Multiple-Scattering Methods in Casimir Calculations

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Time-ordered list of papers

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Green's Function Approach

The multiple scattering approach starts from the well-known formula for the vacuum energy or Casimir energy (for simplicity here we restrict attention to a massless scalar field)(τ is the "infinite" time that the configuration exists) [Schwinger, 1975]

$$E = \frac{i}{2\tau}\mathrm{Tr}\ln G,$$

where G is the Green's function,

$$(-\partial^2 + V)G = 1, \quad +\mathrm{BC}.$$

Ambiguity in formula

The above formula for the Casimir energy is defined up to an infinite constant, which can be at least partially compensated by inserting a factor as do Kenneth and Klich:

$$E = \frac{i}{2\tau} \mathrm{Tr} \ln G G_0^{-1}.$$

Here G_0 satisfies, with the same boundary conditions as G, the free equation

$$-\partial^2 G_0 = 1.$$

T-matrix

Now we define the T-matrix,

$$T = S - 1 = V(1 + G_0 V)^{-1}.$$

If the potential has two disjoint parts,

$$V = V_1 + V_2$$

it is easy to show that

$$T = (V_1 + V_2)(1 - G_0 T_1)(1 - G_0 T_1 G_0 T_2)^{-1}(1 - G_0 T_2),$$

where

$$T_i = V_i(1 + G_0 V_i)^{-1}, \quad i = 1, 2.$$

Interaction in terms of T_i or G_i

Thus, we can write the general expression for the interaction between the two bodies (potentials) in two alternative forms:

$$
\begin{aligned}
E_{12} &= -\frac{i}{2\tau}\mathrm{Tr}\,\ln(1 - G_0 T_1 G_0 T_2) \\
&= -\frac{i}{2\tau}\mathrm{Tr}\,\ln(1 - V_1 G_1 V_2 G_2),
\end{aligned}
$$

where

$$
G_i = (1 + G_0 V_i)^{-1} G_0, \quad i = 1, 2.
$$

Multipole expansion

To proceed to apply this method to general bodies, we use an even older technique, the multipole expansion. Let's illustrate this with a $2+1$ dimensional version, which allows us to describe cylinders with parallel axes. We seek an expansion of the free Green's function

$$G_0(\mathbf{R}+\mathbf{r}'-\mathbf{r}) = \frac{e^{i|\omega||\mathbf{r}-\mathbf{R}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{R}-\mathbf{r}'|}$$
$$= \int \frac{dk_z}{2\pi} e^{ik_z(z-Z-z')} g_0(\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp),$$

Reduced Green's function

$$g_0(\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp) = \int \frac{(d^2k_\perp)}{(2\pi)^2} \frac{e^{-i\mathbf{k}_\perp\cdot\mathbf{R}_\perp} e^{i\mathbf{k}_\perp\cdot(\mathbf{r}_\perp - \mathbf{r}'_\perp)}}{k_\perp^2 + k_z^2 + \zeta^2}.$$

As long as the two potentials do not overlap, so that we have $\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp \neq 0$, we can write an expansion in terms of modified Bessel functions:

$$g_0(\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp) = \sum_{m,m'} I_m(\kappa r) e^{im\phi} I'_m(\kappa r') e^{-im'\phi'} \times \tilde{g}^0_{m,m'}(\kappa R), \quad \kappa^2 = k_z^2 + \zeta^2.$$

Expression for $g^0_{m,m'}$

By Fourier transforming, and using the definition of the Bessel function

$$i^m J_m(kr) = \int_0^{2\pi} \frac{d\phi}{2\pi} \, e^{-im\phi} e^{ikr\cos\phi},$$

we easily find

$$\tilde{g}^0_{m,m'}(\kappa R) = \frac{1}{2\pi} \int \frac{dk\, k}{k^2+\kappa^2} J_{m-m'}(kR) \frac{J_m(kr) J_m(kr')}{I_m(\kappa r) I_m(\kappa r')}$$
$$= \frac{(-1)^{m'}}{2\pi} K_{m-m'}(\kappa R).$$

Discrete matrix realization

Thus we can derive an expression for the interaction between two bodies, in terms of discrete matrices,

$$\mathfrak{E} \equiv \frac{E_{\text{int}}}{L} = \frac{1}{8\pi^2} \int d\zeta \, dk_z \ln \det \left(1 - \tilde{g}^0 \tilde{T}_1 \tilde{g}^{0\top} \tilde{T}_2 \right),$$

where the $\tilde{T}$ matrix elements are given by

$$\tilde{T}_{mm'} = \int dr \, r \, d\phi \int dr' \, r' \, d\phi' I_m(\kappa r) e^{-im\phi} I_{m'}(\kappa r') e^{im'\phi'} \times T(r, \phi; r', \phi').$$

