February 21, 2007

PHYS 6333 (General Relativity)

Assignment #6, Due pm Wednesday February 28, 2007

1. Read Section 9.4 of Hartle,
   and work Problem 16.
2. Read Sections 11.10 & 11.11 of Rindler,
   and work Problem 11.12.
3. Read Ch. 18 of Hughston & Tod,
   and work Problems 18.3 & 18.4 and fill in enough details to arrive at the equation $\Delta\phi = 2r_s/L$ just before equation (18.19) on page 122.

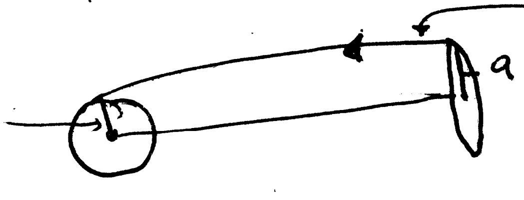
Hartle Ch 9
Prob 16

4/1

$r_{min} = R$ when $\dot{r} = 0$

photon from "∞"

$\frac{dr}{dt} = -c$



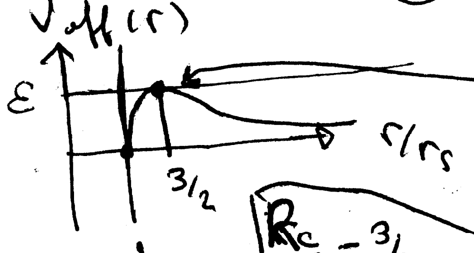
Geodesic eqns

(a) $(1 - r_s/r)\dot{t} = e$

(b) $r^2 \dot{\phi} = l$

(c) $\frac{\dot{r}^2}{2} + \underbrace{\left[\frac{l^2}{2r^2}(1 - r_s/r)\right]}_{V_{eff}(r)} = \frac{e^2c^2}{2} = \mathcal{E}$ ← null condition

"$\cdot$" $\equiv \frac{d}{d\lambda}$ ← arbitrary affine parameter



unstable circular orbit occurs when

$V(R_c) = V_{max} = \frac{2}{27} l^2/r_s^2 = \frac{e^2c^2}{2} = \mathcal{E}$

$\frac{R_c}{r_s} = 3/2$ circular orbit radius

$\therefore \boxed{\frac{l}{e} = \frac{3\sqrt{3}\, c\, r_s}{2}}$ for a circular orbit

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For an incomming photon $\dot{r} = 0$ at $r_{min} = R > R_c$

$$0 + \frac{l^2}{2R^2}(1 - r_s/R) = \frac{e^2c^2}{2} \Rightarrow \frac{l}{e} = \frac{Rc}{\sqrt{1 - r_s/R}} > \frac{3\sqrt{3}\, c\, r_s}{2} \; !$$

required ↑ to escape

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at "∞" this same photon has

$\dot{r} = -ec$

$\dot{t} = e$

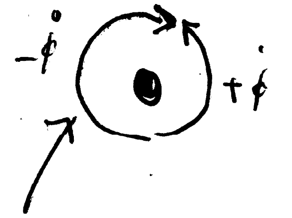
to not escape

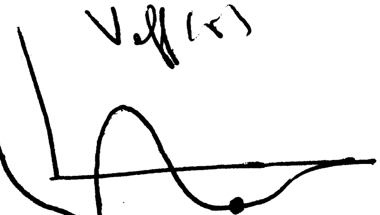
$$\left. r^2 \frac{d\phi}{dt}\right|_{\infty} = \frac{l}{e} = ac \leq \frac{3\sqrt{3}\, c\, r_s}{2} \Rightarrow \boxed{a_{max} = \frac{3\sqrt{3}}{2} r_s}$$

From flat space.

