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Identity of the van der Waals Force and the Casimir Effect and the Irrelevance of these Phenomena to Sonoluminescence

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Abstract

We show that the Casimir, or zero-point, energy of a dilute dielectric ball, or of a spherical bubble in a dielectric medium, coincides with the sum of the van der Waals energies between the molecules that make up the medium. That energy, which is finite and repulsive when self-energy and surface effects are removed, may be unambiguously calculated by either dimensional continuation or by zeta function regularization. This physical interpretation of the Casimir energy seems unambiguous evidence that the bulk self-energy cannot be relevant to sonoluminescence.

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Introduction The Casimir effect has been recognized as a fundamental aspect of quantum field theory for 50 years [1]. This phenomenon, first presented as an attractive force between parallel perfectly conducting plates, may be thought of as a result of changes in the electromagnetic field fluctuations induced by the presence of boundaries. Recently, it has been confirmed to good accuracy by direct measurements [2,3], although the closely related Lifshitz theory [4] was confirmed experimentally 25 years ago [5].

Actually, the history of the effect goes back a bit further. Casimir and Polder had worked out the retarded dispersion force between molecules in 1947 [6], the long range part of the van der Waals force. Niels Bohr shortly thereafter suggested to Casimir that zero-point energy was relevant to the effect [7], and subsequently Casimir presented a derivation of the force between molecules, and between a molecule and a conducting plate, based on such considerations [8]. The derivation of the force between parallel plates followed shortly [1]. It was thus clear from the outset that there was an intimate tie between the van der Waals forces and the Casimir effect.

Identity of van der Waals and Casimir forces The explicit demonstration of the identity of these two forces was given in the case of dilute parallel dielectric slabs, where the Lifshitz formula for the Casimir energy may be easily seen to be equal, if dispersion be neglected, to the sum of pairwise long-range van der Waals energies [4]:

$$V = -\frac{23\alpha_1\alpha_2}{4\pi r^7}, \tag{1}$$

the Casimir-Polder retarded dispersion potential [6]. (See also Ref. [9].) Here the connection between dielectric constant ϵ and polarizability α is $\epsilon = 1 + 4\pi N\alpha$, N being the number density of molecules. The simple geometry makes this calculation easy and unambiguous.

The corresponding calculation for a spherical geometry is fraught with more difficulty. The sum of van der Waals interactions (1) for a spherical ball (the case for a spherical bubble is identical) has been given in Ref. [10]. A sensible procedure for carrying out the calculation is dimensional continuation, which has been advocated in Ref. [11]. That is, we evaluate the integral

$$E_{\rm vdW} = -\frac{23}{8\pi}\alpha^2 N^2 \int d^D r d^D r' (r^2 + r'^2 - 2rr'\cos\theta)^{-\gamma/2}, \tag{2}$$

by first regarding $D > \gamma$ so that the integral exists. The integral may be done exactly in terms of gamma functions, which when evaluated at $D = 3$, $\gamma = 7$ yields [10]

$$E_{\rm vdW} = \frac{23}{1536\pi a}(\epsilon - 1)^2. \tag{3}$$

(Note that the expression (2) is formally negative or attractive, while the continued result is positive or repulsive.)

Of course, the above calculation in three dimensions is divergent. These divergences are of two kinds: "volume" and "surface." The volume divergence is a self-energy effect that would be present if the medium filled all space, and makes no reference to the interface, and therefore is quite unobservable. If the divergences are regulated by inserting a point-splitting cutoff, and the divergent terms are simply omitted, the same result (3) is again obtained.

Now we turn to the Casimir effect. For the case of the dielectric sphere this was first worked out in Ref. [12]. That result has been rederived, for the more general case of a spherical bubble, of radius a, having permittivity ϵ' and permeability μ', surrounded by an infinite medium of permittivity ϵ and permeability μ, in Ref. [13]. Here the volume effect, corresponding to the intrinsic self-energy of either medium, was explicitly removed; a more detailed justification of that procedure is given in Ref. [10]. The general result is rather complicated:

$$E_C = -\frac{1}{4\pi a}\int_{-\infty}^{\infty} dy\, e^{iy\delta}\sum_{l=1}^{\infty}(2l+1)\Big\{x\frac{d}{dx}\ln D_l + 2x'[s_l'(x')e_l'(x') - e_l(x')s_l''(x')] \\ - 2x[s_l'(x)e_l(x) - e_l(x)s_l''(x)]\Big\}, \tag{4}$$