Interaction between cylinders

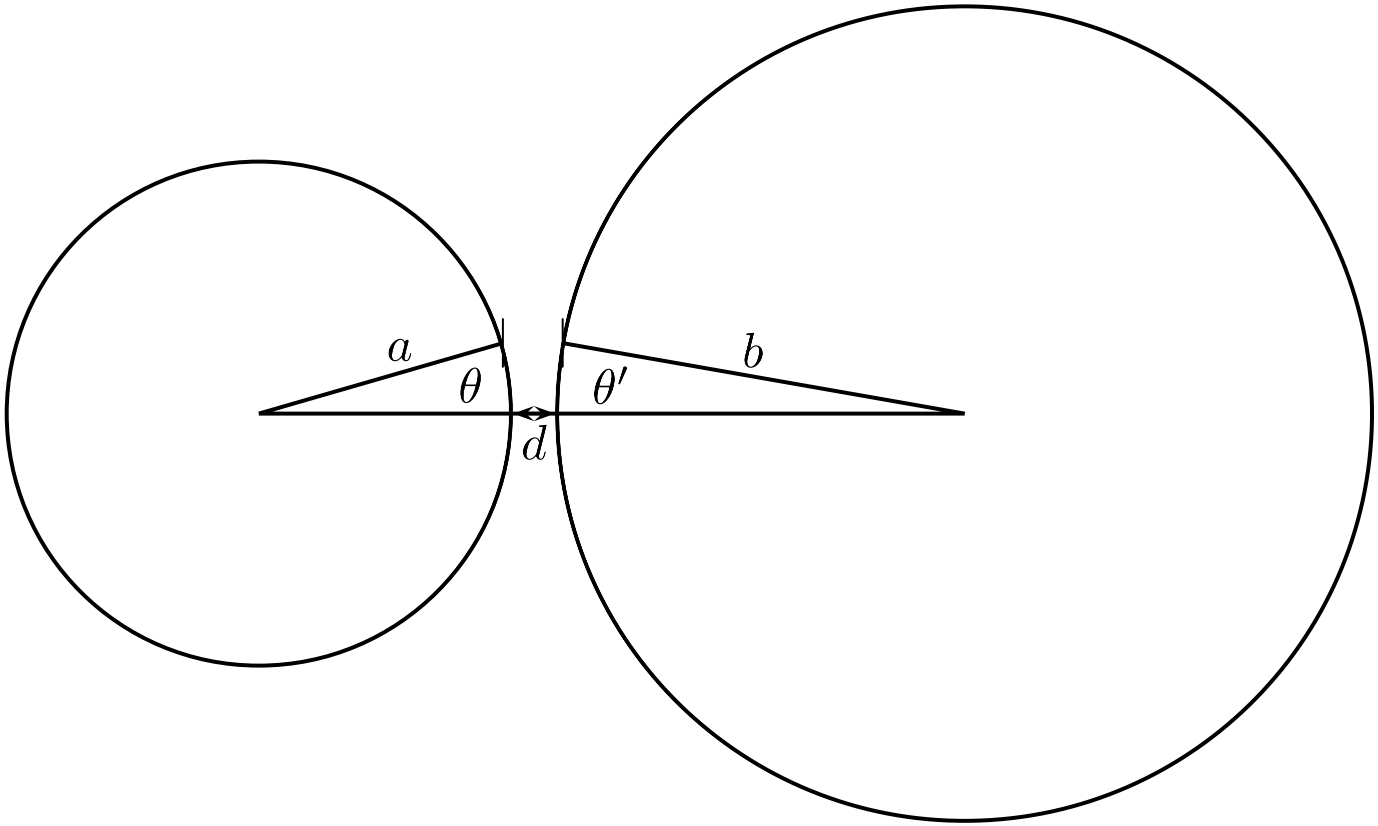


Figure 1: Geometry of two cylinders (or two spheres) with radii a and b, respectively, and distances between their centers of $R > a + b$.

Semitransparent cylinders

Consider two parallel semitransparent cylinders, of radii a and b, respectively, lying outside each other, described by the potentials

$$V_1 = \lambda_1 \delta(r-a), \quad V_2 = \lambda_2 \delta(r'-b),$$

with the separation between the centers R satisfying $R > a+b$. It is easy to work out the scattering matrix in this situation,

$$(t_1)_{mm'} = 2\pi\lambda_1 a \delta_{mm'} \frac{I_m^2(\kappa a)}{1+\lambda_1 a I_m(\kappa a) K_m(\kappa a)}.$$

Cylinder interaction

Thus the Casimir energy per unit length is

$$\mathfrak{E} = \frac{1}{4\pi}\int_0^\infty d\kappa\,\kappa\,\mathrm{tr}\ln(1-A),$$

where $A = B(a)B(b)$, in terms of the matrices

$$B_{mm'}(a) = K_{m+m'}(\kappa R)\frac{\lambda_1 a I_{m'}^2(\kappa a)}{1+\lambda_1 a I_{m'}(\kappa a)K_{m'}(\kappa a)}.$$

Weak-coupling

In weak coupling, the formula for the interaction energy between two cylinders is

$$\mathfrak{E} = -\frac{\lambda_1\lambda_2 ab}{4\pi R^2}\sum_{m,m'=-\infty}^{\infty}\int_0^\infty dx\, x\, K_{m+m'}^2(x)$$
$$\times I_m^2(xa/R) I_{m'}^2(xb/R).$$

Power series expansion

One merely exploits the small argument expansion for the modified Bessel functions $I_m(xa/R)$ and $I_{m'}(xb/R)$:

$$I_m^2(x) = \left(\frac{x}{2}\right)^{2|m|} \sum_{n=0}^{\infty} Z_{|m|,n} \left(\frac{x}{2}\right)^{2n},$$

where the coefficients $Z_{m,n}$ are

$$Z_{m,n} = \frac{2^{2(m+n)}\, \Gamma\left(m+n+\frac{1}{2}\right)}{\sqrt{\pi}\; n!\; (2m+n)!\; \Gamma(m+n+1)}.$$

Closed form result

In this case we get an amazingly simple result

$$\mathfrak{E} = -\frac{\lambda_1 a \lambda_2 b}{4\pi R^2}\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{a}{R}\right)^{2n} P_n(\mu),$$

where $\mu = b/a$, and where by inspection we identify the binomial coefficients

$$P_n(\mu) = \sum_{k=0}^{n}\binom{n}{k}^2 \mu^{2k}.$$

Closed form result (cont.)

