January 31, 2007

PHYS 6333 (General Relativity)

Assignment #3, Due Wednesday pm February 7, 2007

1. Read: Ch. 5 of Hughston & Tod.
   (a) Prove the transformatin law for connection symbols Γ^a_{bc} as given in the first equation (not numbered) of Ch. 5. Prove it by assuming that equation (5.2.1) defines the components of a tensor field of type $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
   (I don't think the J^a_b notation is very helpful, e.g. equation (5.1.1). I would stick with the usual partial derivatives.)
   (b) Given a flat connection for Minkowski space (i.e., $\Gamma^a_{bc} = 0$ for all a, b, c in rectangular Cartesian coordinates) compute the connection symbols in spherical polar coordinates.
   (c) Work problem 5.1
2. Read: Ch. 8 of Rindler and Ch. 6 of Hartle.

HW#3 Solutions

(a) Assume that if

(5.2.1) $\nabla_b V^a = \partial_b V^a + \Gamma^a_{bc} V^c$ transforms as a $\left[{1 \atop 1}\right]$ tensor, i.e., if

(i) $$\nabla'_b V'^a = \frac{\partial x^d}{\partial x'^b}\frac{\partial x'^a}{\partial x^c}\nabla_d V^c ,$$

then $$\Gamma'^a_{bc} = \frac{\partial x'^a}{\partial x^p}\frac{\partial x^q}{\partial x'^b}\frac{\partial x^r}{\partial x'^c}\Gamma^p_{qr} + \frac{\partial x'^a}{\partial x^p}\frac{\partial^2 x^p}{\partial x'^b \partial x'^c} .$$

$$\frac{\partial}{\partial x'^b} V'^a + \Gamma'^a_{bq} V'^q = \frac{\partial x^d}{\partial x'^b}\frac{\partial x'^a}{\partial x^c}\left(\frac{\partial}{\partial x^d} V^c + \Gamma^c_{de} V^e\right)$$

V'^q: arbitrary

$\frac{\partial}{\partial x'^b}$

substitute $\left(\frac{\partial x^c}{\partial x'^q} V'^q\right)$, $\left(\frac{\partial x^e}{\partial x'^q} V'^q\right)$

$$= \frac{\partial x'^a}{\partial x^c}\left(\frac{\partial^2 x^c}{\partial x'^b \partial x'^q}\right)V'^q + \underbrace{\left(\frac{\partial x'^a}{\partial x^c}\frac{\partial x^c}{\partial x'^q}\right)}_{\delta'^a_q}\frac{\partial}{\partial x'^b}V'^q$$

$$+ \frac{\partial x^d}{\partial x'^b}\frac{\partial x'^a}{\partial x^c}\Gamma^c_{de}\frac{\partial x^e}{\partial x'^q}V'^q$$

cancel

$$\therefore \frac{\partial}{\partial x'^b}V'^a + \underbrace{\left[\frac{\partial x'^a}{\partial x^c}\left(\frac{\partial^2 x^c}{\partial x'^b \partial x'^q}\right) + \frac{\partial x^d}{\partial x'^b}\frac{\partial x'^a}{\partial x^c}\Gamma^c_{de}\frac{\partial x^e}{\partial x'^q}\right]}_{=\Gamma'^a_{bq}} V'^q$$

arbitrary

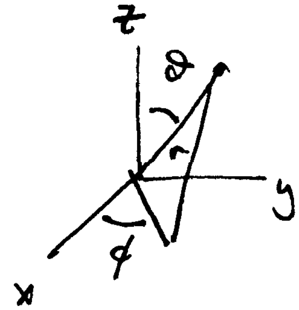
$\frac{4}{5}$

(b) $t' = t$

$r = \sqrt{x^2+y^2+z^2}$

$\theta = \cos^{-1} \frac{z}{\sqrt{x^2+y^2+z^2}}$

$\phi = \cot^{-1}\left(\frac{x}{y}\right)$ $\Gamma = 0$ $\Gamma' = ?$



$x = r\cos\phi\sin\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\theta$

$x^i = (x, y, z)$

$\Gamma'^a_{bc} = \frac{\partial x'^a}{\partial x^g} \frac{\partial^2 x^g}{\partial x'^b \partial x'^c}$

$$\boxed{\Gamma^t_{ab} = \Gamma^a_{tb} = \Gamma^a_{bt} = 0}$$

$\frac{\partial x'^a}{\partial t} = 0$ unless $x'^a = t'$

$\frac{\partial x^a}{\partial t'} = 0$ unless $x^a = t$

$\frac{\partial r}{\partial x^i} = \frac{x^i}{r}$

$\frac{\partial x^i}{\partial r} = \frac{x^i}{r}$

$\frac{\partial x}{\partial \phi} = -y$ $\frac{\partial x}{\partial \theta} = x\cot\theta = r\cos\phi\cos\theta$

$\frac{\partial y}{\partial \phi} = x$ $\frac{\partial y}{\partial \theta} = y\cot\theta = r\sin\phi\cos\theta$

$\frac{\partial z}{\partial \phi} = 0$ $\frac{\partial z}{\partial \theta} = -z\tan\theta = -r\sin\theta$

