{angle } \phi$

$$\therefore \sqrt{(u-u_+)(u_--u)} = \sqrt{(1-u_+)(1-\cos^2\Phi)(u_--u_+-(1-u_+)\sin^2\Phi)}$$

$$\sqrt{1-u} = \sqrt{(1-u_+)\cos^2\Phi}$$

$$K(k) \equiv \int_0^{\pi/2} \frac{d\Phi}{\sqrt{1-k^2\sin^2\Phi}}$$

$$\therefore \frac{2}{\sqrt{\frac{r_s}{R}}}\int_0^{\pi/2}\frac{d\Phi}{\sqrt{u_--u_+-(1-u_+)\sin^2\Phi}} = \pi + \tfrac{1}{2}\Delta\phi$$

complete elliptic int. of 1st kind

$$\downarrow$$

$$\frac{2}{\sqrt{r_s/R}\sqrt{u_--u_+}} \qquad K(k) = \frac{2}{\sqrt{\Box}} \qquad K(k) = \pi + \tfrac{1}{2}\Delta\phi$$

$$k^2 \equiv \frac{1-u_+}{u_--u_+} = \frac{r_-}{r_+-r_-}\left(r_+/R - 1\right)$$

$$\frac{r_\pm}{R} = \frac{(\ ) \pm \sqrt{(\ )^2 - \frac{4r_s}{R}\left(\frac{r_s/R}{\beta_R^2}-1\right)}}{2\left(\frac{r_s/R}{\beta_R^2}-1\right)} \qquad (\ ) \equiv 1 - r_s/R$$

$$k^2 = \left(\frac{\frac{3r_s}{R} + \sqrt{\Gamma} - 1}{2\sqrt{\Gamma}}\right) \qquad \Gamma \equiv \sqrt{\left(1+\frac{r_s}{R}\right)^2 - 4\left(\frac{r_s}{R}\right)^2\frac{1}{\beta_R^2}}$$

$$\frac{r_s}{R} \equiv \delta < 1 \qquad \frac{r_s}{R\beta_R^2} \equiv \alpha > 1 \text{ (bound orbit)}$$

$$\therefore \boxed{\Delta\phi = \frac{4}{\left[1-2\delta(2\alpha-1)+\delta^2\right]^{1/4}} K\left(\frac{\sqrt{1-2\delta(2\alpha-1)+\delta^2}-1+3\delta}{2\sqrt{1-2\delta(2\alpha-1)+\delta^2}}\right) - 2\pi}$$

$\downarrow k^2$

$=$ exact! $\delta \equiv r_s/r_0 < 1$

for bound motion

$$\delta < \alpha \equiv \frac{r_s}{r_0}\frac{c^2}{V_0^2} = \frac{GM}{r_0 \frac{1}{2}V_0^2} > 1$$

Expand to first order in δ - keep lowest order term

$$\Delta\phi \cong 4\left[1+\frac{\delta(2\alpha-1)}{2}\right] K\left(\sqrt{\frac{\delta(4-2\alpha)}{2}}\right) - 2\pi$$

$K \downarrow$ series to first order

$$\cong 4\left[1+\delta\left(\alpha-\tfrac{1}{2}\right)\right]\frac{\pi}{2}\left[1+\left(\tfrac{1}{2}\right)^2\delta(2-\alpha)\right] - 2\pi$$

$$= 2\pi\,\delta\left(\tfrac{3}{4}\alpha\right) + \mathcal{O}(\delta^2)$$

For Mercury $r_0 = 4.60\times10^7$ km
$V_{r_0} = 59$ km/s
$\delta = 6.4\times10^{-8}$, $\alpha = 1.65$
$\Delta\phi = 5.0\times10^{-7}$ rad/rev $= 0.103''/100y$

$$\Delta\phi \cong \frac{3\pi}{2}\delta\alpha = \frac{3\pi}{2}\left(\frac{r_s}{r_0}\right)\left(\frac{r_s}{r_0}\frac{c^2}{V_0^2}\right)$$

$$\boxed{\Delta\phi \cong \frac{3\pi}{2}\left(\frac{2GM}{c^2 r_0}\right)\frac{\left(\frac{GM}{r_0}\right)}{\frac{1}{2}V_0^2}} > 0$$

Perihelian Advance

Checks with eqn. 11.51 of Rindler
$h = V_0 r_0 \gamma\sqrt{1-r_s/r_0}$
$\cong V_0 r_0$