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Vector Casimir effect for a D-dimensional sphere

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Abstract

The Casimir energy or stress due to modes in a D-dimensional volume subject to TM (mixed) boundary conditions on a bounding spherical surface is calculated. Both interior and exterior modes are included. Together with earlier results found for scalar modes (TE modes), this gives the Casimir effect for fluctuating "electromagnetic" (vector) fields inside and outside a spherical shell. Known results for three dimensions, first found by Boyer, are reproduced. Qualitatively, the results for TM modes are similar to those for scalar modes: Poles occur in the stress at positive even dimensions, and cusps (logarithmic singularities) occur for integer dimensions $D \leq 1$. Particular attention is given the interesting case of $D = 2$.

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I. INTRODUCTION

The dependence of physical quantities on the number of dimensions is of considerable interest [1,2]. In particular, by expanding in the number of dimensions one can obtain nonperturbative information about the coupling constant [3–6]. Useful expansions have also been obtained in inverse powers of the dimension [7].

In a previous paper we investigated the dimensional dependence of the Casimir stress on a spherical shell of radius a in D space dimensions [8]. Specifically, we studied the Casimir stress (the stress on the sphere is equal to the Casimir force per unit area multiplied by the area of the sphere) that is due to quantum fluctuations of a free massless scalar field satisfying Dirichlet boundary conditions on the shell. That is, the Green's functions satisfy the boundary conditions

$$G(\mathbf{x},t;\mathbf{x}',t')\Big|_{|\mathbf{x}|=a}=0. \tag{1.1}$$

Here, following the suggestion of [9], we calculate the TM modes (for which $H_r=0$) in the same situation. The TM modes are modes which satisfy mixed boundary conditions on the surface [10,11],

$$\frac{\partial}{\partial r}r^{D-2}G(\mathbf{x},t;\mathbf{x}',t')\Big|_{|\mathbf{x}|=r=a}=0, \tag{1.2}$$

as opposed to the TE modes (for which $E_r=0$), which satisfy Dirichlet boundary conditions (1.1) on the surface, and are equivalent to the scalar modes found in [8].

The organization of this paper is straightforward. In Sec. II we construct the Green's functions in D-dimensional space by direct solution of the differential equation, subject to the boundary condition (1.2). Then, the Casimir stress is computed from the vacuum expectation value of the energy-momentum-stress tensor, expressed as derivatives of the Green's functions. The resulting expression for the Casimir stress on a D-dimensional spherical shell takes the form of an infinite sum of integrals over modified Bessel functions; the dimension D appears explicitly as well as in the orders of the Bessel functions. By combining the results for the TM modes found here with those for the TE modes found earlier [8], we obtain a general expression for vector modes subject to perfect conductor boundary conditions on the spherical shell, which expression agrees with that found long ago for three dimensions [12]. As a check, in Sec. III we rederive the same result from the vacuum expectation value of the energy density of the field. In Sec. IV we examine this expression for the TM Casimir stress in detail. We show that for $D < 1$ a constant can be added to the series without effect; a suitable constant is chosen so that each term in the series exists (each of the integrals converges). We show how to evaluate the sum of the series numerically for all real D, using two methods: one involving Riemann zeta functions, and the second involving continuation in dimension. Both methods give the same numerical results. When $D > 2$ the TM Casimir stress is real, and finite except when D is an even integer. The well-known $D = 3$ result [10,13,14,12] is reproduced when the $n = 0$ mode is removed. When $D \leq 2$ the Casimir stress is complex; there are logarithmic singularities in the complex-D plane at $D = 2,\ 1,\ 0,\ -1,\ -2,\ \ldots$. In the Appendix, the important case of $D = 2$ is discussed [15–17].

II. STRESS TENSOR FORMALISM

The calculation given in this paper for the Casimir stress on a spherical shell follows very closely the Green's function technique described in [8], and we will therefore be concise, and emphasize the significant differences.