Rindler Problem 11.12

Stable circular orbits at radius R





For stable circular orbits

(a) $$\frac{R}{r_s} = \frac{l^2}{c^2 r_s^2}\left[1+\sqrt{1-3\left(\frac{r_s c}{l}\right)^2}\right]$$

$\Updownarrow$

(b) $$\frac{l}{c\, r_s} = \frac{\sqrt{R/r_s}}{\sqrt{2-3r_s/R}}$$

$$P^a_\pm = m\frac{dx^a}{d\tau} = m\left(\dot t, 0, 0, \pm\dot\phi\right)$$

Geodesic eqns. → $\frac{e}{1-r_s/R}$ $\qquad$ l/R^2

normalisation

$$-(1-r_s/R)\,c^2\dot t^2 + R^2\dot\phi^2 = -c^2$$

$$\Rightarrow \quad e = \sqrt{1-r_s/R}\,\sqrt{1+\left(\frac{l}{cR}\right)^2}$$

(c) $$\boxed{\therefore \quad e = \left(1-\frac{r_s}{R}\right)\Big/\sqrt{1-\frac{3}{2}r_s/R}}$$

Energy seen by an observer $\mathcal{E} = -\left(P^a_+ + P^a_-\right)u_a = 2mc^2\gamma_R$

where $u^a = \left(\frac{1}{\sqrt{1-r_s/R}}, 0, 0, 0\right)$ ← rest observers!

$$\therefore \quad \boxed{\mathcal{E}} = \frac{2m\dot t\,(1-r_s/R)\,c^2}{\sqrt{1-r_s/R}} = \frac{2mc^2 e}{\sqrt{1-r_s/R}} = \boxed{2mc^2\sqrt{\frac{(1-r_s/R)}{\left(1-\frac{3}{2}r_s/R\right)}}}$$

Hughston + Tod 18.3

1/1

$p^a u_a = -h\nu \leftarrow$ frequency seen by observer u^a.

$\uparrow$ photon 4-momentum, $\nwarrow$ observer 4-velocity

$$p^a \propto k^a = (\dot t, \dot r, 0, 0)$$

$$\dot{(\;)} = \frac{d}{d\lambda} \leftarrow \text{affine parameter}$$

Chose λ so that $e = \frac{h\nu_\infty}{c^2}$ makes $p^a = k^a$!

$$\dot t = \frac{e}{(1 - r/r_s)}, \quad -(1 - r/r_s)\dot t^2 c^2 + \frac{\dot r^2}{(1 - r/r_s)} = 0$$

$$\underset{ss}{u^a}(r=b) = \left(\frac{1}{\sqrt{1 - r_s/b}}, 0, 0, 0\right) \leftarrow \text{observer at rest in SS}$$

$$\underset{cir}{u^a}(r=a) = (\dot t, 0, 0, \dot\phi) \leftarrow \text{circular orbit}$$

$$= \left(\frac{1}{\sqrt{1 - \frac{3}{2}\frac{r_s}{a}}}, 0, 0, \frac{\frac{c}{a}\sqrt{\frac{r_s}{2a}}}{\sqrt{1 - \frac{3}{2}\frac{r_s}{a}}}\right)$$

$$p^a \underset{ss}{u_a}(b) = \frac{-h\nu_\infty}{\sqrt{1 - r_s/b}} = -h\nu_b$$

$$p^a \underset{cir}{u_a}(a) = \frac{-h\nu_\infty}{\sqrt{1 - \frac{3}{2}r_s/a}} = -h\nu_a$$

$$\Rightarrow \boxed{\frac{\nu_a}{\nu_b} = \sqrt{\frac{1 - r_s/a}{1 - \frac{3}{2}r_s/b}}}$$

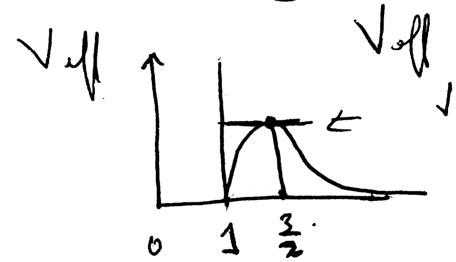
Photon orbits (Hughston + Tod 18.4)

(a) $(1-r_s/r)\,\dot t = e$ $\qquad$ "$\dot{\ }$" $\equiv \frac{d}{d\lambda}$ ← affine parameter