where

$$D_l = [s_l(x')e_l'(x) - s_l'(x')e_l(x)]^2 - \xi^2[s_l(x')e_l'(x) + s_l'(x')e_l(x)]^2, \tag{5}$$

which uses the abbreviation

$$\xi = \frac{\left(\frac{\epsilon'}{\epsilon}\frac{\mu}{\mu'}\right)^{1/2} - 1}{\left(\frac{\epsilon'}{\epsilon}\frac{\mu}{\mu'}\right)^{1/2} + 1}. \tag{6}$$

The integration variables are $x = \sqrt{\mu\epsilon}|y|$, $x' = \sqrt{\mu'\epsilon'}x$, and the Ricatti-Bessel functions are

$$s_l(x) = \left(\frac{\pi x}{2}\right)^{1/2} I_{l+1/2}(x), \quad e_l(x) = \left(\frac{2x}{\pi}\right)^{1/2} K_{l+1/2}(x). \tag{7}$$

The formula (4) has been regulated by a time-splitting parameter, $\delta = \tau/a \to 0$, where τ is a Euclidean time separation between field points.

The general expression (4) is rather opaque. Therefore, we consider a dilute dielectric ball, which was already considered in Ref. [12]. (That is, we consider $\mu = 1$ everywhere, and $\epsilon = 1$ outside of the ball.) The formula, which still admits of dispersion, becomes in that case

$$E_C \approx -\frac{1}{8\pi a}\sum_{l=1}^{\infty}(2l+1)\frac{1}{2}\int_{-\infty}^{\infty} dy\, e^{iy\delta}(\epsilon(y)-1)^2 x\frac{d}{dx}F_l(x), \tag{8}$$

where

$$F_l(x) = x^2\left(1 + \frac{l(l+1)}{x^2}\right) - \frac{1}{4}\left(\frac{d}{dx}e_l s_l\right)^2 - x^2\left[2\left(1 + \frac{l(l+1)}{x^2}\right)e_l s_l - \frac{1}{2}\frac{d^2}{dx^2}e_l s_l\right]^2. \tag{9}$$

The integrand here may be approximated by the uniform asymptotic approximation [14]:

$$e_l(x)s_l(x) \sim \frac{1}{2}zt\left(1 + \frac{a_1(t)}{\nu^2} + \frac{a_2(t)}{\nu^4} + \dots\right), \tag{10}$$

where $\nu = l + 1/2$, $x = \nu z$, and $t = (1+z^2)^{-1/2}$. The coefficients $a_k(t)$ are polynomials in t of degree $3k$. If we ignore dispersion, and set the time-splitting parameter $\delta = 0$, we obtain [15] the leading uniform asymptotic approximation to (8),

$$E_C \sim \frac{(\epsilon-1)^2}{64a}\sum_{l=1}^\infty \left\{\nu^2 - \frac{65}{128} + \frac{927}{16384\nu^2} + O(\nu^{-4})\right\}. \tag{11}$$

The first two terms are formally divergent, but may be evaluated by the zeta-function definition, $\sum_{l=0}^\infty \nu^{-s} = (2^s-1)\zeta(s)$. Note that if only the leading term were kept, the result given in Ref. [13] would be obtained, $E_1 \sim -(\epsilon-1)^2/(256a)$, while including two terms reverses the sign and hardly changes the magnitude [15]: $E_2 \sim +33(\epsilon-1)^2/(8192a)$. This would seem to resolve the conundrum found in Ref. [13], the apparent sign disparity between the Casimir effect and the van der Waals energy.

