Casimir torque between corrugated surfaces:II
Non-contact gears

March 15, 2008

Introduction

Recently a non-contact rack and pinion arrangement has been proposed by Ashourvan etal.[1]. We generalize this proposal to the design of a non-contact gear consisting of two corrugated concentric cylinders. We derive an analytic expression for the torque on the cylinders in this arrangement for the case when mean of the corrugation amplitudes are small compared to difference between the mean radii of individual cylinders.

Corrugated cylinders: Casimir torque

1. We consider two concentric, semi-transparent, corrugated cylinders described by the potentials,

$$V_i(r,\theta) = \lambda_i \, \delta(r - a_i - h_i(\theta)), \tag{1}$$

where $i = 1, 2$ and we shall have $a = a_2 - a_1 > 0$.

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3. This is described by an angular translation of the corrugations on the inner cylinder as, $h_1(\theta + \theta_0(t))$, with the initial condition, $\theta_0(0) = 0$.
4. We confine to simple static situation $\theta_0(t) \to \theta_0$.
5. The torque on the cylinders for this case will be

$$\tau = -\frac{\partial E}{\partial \theta_0}, \tag{2}$$

where E is the total Casimir energy associated with the concentric cylinders.

Corrugated cylinders

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2. We fix this situtation as our background. Denote quantities associated with it by superscript (0).
3. Thus the potential for the background is

$$V_i^{(0)}(r) = \lambda_i \, \delta(r - a_i), \tag{3}$$

   which has no angular dependence.
4. The total Casimir energy associated with the background due to the two uncorrugated cylinders will be denoted as $E^{(0)}$, which will include the divergent contributions associated with the single cylinders.
5.

$$\tau^{(0)} = -\frac{\partial E^{(0)}}{\partial \theta_0} = 0, \tag{4}$$

   which means that this configuration does not contribute to the Casimir torque.

Corrugated cylinders

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$$\Delta E = E - E^{(0)} = E_1 + E_2 + E_{12} \tag{5}$$

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3. E_{12} is the contribution to the total energy due to the interaction between the corrugations in the cylinders. Only this part of the energy contributes to the Casimir torque. Thus,

$$\tau = \frac{\partial}{\partial \theta_0} \Delta E = -\frac{\partial E_{12}}{\partial \theta_0} \tag{6}$$

Casimir energy

1. Casimir energy can be obtained using

$$\Delta E = E - E^{(0)} = \frac{i}{2T} \operatorname{Tr} \ln G \, {G^{(0)}}^{-1}, \tag{7}$$

where, T is the infinite time associated with the system.

2. The Green's function G satisfies the differential equation, which can be written in the matrix notation as,

$$(-\partial^2 + V_1 + V_2) G = 1, \tag{8}$$

3. Corresponding Green's function associated with the background satisfies the differential equation,

$$(-\partial^2 + V_1^{(0)} + V_2^{(0)}) G^{(0)} = 1. \tag{9}$$

4. Using above equations we can obtain interaction energy term as

$$\Delta E_{12} = -\frac{i}{2T} \operatorname{Tr} \ln \Big[1 - G_1 \Delta V_1 G_2 \Delta V_2 \Big]. \tag{10}$$

Perturbative evaluation of ΔE_{12}

1. Let us define the function

$$a(\theta) = a_2 - a_1 + h_2(\theta) - h_1(\theta) \tag{11}$$

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2. When the corrugations can be treated as small pertubations, $|h_i(\theta)| \ll a < a_i$, we can approximate the potentials as

$$\Delta V_i(r,\theta) = V_i^{(1)}(r,\theta) + \mathcal{O}(h)^2, \tag{12}$$

where

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3. Thus, to the leading order the interaction energy of the corrugations takes the form

$$\Delta E_{12} = \frac{i}{2T}\,\mathrm{Tr}\left[G^{(0)}\Delta V_1^{(1)} G^{(0)}\Delta V_2^{(1)}\right] + \mathcal{O}(h)^3. \tag{14}$$

Evaluation of ΔE_{12}

1. Evaluation of ΔE_{12} to the leading order involves solving for the Green's function for the configuration involving the background alone, which is given by

$$G^{(0)}(x,x') = \int \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')} \int \frac{dk}{2\pi}\, e^{ik(z-z')} \sum_{m=-\infty}^{+\infty} \frac{1}{2\pi}\, e^{im(\theta-\theta')}\, g_m^{(0)}(r,r';\kappa), \tag{15}$$

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2. Reduced Green's function $g_m^{(0)}(r,r';\kappa)$, satisfies the equation

$$-\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{m^2}{r^2} - \kappa^2 - \lambda_1 \delta(r-a_1) - \lambda_2 \delta(r-a_2)\right] g_m^{(0)}(r,r';\kappa) = \frac{\delta(r-r')}{r} \tag{16}$$

which can be solved explicitly in terms of Bessel functions.