Remarkably, it is possible to perform the sums, so we obtain the following closed form for the interaction between two weakly-coupled cylinders:

$$\mathfrak{E} = -\frac{\lambda_1 a \lambda_2 b}{8\pi R^2}\left[\left(1-\left(\frac{a+b}{R}\right)^2\right)\left(1-\left(\frac{a-b}{R}\right)^2\right)\right]^{-1/2}.$$

PFA

We note that in the limit $R - a - b = d \to 0$, d being the distance between the closest points on the two cylinders, we recover the proximity force theorem in this case

$$U(d) = -\frac{\lambda_1\lambda_2}{32\pi}\sqrt{\frac{2ab}{R}}\frac{1}{d^{1/2}}, \quad d \ll a, b.$$

The rate of approach is given by

$$\frac{\mathfrak{E}}{U} \approx 1 - \frac{1+\mu+\mu^2}{4\mu}\frac{d}{R} \approx 1 - \frac{R^2 - aR + a^2}{4a(R-a)}\frac{d}{R}.$$

$a = b$

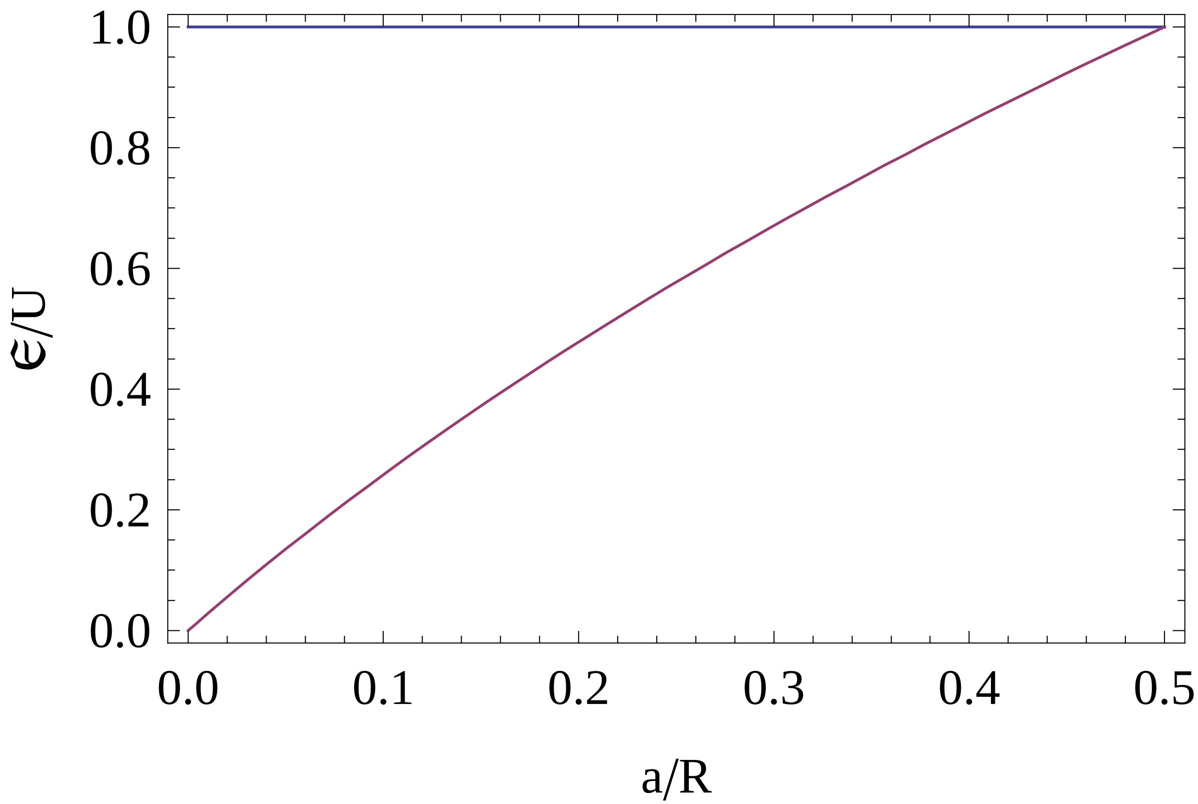


Figure 2: Plotted is the ratio of the exact interaction energy of two weakly-coupled cylinders to the proximity force approximation