$\frac{\partial \theta}{\partial x} = + \frac{zx}{r^3\sin\theta}$

$\frac{\partial \theta}{\partial y} = + \frac{zy}{r^3\sin\theta}$

$\frac{\partial \theta}{\partial z} = -\frac{x^2+y^2}{r^3\sin\theta}$

$\frac{\partial \phi}{\partial x} = -\frac{\sin^2\phi}{y}$

$\frac{\partial \phi}{\partial y} = \frac{\sin^2\phi\, x}{y^2}$

$\frac{\partial \phi}{\partial z} = 0$

$$\Gamma'^{a}_{bc} = \left[\frac{\partial x'}{\partial x}\right]\frac{\partial}{\partial x'^{c}}\left[\frac{\partial x}{\partial x'}\right] = \left[\frac{\partial x'}{\partial x}\right]\left[\frac{\partial x}{\partial x'}\right]_{,c}$$

$$\frac{\partial x'^{a}}{\partial x^{b}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x/r & y/r & z/r \\ 0 & \frac{zx}{\sqrt{x^2+y^2}\,r^2} & \frac{zy}{\sqrt{x^2+y^2}\,r^2} & -\frac{\sqrt{x^2+y^2}}{r^2} \\ 0 & -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix}$$

row

col

$$\left[\frac{\partial x'}{\partial x}\right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & \frac{\cos\theta\cos\phi}{r} & \frac{\cos\theta\sin\phi}{r} & -\frac{\sin\theta}{r} \\ 0 & -\frac{\sin\phi}{\sin\theta\, r} & \frac{\cos\phi}{\sin\theta\, r} & 0 \end{pmatrix}$$

row

col

$$\left[\frac{\partial x}{\partial x'}\right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

$$\left[\frac{\partial x}{\partial x'}\right]_{,r} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \cos\theta\cos\phi & -\sin\theta\sin\phi \\ 0 & 0 & \cos\theta\sin\phi & \sin\theta\cos\phi \\ 0 & 0 & -\sin\theta & 0 \end{array}\right)$$

$$\left[\frac{\partial x}{\partial x'}\right]_{,\theta} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & \cos\theta\cos\phi & -r\sin\theta\cos\phi & -r\cos\theta\sin\phi \\ 0 & \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ 0 & -\sin\theta & -r\cos\theta & 0 \end{array}\right)$$

$$\left[\frac{\partial x}{\partial x'}\right]_{,\phi} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & -\sin\theta\sin\phi & -r\cos\theta\sin\phi & -r\sin\theta\cos\phi \\ 0 & \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Gamma^{\bullet}{}_{r\bullet} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r} \end{array}\right)$$

row

$$\Gamma^{\cdot}_{r\cdot} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r} \end{array}\right)$$

col

$\Gamma^{\theta}_{r\theta}$

$\Gamma^{\phi}_{r\phi}$

$$\Gamma^{\cdot}_{\theta\cdot} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -r & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & \cot\theta \end{array}\right)$$

$\Gamma^{r}_{\theta\theta}$

$\Gamma^{\theta}_{\theta r}$

$\Gamma^{\phi}_{\theta\phi}$

$$\Gamma^{\cdot}_{\phi\cdot} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -r\sin^2\theta \\ 0 & 0 & 0 & -\sin\theta\cos\theta \\ 0 & \frac{1}{r} & \cot\theta & 0 \end{array}\right)$$

$\Gamma^{r}_{\phi\phi}$

$\Gamma^{\theta}_{\phi\phi}$

$\Gamma^{\phi}_{\phi r}$ $\Gamma^{\phi}_{\phi\theta}$

(b)

Non-Vanishing Γ's

$$\Gamma^{r}_{\theta\theta} = -r \quad , \quad \Gamma^{r}_{\phi\phi} = -r\sin^2\theta$$

$$\Gamma^{\theta}_{\theta r} = \frac{1}{r} = \Gamma^{\theta}_{r\theta}$$

$$\Gamma^{\phi}_{\phi r} = \frac{1}{r} = \Gamma^{\phi}_{r\phi}$$

$$\Gamma^{\phi}_{\theta\phi} = \cot\theta = \Gamma^{\phi}_{\phi\theta}$$

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$$

y_1

$\boxed{5.1}$

$$\nabla'_b W'_a = \partial'_b W'_a - \Gamma'^c_{ba} W'_c$$

$$\nabla'_b W'_a = \partial'_b\left(\frac{\partial x^c}{\partial x'^a} W_c\right) - \left(\frac{\partial x^d}{\partial x'^b}\frac{\partial x^e}{\partial x'^a}\Gamma^f_{de}\frac{\partial x'^c}{\partial x^f} + \frac{\partial x'^c}{\partial x^e}\frac{\partial^2 x^e}{\partial x'^b \partial x'^a}\right)\frac{\partial x^g}{\partial x'^c} W_g$$

$$= \frac{\partial^2 x^c}{\partial x'^b \partial x'^a} W_c + \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}\frac{\partial W_c}{\partial x^d}$$

cancel

$$- \frac{\partial^2 x^e}{\partial x'^b \partial x'^a}\delta^g_e W_g - \frac{\partial x^d}{\partial x'^b}\frac{\partial x^c}{\partial x'^a}\Gamma^f_{dc}\delta^g_f W_g$$

$$= \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}\left(\frac{\partial}{\partial x^d} W_c - \Gamma^f_{dc} W_f\right)$$

$$\therefore \boxed{\left(\nabla'_b W'_a\right) = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}\left(\nabla_d W_c\right)}$$