A. Green's function

The two-point Green's function $G(\mathbf{x}, t; \mathbf{x}', t')$ satisfies the inhomogeneous Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) G(\mathbf{x}, t; \mathbf{x}', t') = -\delta^{(D)}(\mathbf{x} - \mathbf{x}')\delta(t - t'), \tag{2.1}$$

subject to the boundary condition (1.2) on the surface $|\mathbf{x}| = a$. We solve this equation by the standard discontinuity method. In particular, we divide space into two regions, *region I*, the interior of a sphere of radius a and *region II*, the exterior of the sphere. In addition, in region I we will require that G be finite at the origin $\mathbf{x} = 0$ and in region II we will require that G satisfy outgoing-wave boundary conditions at $|\mathbf{x}| = \infty$.

We begin by taking the time Fourier transform of G:

$$\mathcal{G}_\omega(\mathbf{x};\mathbf{x}') = \int_{-\infty}^{\infty} dt\, e^{-i\omega(t-t')} G(\mathbf{x},t;\mathbf{x}',t'). \tag{2.2}$$

The differential equation satisfied by $\mathcal{G}_\omega$ is

$$\left(\omega^2 + \nabla^2\right) \mathcal{G}_\omega(\mathbf{x};\mathbf{x}') = \delta^{(D)}(\mathbf{x}-\mathbf{x}'). \tag{2.3}$$

To solve this equation we introduce polar coordinates and seek a solution that has cylindrical symmetry; i.e., we seek a solution that is a function only of the three variables $r = |\mathbf{x}|$, $r' = |\mathbf{x}'|$, and θ, the angle between $\mathbf{x}$ and $\mathbf{x}'$ so that $\mathbf{x}\cdot\mathbf{x}' = rr'\cos\theta$. In terms of these polar variables (2.3) becomes

$$\left(\omega^2 + \frac{\partial^2}{\partial r^2} + \frac{D-1}{r}\frac{\partial}{\partial r} + \frac{\sin^{2-D}\theta}{r^2}\frac{\partial}{\partial\theta}\sin^{D-2}\theta\frac{\partial}{\partial\theta}\right)\mathcal{G}_\omega(r,r',\theta) = \frac{\delta(r-r')\delta(\theta)\Gamma\left(\frac{D-1}{2}\right)}{2\pi^{(D-1)/2}r^{D-1}\sin^{D-2}\theta}. \tag{2.4}$$

We solve (2.4) using the method of separation of variables. The angular dependence is given in terms of the ultraspherical (Gegenbauer) polynomial [18]

$$C_n^{(-1+D/2)}(z) \quad (n = 0,\, 1,\, 2,\, 3,\, \ldots), \tag{2.5}$$

where $z = \cos\theta$. The general solution to (2.4) is an arbitrary linear combination of separated-variable solutions; in region I the Green's function has the form (with $\nu = n - 1 + D/2$ and $k = |\omega|$)

$$\mathcal{G}_\omega(r,r',\theta) = \sum_{n=0}^{\infty} a_n r^{1-D/2} J_\nu(kr) C_n^{(-1+D/2)}(z) \quad (r < r' < a) \tag{2.6a}$$

and

$$\mathcal{G}_\omega(r,r',\theta) = \sum_{n=0}^{\infty} r^{1-D/2}\left[b_n J_\nu(kr) + c_n J_{-\nu}(kr)\right] C_n^{(-1+D/2)}(z) \quad (r' < r < a). \tag{2.6b}$$

[Note that $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent so long as ν is not an integer. Thus, in writing (2.6b), we assume explicitly that D is not an even integer. We also assumed $D > 2$ in writing down (2.6a), so that $J_{-\nu}$ is excluded because it is singular at $r = 0$.] The general solution to (2.4) in region II has the form

$$\mathcal{G}_\omega(r,r',\theta) = \sum_{n=0}^{\infty} d_n r^{1-D/2} H_\nu^{(1)}(kr) C_n^{(-1+D/2)}(z) \quad (r > r' > a) \tag{2.7a}$$

and

$$\mathcal{G}_\omega(r,r',\theta) = \sum_{n=0}^{\infty} r^{1-D/2}\left[e_n H_\nu^{(1)}(kr) + f_n H_\nu^{(2)}(kr)\right] C_n^{(-1+D/2)}(z) \quad (r' > r > a). \tag{2.7b}$$