$b/a = 99$

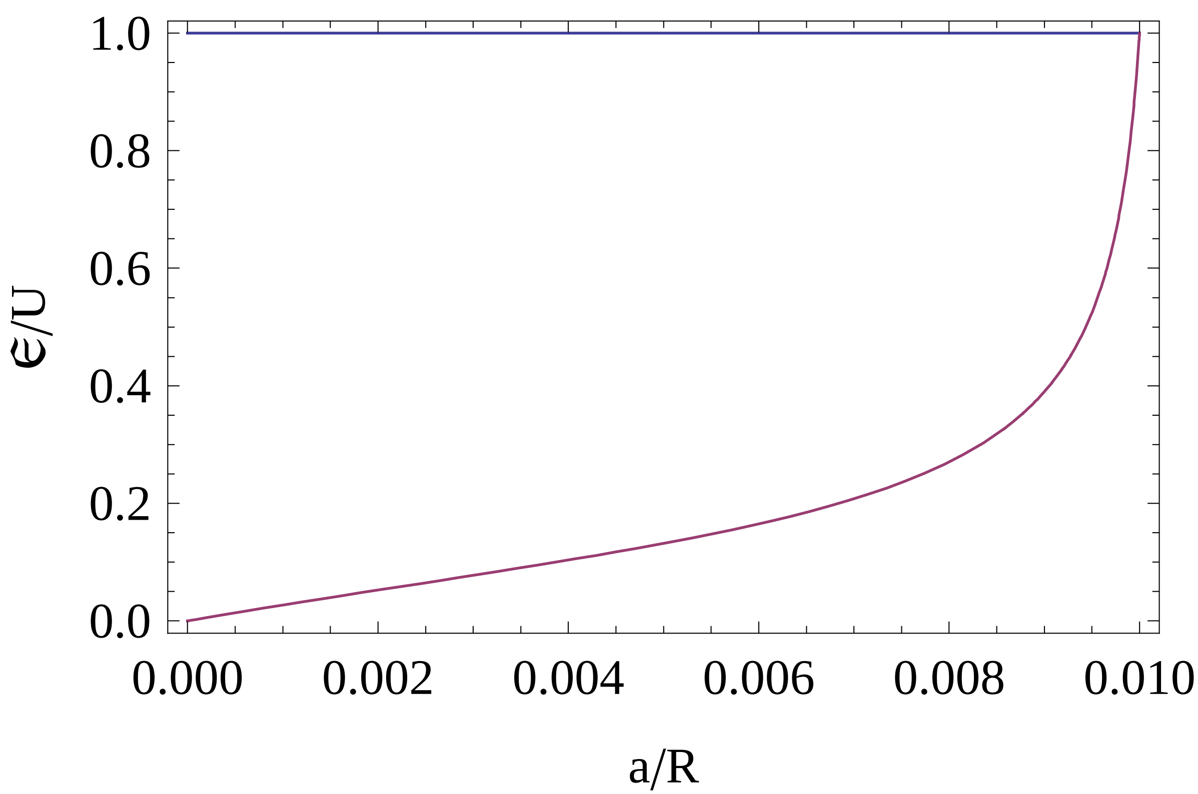


Figure 3: Plotted is the ratio of the exact interaction energy of two weakly-coupled cylinders to the proximity force approximation

Cylinder/plane interaction

By the method of images, we can find the interaction between semitransparent cylinder and a Dirichlet plane is

$$\mathfrak{E} = \frac{1}{4\pi}\int_0^\infty \kappa\, d\kappa\, \mathrm{tr}\, \ln(1 - B(a)),$$

where $B(a)$ is given above. In the strong-coupling limit this result agrees with that given by Bordag, because

$$\mathrm{tr}\, B^s = \mathrm{tr}\, \tilde{B}^s, \quad \tilde{B}_{mm'} = \frac{1}{K_m(\kappa a)} K_{m+m'}(\kappa R) I_{m'}(\kappa a).$$

Exact cylinder/plane energy

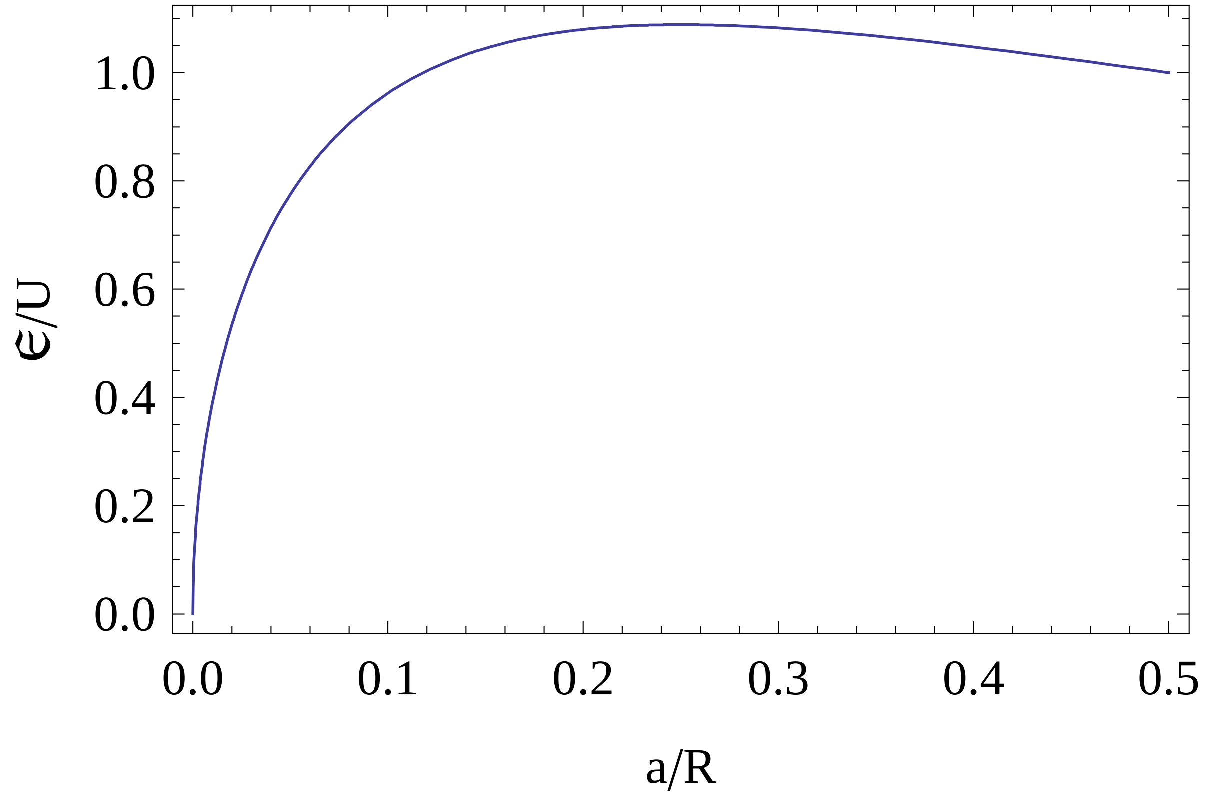
In exactly the same way, we can obtain a closed-form result for the interaction energy between a Dirichlet plane and a weakly-coupled cylinder of radius a separated by a distance $R/2$. The result is again quite simple:

$$\mathfrak{E} = -\frac{\lambda a}{4\pi R^2}\left[1-\left(\frac{2a}{R}\right)^2\right]^{-3/2}.$$

In the limit as $d \to 0$, this agrees with the PFA:

$$U(d) = -\frac{\lambda}{64\pi}\frac{\sqrt{2a}}{d^{3/2}}.$$

Comparison of PFA and exact



3-dimensional formalism

The three-dimensional formalism is very similar. In this case, the free Green's function has the representation

$$G_0(\mathbf{R}+\mathbf{r}'-\mathbf{r}) = \sum_{lm,l'm'} j_l(i|\zeta|r) j_{l'}(i|\zeta|r') Y^*_{lm}(\hat{\mathbf{r}}) Y_{l'm'}(\hat{\mathbf{r}}') \times g_{lm,l'm'}(\mathbf{R}).$$

Reduced Green's function

The reduced Green's function can be written in the form

$$g^0_{lm,l'm'}(\mathbf{R}) = (4\pi)^2 i^{l'-l} \int \frac{(d\mathbf{k})}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{R}}}{k^2+\zeta^2} \frac{j_l(kr)j_{l'}(kr')}{j_l(i|\zeta|r)j_{l'}(i|\zeta|r')}$$

$$\times Y_{lm}(\hat{\mathbf{k}})Y^*_{l'm'}(\hat{\mathbf{k}}).$$

Now we use the plane-wave expansion once again, this time for $e^{i\mathbf{k}\cdot\mathbf{R}}$,

$$e^{i\mathbf{k}\cdot\mathbf{R}} = 4\pi \sum_{l''m''} i^{l''} j_{l''}(kR) Y_{l''m''}(\hat{\mathbf{R}}) Y^*_{l''m''}(\hat{\mathbf{k}}),$$

so now we encounter something new, an integral over three spherical harmonics,

$$\int d\hat{\mathbf{k}} Y_{lm}(\hat{\mathbf{k}}) Y^*_{l'm'}(\hat{\mathbf{k}}) Y^*_{l''m''}(\hat{\mathbf{k}}) = C_{lm,l'm',l''m''},$$

Wigner coefficients

where

$$C_{lm,l'm',l''m''} = (-1)^{m'+m''}\sqrt{\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}} \times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix}.$$

The three-j symbols (Wigner coefficients) here vanish unless $l + l' + l''$ is even.

Reduced Green's function

This fact is crucial, since because of it we can follow the previous method of writing $j_{l''}(kR)$ in terms of Hankel functions of the first and second kind, using the reflection property of the latter, $h^{(2)}_{l''}(kR) = (-1)^{l''} h^{(1)}_{l''}(-kR)$, and then extending the k integral over the entire real axis to a contour integral closed in the upper half plane.

$$g^0_{lm,l'm'}(\mathbf{R}) = 4\pi i^{l'-l}\sqrt{\frac{2|\zeta|}{\pi R}}\sum_{l''m''} C_{lm,l'm',l''m''} \times K_{l''+1/2}(|\zeta|R) Y_{l''m''}(\hat{\mathbf{R}}).$$

Casimir interaction of spheres

For the case of two semitransparent spheres that are totally outside each other,

$$V_1(r) = \lambda_1 \delta(r-a), \quad V_2(r') = \lambda_2 \delta(r'-b),$$

in terms of spherical coordinates centered on each sphere, it is again very easy to calculate the scattering matrices,

$$T_1(\mathbf{r},\mathbf{r}') = \frac{\lambda_1}{a^2}\delta(r-a)\delta(r'-a)$$
$$\times \sum_{lm} \frac{Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')}{1+\lambda_1 a K_{l+1/2}(|\zeta|a) I_{l+1/2}(|\zeta|a)},$$