Indeed, let us do the result exactly. (Probably it is possible to do the integrals analytically, but we have not immediately seen how to do this.) We simply add and subtract the two leading asymptotic terms from the integrand in (8), so that $E = E_2 + E_R$, where the remainder is

$$E_R = \frac{(\epsilon-1)^2}{4\pi a}\sum_{l=1}^\infty \nu^2 \int_0^\infty dz \left[F_l(\nu z) - \frac{t^4}{4} + \frac{t^{10}}{8\nu^2}(1+8z^2-5z^4+z^6)\right], \tag{12}$$

According to the third term in Eq. (11), the z integral here is asymptotic to $927\pi/262144\nu^4$; we evaluate the l sum by doing the integral numerically for the first ten terms, and using the asymptotic approximant thereafter. The result is

$$E_C = (\epsilon-1)^2\frac{0.004767}{a}. \tag{13}$$

This precisely agrees with the van der Waals result (3).[1] (The approximation E_2 is 15% too low, whereas if the first three terms in Eq. (11) are kept, the estimate is 1.8% high.)

The Irrelevance to Sonoluminescence There has recently been considerable controversy concerning the possible relevance of the Casimir effect to sonoluminescence [17]. The idea that the "dynamical Casimir effect" might be relevant to sonoluminescence originated in the work of Schwinger [18]. Recently, a series of papers has strongly advocated Schwinger's point of view [19,20]. This view has been criticized elsewhere [10]. However, now that we clearly see that the Casimir energy may be identified with van der Waals interactions, it seems perfectly plain that the volume effect they consider, proportional to $\epsilon - 1$, simply cannot be present, because such cannot arise from pairwise interactions. (This point was already made in Ref. [12].) Our interpretation stands vindicated: an effect proportional

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[1] As this paper was being written, we received a manuscript from G. Barton [16], in which exactly the same identity between the van der Waals and Casimir forces was noted. He uses an elementary method of summing zero-point energies directly in powers of $(\epsilon - 1)$, using ordinary perturbation theory. Of course, our approach is in principle more general, in that it allows for arbitrary ϵ and permits inclusion of dispersion.

to the volume represents a contribution to the mass density of the material, and cannot give rise to observable effects. (A direct refutation of the photon production calculation of Ref. [20] will appear elsewhere [21].)

More subtle is the role of surface divergences [13]. The zeta function regularization calculation we presented above simply discards such terms; but they appear in more physical regularization schemes. For example, if the time splitting parameter in Eq. (4) is retained, we get from the leading asymptotic expansion

$$E_{\rm div} = -\frac{(\epsilon-1)^2}{4a}\frac{1}{\delta^3}; \tag{14}$$

and if a simple model for dispersion is used, with characteristic frequency ω_0, the same result is obtained with $1/\delta \to \omega_0 a/4$ [13]. (A very similar result is given in Ref. [15].) We believe these terms are probably also unobservable, for they modify the surface tension of the liquid, which, like the bulk energy, is already phenomenologically described. (That surface tension has its origin in the Casimir effect was proposed in Ref. [9].) In any case, this surface energy is probably too small, and definitely of the wrong sign, to be relevant to sonoluminescence. (The flash of light is emitted at the minimum radius.)

We note that Barton in his recent work [16] seems to concur with our assessment: The terms "proportional to V [the volume] and to S [the surface] would be combined with other contributions to the bulk and to the surface energies of the material, and play no further role if one uses the measured values."

It is truly remarkable that however the (true) divergences in the theory are regulated, and subsequently discarded, the finite result is unchanged. That is, in the van der Waals energy, we can simply omit the point-split divergences, or proceed through dimensional continuation, where no divergences are explicit; in either case, the same result (3) is obtained. Likewise, the same result is obtained for the Casimir energy using either a temporal point-splitting, or an exponential wavenumber cutoff [16], and omitting the divergent terms; or through the formal trick of zeta-function regularization. It is worth re-emphasizing that we are not claiming that the Casimir effect for a dielectric ball is finite, unlike the classic case of a spherical conducting shell [22]. It is merely that those divergent terms serve to renormalize phenomenological parameters in the condensed matter system.

So, finally, we are left with the finite term, which in the dilute approximation is given by (3) or (13). For a bubble of minimum radius $\sim 10^{-4}$ cm, the corresponding Casimir energy is only $E_C \sim 10^{-3}$ eV. This is ten orders of magnitude to small to be relevant to sonoluminescence, where about one million optical photons are emitted per flash, and again the sign is wrong. (As to the relevance of a static calculation to the dynamical regime of sonoluminescence, we note that the adiabatic approximation seems favorable, since the time scale for the flash, $\sim 10^{-11}$ s is far longer than the time scale for optical photons, $\sim 10^{-15}$ s.)

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