(b) $r^2\dot\phi = l$

(c) $-(1-r_s/r)\,\dot t^2 c^2 + \frac{\dot r^2}{(1-r_s/r)} + r^2\dot\phi^2 = 0$

$$\therefore \quad \frac{\dot r^2}{2} + \underbrace{\frac{l^2}{2r^2}\left(1-\frac{r_s}{r}\right)}_{V_{eff}} = \frac{e^2c^2}{2}$$



$V_{max} = \frac{2l^2}{27 r_s^2} \Rightarrow \frac{l}{ec} = \frac{3\sqrt{3}}{2} r_s$

for circular orbit

↑

unstable = max

$$\left.\frac{dV_{eff}}{dr}\right|_{R_c} = 0 \Rightarrow \boxed{R_c = 3/2\, r_s}$$

Steps to get eq (18.19) on page 122 of Hughston + Tod

$$L = -(1 - r_s/r)\,\dot{t}\,c^2 + \frac{\dot{r}^2}{1 - r_s/r} + r^2\dot{\phi}^2 \qquad \dot{} = \frac{d}{d\lambda} \leftarrow \text{affine param}$$

geodesic eqns $(1 - r_s/r)\,\dot{t} = e = E,\quad r^2\dot{\phi} = l = J$

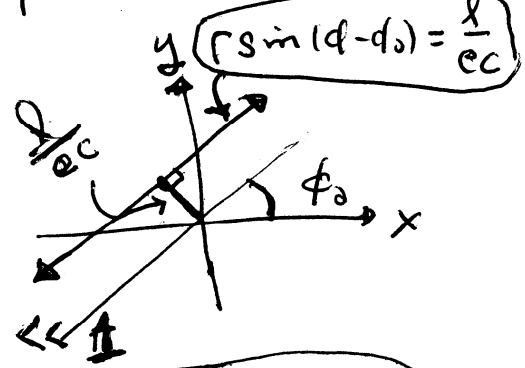
$$0 = -(\;)\left(\frac{e}{(\;)}\right)^2 c^2 + \frac{\dot{r}^2}{(\;)} + \left(\frac{l}{r}\right)^2 \Rightarrow \dot{r}^2 = e^2c^2 - \left(\frac{l}{r}\right)^2\left(1 - \frac{r_s}{r}\right)$$

$$\dot{r}^2 = \left(\frac{dr}{d\phi}\right)^2\dot{\phi}^2 = \left(\frac{dr}{d\phi}\right)^2\left(\frac{l}{r^2}\right)^2 \Rightarrow \left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 = \left(\frac{ec}{l}\right)^2 - \left(\frac{1}{r}\right)^2(1 - r_s/r)$$

$$u \equiv \frac{1}{r} \Rightarrow \left(\frac{du}{d\phi}\right)^2 = \left(\frac{ec}{l}\right)^2 - u^2 + \underbrace{r_s u^3}_{r_s u \ll 1} \qquad (18.15)$$

$$r_s = 0 \Rightarrow \frac{du}{\sqrt{\left(\frac{ec}{l}\right)^2 - u^2}} = d\phi \Rightarrow u = \frac{ec}{l}\sin(\phi - \phi_0)$$

$u = \frac{1}{r}$, ϕ_0 ↑ integration const.



Now include $r_s u^3 \ll u^2$ term

Put $\phi_0 = 0$, i.e. straight line ∥ to x axis!

$$u = \frac{ec}{l}\left(\sin\phi + f(\phi)\right) \qquad f(\phi) \ll 1$$

$$\left(\frac{du}{d\phi}\right)^2 = \left[\frac{ec}{l}\left(\cos\phi + \frac{df}{d\phi}\right)\right]^2$$

$$\cong \left(\frac{ec}{l}\cos\phi\right)^2 + 2\left(\frac{ec}{l}\right)^2\cos\phi\,\frac{df}{d\phi}$$

$$V \equiv \frac{ec}{l}f = \frac{f}{b}$$

f is unitless.