Scattering matrix element

and then the harmonic transform is very similar to that seen for the cylinder, ($k = i|\zeta|$)

$$(t_1)_{lm,l'm'} = \int (d\mathbf{r})(d\mathbf{r}')\, j_l(kr) Y^*_{lm}(\hat{\mathbf{r}}) j_{l'}(kr') Y_{l'm'}(\hat{\mathbf{r}}') T_1(\mathbf{r},\mathbf{r}'$$

$$= \delta_{ll'}\delta_{mm'}(-1)^l \frac{\lambda_1 a\pi}{2|\zeta|} \frac{I^2_{l+1/2}(|\zeta|a)}{1+\lambda_1 a K_{l+1/2}(|\zeta|a) I_{l+1/2}(|\zeta|a)}.$$

Interaction energy

Let us suppose that the two spheres lie along the z-axis, that is, $\mathbf{R} = R\hat{\mathbf{z}}$. Then we can simplify the expression for the energy somewhat by using $Y_{lm}(\theta = 0) = \delta_{m0}\sqrt{(2l+1)/4\pi}$. The formula for the energy of interaction becomes

$$E = \frac{1}{2\pi}\int_0^\infty d\zeta\, \mathrm{tr}\, \ln(1 - A),$$

where the matrix

$$A_{lm,l'm'} = \delta_{m,m'} \sum_{l''} B_{ll''m}(a) B_{l''l'm}(b)$$

$B_{ll'm}$

$$B_{ll'm}(a) = \frac{\sqrt{\pi}}{\sqrt{2\zeta R}} i^{-l+l'} \sqrt{(2l+1)(2l'+1)} \sum_{l''} (2l''+1)$$

$$\times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix} \frac{K_{l''+1/2}(\zeta R) \lambda_1 a I^2_{l'+1/2}(\zeta a)}{1 + \lambda_1 a I_{l'+1/2}(\zeta a) K_{l'+1/2}(\zeta a}$$

Note that the phase always cancels in the trace.

Weak coupling

For weak coupling, a major simplification results because of the orthogonality property,

$$\sum_{m=-l}^{l} \begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix} \begin{pmatrix} l & l' & l''' \\ m & -m & 0 \end{pmatrix} = \delta_{l''l'''} \frac{1}{2l''+1}, l \le l'.$$

$$E = -\frac{\lambda_1 a \lambda_2 b}{4R} \int_0^\infty \frac{dx}{x} \sum_{ll'l''} (2l+1)(2l'+1)(2l''+1)$$

$$\times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix}^2 K^2_{l''+1/2}(x) I^2_{l+1/2}(xa/R) I^2_{l'+1/2}(xb/R).$$

Power series expansion

As with the cylinders, we expand the modified Bessel functions of the first kind in power series in $a/R, b/R < 1$. This expansion yields the infinite series

$$E = -\frac{\lambda_1 a \lambda_2 b}{4\pi R}\frac{ab}{R^2}\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{m=0}^{n} D_{n,m}\left(\frac{a}{R}\right)^{2(n-m)}\left(\frac{b}{R}\right)^{2m}$$

where by inspection of the first several $D_{n,m}$ coefficients we can identify them as

$$D_{n,m} = \frac{1}{2}\binom{2n+2}{2m+1},$$

Closed form

and now we can immediately sum the expression for the Casimir interaction energy to give the closed form

$$E = \frac{\lambda_1 a \lambda_2 b}{16\pi R} \ln\left(\frac{1-\left(\frac{a+b}{R}\right)^2}{1-\left(\frac{a-b}{R}\right)^2}\right).$$

PFA

Again, when $d = R - a - b \ll a, b$, the proximity force theorem is reproduced:

$$U(d) \sim \frac{\lambda_1 \lambda_2 ab}{16\pi R} \ln(d/R), \quad d \ll a, b.$$

However, as the figures demonstrate, the approach is not very smooth, even for equal-sized spheres. The ratio of the energy to the PFA is ($b/a = \mu$)