The arbitrary coefficients a_n, b_n, c_n, d_n, e_n, and f_n are uniquely determined by six conditions; namely, the mixed boundary condition (1.2) at $r = a$,

$$\left(\frac{D}{2} - 1\right)\left[b_n J_\nu(ka) + c_n J_{-\nu}(ka)\right] + ka\left[b_n J_\nu'(ka) + c_n J_{-\nu}'(ka)\right] = 0 \tag{2.8a}$$

and

$$\left(\frac{D}{2} - 1\right)\left[e_n H_\nu^{(1)}(ka) + f_n H_\nu^{(2)}(ka)\right] + ka\left[e_n H_\nu^{(1)\prime}(ka) + f_n H_\nu^{(2)\prime}(ka)\right] = 0, \tag{2.8b}$$

the condition of continuity at $r = r'$,

$$a_n J_\nu(kr') = b_n J_\nu(kr') + c_n J_{-\nu}(kr') \tag{2.8c}$$

and

$$d_n H_\nu^{(1)}(kr') = e_n H_\nu^{(1)}(kr') + f_n H_\nu^{(2)}(kr'), \tag{2.8d}$$

and the jump condition in the first derivative of the Green's function at $r = r'$,

$$b_n J_\nu'(kr') + c_n J_{-\nu}'(kr') - a_n J_\nu'(kr') = \frac{2\nu\Gamma\left(\frac{D-2}{2}\right)}{4(\pi r')^{\frac{D}{2}} k} \tag{2.8e}$$

and

$$e_n H_\nu^{(1)\prime}(kr') + f_n H_\nu^{(2)\prime}(kr') - d_n H_\nu^{(1)\prime}(kr') = -\frac{2\nu\Gamma\left(\frac{D-2}{2}\right)}{4(\pi r')^{\frac{D}{2}}k}. \tag{2.8f}$$

Solving these equations for the coefficients, we easily find the Green's function to be, in region I,

$$\mathcal{G}_\omega(r,r',\theta) = \sum_{n=0}^{\infty} \frac{2\nu\Gamma\left(\frac{D}{2}-1\right)}{8(\pi r r')^{D/2-1}\sin\pi\nu} C_n^{(D/2-1)}(\cos\theta)\left[J_\nu(kr_<)J_{-\nu}(kr_>) - \beta J_\nu(kr)J_\nu(kr')\right], \tag{2.9a}$$

where

$$\beta = \frac{\left(\frac{D}{2}-1\right)J_{-\nu}(ka) + kaJ'_{-\nu}(ka)}{\left(\frac{D}{2}-1\right)J_\nu(ka) + kaJ'_\nu(ka)}, \tag{2.9b}$$

and, in Region II,

$$\begin{aligned}\mathcal{G}_\omega(r,r',\theta) = -i\sum_{n=0}^{\infty} \frac{2\nu\Gamma\left(\frac{D}{2}-1\right)}{16(\pi r r')^{D/2-1}} C_n^{(D/2-1)}(\cos\theta)\Big[&H_\nu^{(1)}(kr_<)H_\nu^{(2)}(kr_>)\\ &- \gamma H_\nu^{(1)}(kr)H_\nu^{(1)}(kr')\Big],\end{aligned} \tag{2.10a}$$

where

$$\gamma = \frac{\left(\frac{D}{2}-1\right)H_\nu^{(2)}(ka) + kaH_\nu^{(2)\prime}(ka)}{\left(\frac{D}{2}-1\right)H_\nu^{(1)}(ka) + kaH_\nu^{(1)\prime}(ka)}. \tag{2.10b}$$