$$u^2 \cong \left(\frac{ec}{l}\sin\phi\right)^2 + 2\left(\frac{ec}{l}\right)^2\sin\phi\, f$$

$$r_s u^3 \cong r_s\left(\frac{ec}{l}\right)^3\sin^3\phi$$

eqn (18.19) of H+T

Put into (18.15)

$$2\left(\frac{ec}{\ell}\right)^2 \cos\phi\,\frac{df}{d\theta} = -2\left(\frac{ec}{\ell}\right)^2 \sin\phi\, f + r_s\left(\frac{ec}{\ell}\right)^3 \sin^3\phi$$

$$\frac{\cos\phi}{\sin\phi}\frac{df}{d\theta} + f = \left(\frac{r_s ec}{2\ell}\right)\sin^2\phi \quad \Rightarrow f \overset{\text{order of}}{\sim} \frac{r_s ec}{\ell} = \frac{r_s}{b}$$

$$\underbrace{-\frac{\cos\phi\, df}{d\cos\phi}}_{} + f = \left(\frac{r_s ec}{2\ell}\right)(1-\cos^2\phi)$$

↑ doesn't change order of $\cos^n\phi$ in taylor series ∴ $f = A + B\cos^2\phi$ are only powers present on right hand side.

$$-2B\cos^2\phi + A + B\cos^2\phi = \left(\frac{r_s ec}{2\ell}\right)(1-\cos^2\phi)$$

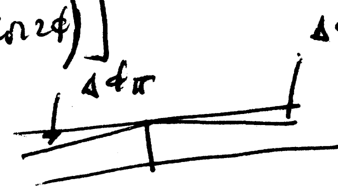
$$\Rightarrow \quad B = \frac{r_s ec}{2\ell}, \quad A = \frac{r_s ec}{2\ell}$$

$$\therefore \quad f = \frac{r_s ec}{2\ell}(1+\cos^2\phi) = \frac{r_s ec}{2\ell}\left(\frac{3}{2} + \frac{\cos 2\phi}{2}\right)$$

$\hookrightarrow \frac{1+\cos 2\phi}{2}$

$$f = \frac{3}{4}\frac{r_s ec}{\ell}\left(1 + \frac{1}{3}\cos 2\phi\right) \qquad (18.18)$$

$$u = \frac{ec}{\ell}\left[\sin\phi + \frac{3}{4}\frac{r_s ec}{\ell}\left(1+\frac{1}{3}\cos 2\phi\right)\right]$$



at $u = 0$ $\quad \phi = \begin{cases} 0 - \Delta\phi_0 \\ \pi + \Delta\phi_\pi \end{cases}$

eqn 18.19 of H+T

$$0 = -\sin\Delta\phi_0 + \frac{3}{4}\frac{r_s e c}{\ell}\left(1+\frac{1}{3}\cos 2\Delta\phi_0\right)$$

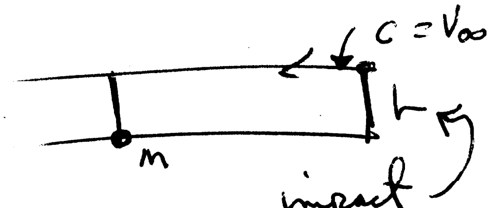
$$0 = -\sin\Delta\phi_\pi + \frac{3}{4}\frac{r_s e c}{\ell}\left(1+\frac{1}{3}\cos 2\Delta\phi_\pi\right)$$

If $\frac{3}{4}\frac{r_s e c}{\ell} \ll 1$ then $\Delta\phi_0 \ll \pi$
$\Delta\phi_\pi \ll \pi$

and $\Delta\phi_0 = \Delta\phi_\pi \cong \frac{r_s e c}{\ell}$

Total scattering angle $\Delta\phi_0 + \Delta\phi_\pi = \left(\frac{2 r_s e c}{\ell}\right)$

but $r^2 \frac{d\phi}{dt}\Big|_{\text{at}\,\infty} = \frac{\ell}{e} = b V_\infty = bc$



$$\therefore \quad \boxed{\Delta\phi_{TOT} = \left(\frac{2 r_s}{b}\right)}$$

For just grazing Sun $b = r_\odot$

$$\boxed{\Delta\phi = \frac{2(2.9\text{ km})}{6.96\times10^{5}\text{ km}} = 0.83\times10^{-5}\text{ rad} = 1.7''}$$

$1'' = 4.85\times10^{-6}$ rad