Evaluation of ΔE_{12}

1. Interaction energy in terms of reduced Green's function is

$$\frac{\Delta E_{12}}{L_z} = \frac{1}{(2\pi)^2} \sum_{m=-\infty}^{+\infty} \sum_{m'=-\infty}^{+\infty} (\tilde{h}_1)_{m-m'} (\tilde{h}_2)_{m'-m} L_{mm'}, \tag{17}$$

where, $(\tilde{h}_i)_m$ are the Fourier transforms of the functions $h_i(\theta)$

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2. The kernel $L_{mm'}$ is given in terms of the reduced Green's function as

$$L_{mm'} = -\frac{\lambda_1 \lambda_2}{4\pi} \int_0^\infty \kappa \, d\kappa \left[\frac{\partial}{\partial r} \frac{\partial}{\partial \bar{r}} r \, \bar{r} \, g_m^{(0)}(r, \bar{r}; \kappa) \, g_{m'}^{(0)}(\bar{r}, r; \kappa) \right]_{\bar{r}=a_1, r=a_2} . \tag{19}$$

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3. Note that

$$L_{mm'} = L_{m'm} \tag{20}$$

Evaluation of $L_{mm'}$

Evaluation of $L_{mm'}$ involves derivatives of green's functions on the cylinder boundaries, which are discontinous.

Evaluation of $L_{mm'}$: Dirichlet limit

For the case of Dirichlet limit ($\lambda_{1,2} \to \infty$) we get the relatively simple expression to be

$$L_{mm'} = \frac{1}{4\pi}\frac{1}{a_1 a_2}\int_0^\infty \kappa\, d\kappa \frac{1}{w_m(a_1, a_2; \kappa)}\frac{1}{w_{m'}(a_1, a_2; \kappa)}, \tag{21}$$

where

$$w_m(a_1, a_2; \kappa) = I_m(\kappa a_1)K_m(\kappa a_2) - I_m(\kappa a_2)K_m(\kappa a_1). \tag{22}$$

Sinusoidal corrugations

For the case of sinusoidal corrugations we have

$$h_1(\theta) = h_1 \sin[\nu_1(\theta + \theta_0)]$$
$$h_2(\theta) = h_2 \sin[\nu_2 \theta]$$

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$$\begin{aligned} h_1(\theta) &= h_1 \sin[\nu_1(\theta+\theta_0)] \\ h_2(\theta) &= h_2 \sin[\nu_2\theta] \end{aligned}$$

We can evaluate the Fourier transforms ($\tilde{h}_i$)

$$\begin{aligned} \tilde{h}_{1,m} &= h_1 \frac{2\pi}{2i}\left[e^{i\nu_1\theta_0}\delta_{m,\nu_1} - e^{-i\nu_1\theta_0}\delta_{m,-\nu_1}\right] \\ \tilde{h}_{2,m} &= h_2 \frac{2\pi}{2i}\left[\delta_{m,\nu_2} - \delta_{m,-\nu_1}\right] \end{aligned}$$

Casimir torque: Dirichlet limit

We can write Casimir energy as

$$\frac{\Delta E_{12}}{L_z} = \delta_{\nu_1,\nu_2} \, A(a_1, a_2, \nu) \, \frac{h_1 h_2}{a^4} \, \cos \nu\theta_0, \tag{24}$$

where $\nu = \nu_1 = \nu_2$ and

$$A(a_1, a_2, \nu) = \frac{1}{8\pi} \frac{a^4}{a_1 a_2} \sum_{m=-\infty}^{+\infty} \int_0^\infty \kappa \, d\kappa \frac{1}{w_m(a_1, a_2; \kappa)} \frac{1}{w_{m+\nu}(a_1, a_2; \kappa)}. \tag{25}$$

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Using above results we get the expression for the torque per unit length on the cylinders to be

$$\frac{\tau}{L_z} = \delta_{\nu_1,\nu_2}\, \nu\, A(a_1, a_2, \nu)\, \frac{h_1 h_2}{a^4}\, \sin \nu\theta_0. \tag{26}$$

To do: in cylinders

1. Plots!
2. Complete the calculation using uniform asymptotic approximation get the result for parallel plates. This seems straight forward to do.
3. Evaluate force in the PFA as done for parallel plates.
4. Higher order calculation.

References

Arash Ashourvan, MirFaez Miri, and Ramin Golestanian, Phys. Rev. Lett. **98**, 140801 (2007).