$$\frac{E}{U} = 1 + \frac{\ln[(1+\mu)^2/2\mu]}{\ln d/R}, \quad d \ll a, b.$$

$a = b$; truncation at 100 shown

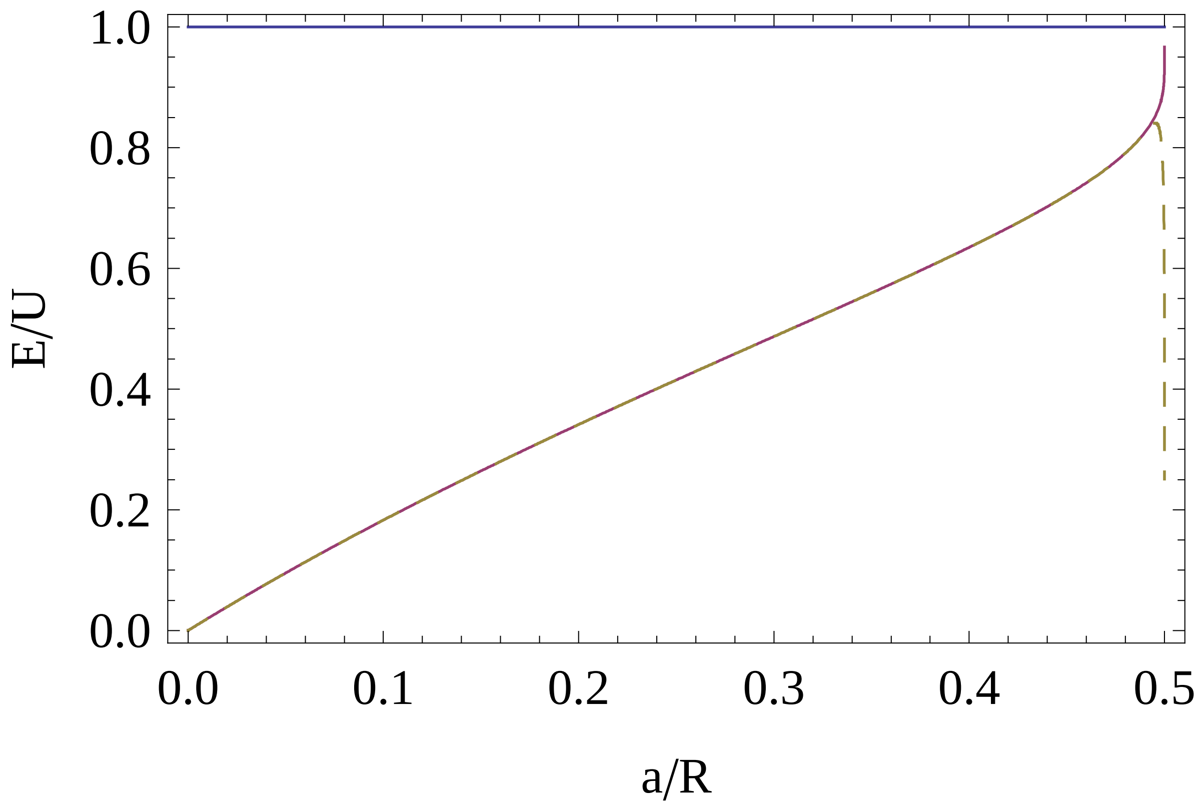


Figure 4: Plotted is the ratio of the exact interaction energy of two weakly-coupled spheres to the proximity force approximation

$b/a = 49$

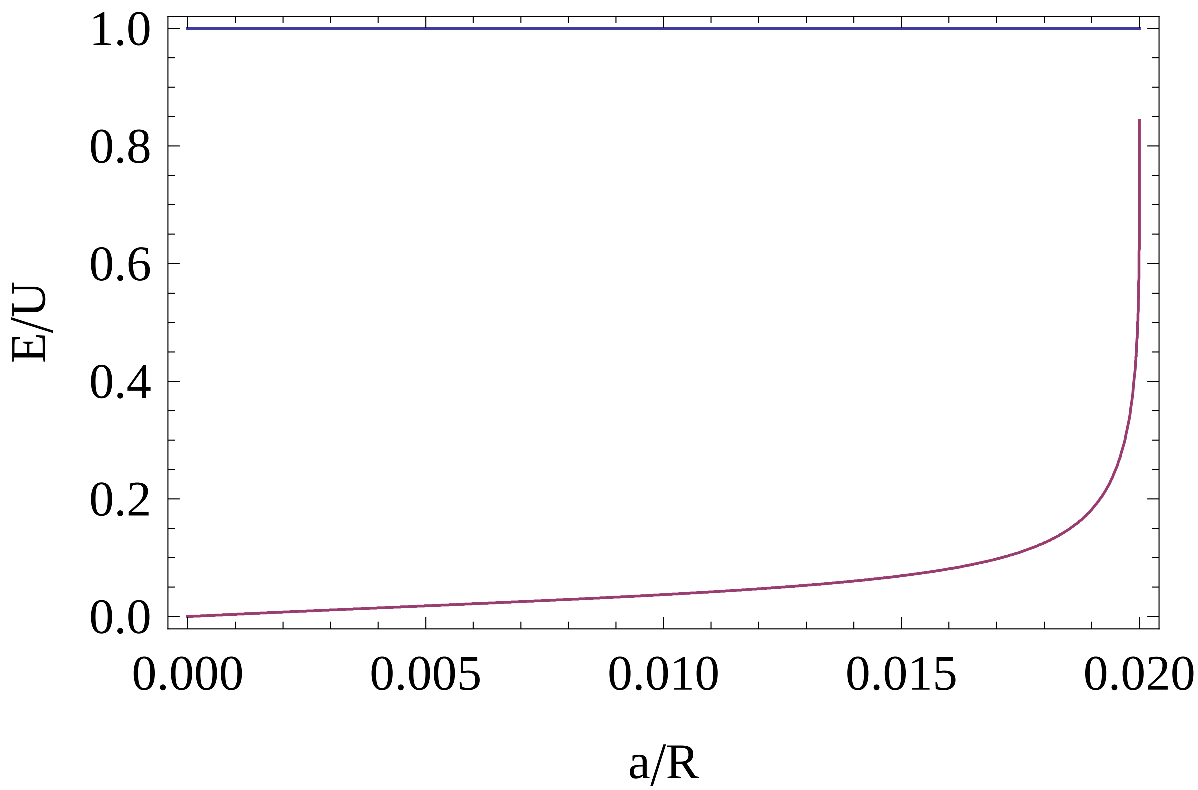


Figure 5: Plotted is the ratio of the exact interaction energy of two weakly-coupled spheres to the proximity force approximation

Exact plane/sphere energy

In just the way indicated above, we can obtain a closed-form result for the interaction energy between a weakly-coupled sphere and a Dirichlet plane. Using the simplification that

$$\sum_{m=-l}^{l}(-1)^m \begin{pmatrix} l & l & l' \\ m & -m & 0 \end{pmatrix} \begin{pmatrix} l & l & l' \\ 0 & 0 & 0 \end{pmatrix} = \delta_{l'0},$$

we can write the interaction energy in the form

$$E = -\frac{\lambda a}{2\pi R}\int_0^\infty dx \sum_{l=0}^{\infty} \sqrt{\frac{\pi}{2x}}(2l+1)K_{1/2}(x)I^2_{l+1/2}\left(x\frac{a}{R}\right)$$

Then in terms of $R/2$ as the distance between the center of the sphere and the plane, the exact interaction energy is

$$E = -\frac{\lambda}{2\pi}\left(\frac{a}{R}\right)^2 \frac{1}{1-(2a/R)^2},$$

which as $a \to R/2$ reproduces the proximity force limit, contained in the (ambiguously defined) PFA formula

$$U = -\frac{\lambda}{8\pi}\frac{a}{d}.$$

Exact energy vs. PFA

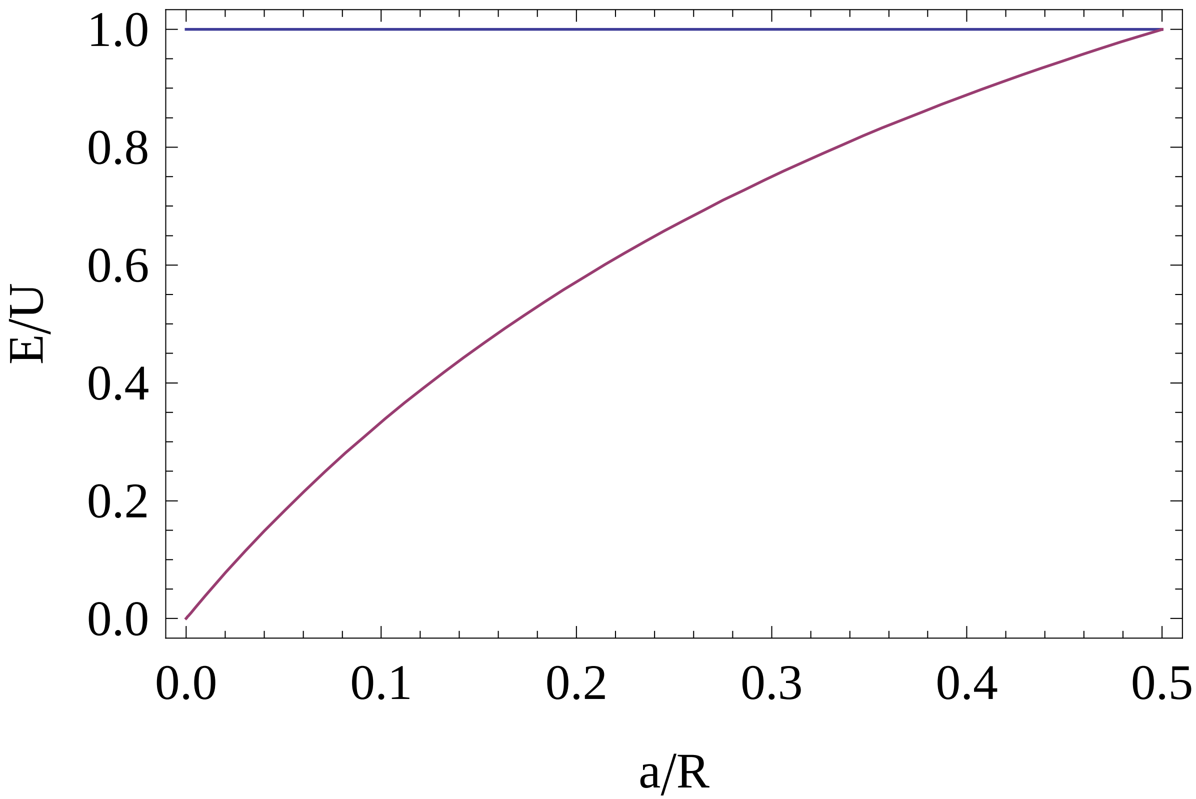


Figure 6: Plotted is the ratio of the exact interaction energy of a weakly-coupled sphere above a Dirichlet plane to the PFA.

Comments and Prognosis

- The methods proposed are in fact not particularly novel, and illustrate the ability of physicists to continually rediscover old methods.
- What is new is the ability, partly due to enhancement in computing power and flexibility, to evaluate continuum determinants (or infinitely dimensional discrete ones) accurately numerically.
- This will make it possible to compute Casimir forces for geometries previously inaccessible.

New results

- It is indeed remarkable, if perhaps not surprising in retrospect, to see that closed form expressions can be obtained for the interaction between spheres and between parallel cylinders in weak coupling.
- These results demonstrate most conclusively the unreliability of the proximity force approximation (of course, the proximity force theorem holds true).
- Further applications of our method will be given in the talks by Shajesh and Prachi. (See also Jef’s talk.)