B. Stress Tensor

For a scalar field, we can calculate the induced force per unit area on the sphere from the stress-energy tensor $T^{\mu\nu}(\mathbf{x},t)$, defined by

$$T^{\mu\nu}(\mathbf{x},t) \equiv \partial^\mu\varphi(\mathbf{x},t)\partial^\nu\varphi(\mathbf{x},t) - \frac{1}{2}g^{\mu\nu}\partial_\lambda\varphi(\mathbf{x},t)\partial^\lambda\varphi(\mathbf{x},t). \tag{2.11}$$

The radial scalar Casimir force per unit area f on the sphere is obtained from the radial-radial component of the vacuum expectation value of the stress-energy tensor [12]:

$$f = \langle 0 | T^{rr}_{\rm in} - T^{rr}_{\rm out} | 0 \rangle \,|_{r=a} \,. \tag{2.12}$$

To calculate f we exploit the connection between the vacuum expectation value of the fields and the Green's function,

$$\langle 0 | T\phi(\mathbf{x},t)\phi(\mathbf{x}',t')|0\rangle = iG(\mathbf{x},t;\mathbf{x}',t'), \tag{2.13}$$

so that the force density is given by the derivative of the Green's function at equal times, $G(\mathbf{x},t;\mathbf{x}',t)$:

$$f = \frac{i}{2}\left[\frac{\partial}{\partial r}\frac{\partial}{\partial r'}G(\mathbf{x},t;\mathbf{x}',t)_{\rm in} - \frac{\partial}{\partial r}\frac{\partial}{\partial r'}G(\mathbf{x},t;\mathbf{x}',t)_{\rm out}\right]\Bigg|_{\mathbf{x}=\mathbf{x}',\ |\mathbf{x}|=a} . \tag{2.14}$$

It is a bit more subtle to calculate the force density for the TM modes. For a given frequency, we write

$$\langle T_{rr}\rangle = \frac{i}{2}\left[\nabla_r\nabla_{r'} + \omega^2 - \boldsymbol{\nabla}_\perp\cdot\boldsymbol{\nabla}_{\perp'}\right]\mathcal{G}_\omega, \tag{2.15}$$

where, if we average over all directions, we can integrate by parts on the transverse derivatives,

$$-\boldsymbol{\nabla}_\perp\cdot\boldsymbol{\nabla}_{\perp'} \to \nabla^2_\perp \to -\frac{n(n+D-2)}{r^2}, \tag{2.16}$$

where the last replacement, involving the eigenvalue of the Gegenbauer polynomial, is appropriate for a given mode n [see (2.14) of [8]]. As for the radial derivatives, they are[1]

$$\nabla_r = r^{2-D}\partial_r r^{D-2}, \quad \nabla_{r'} = r'^{2-D}\partial_{r'}r'^{D-2}, \tag{2.17}$$

which, by virtue of (1.2), implies that the $\nabla_r\nabla_{r'}$ term does not contribute to the stress on the sphere. In this way, we easily find the following formula for the contribution to the force per unit area for interior modes,

[1]In the TM mode, the radial derivatives correspond to tangential components of $\mathbf{E}$, which must vanish on the surface. See [11].

$$f^{\rm TM}_{\rm in}=-\frac{i}{\pi^{(D+1)/2}2^Da^{D+1}\Gamma(\frac{D-1}2)}\int_0^\infty\frac{dx}{x}\sum_{n=0}^\infty w(n,D)(x^2-n(n+D-2))\frac{s_n(x)}{s_n'(x)},\qquad(2.18)$$

and for exterior modes,

$$f^{\rm TM}_{\rm out}=-\frac{i}{\pi^{(D+1)/2}2^Da^{D+1}\Gamma(\frac{D-1}2)}\int_0^\infty\frac{dx}{x}\sum_{n=0}^\infty w(n,D)(x^2-n(n+D-2))\frac{e_n(x)}{e_n'(x)},\qquad(2.19)$$

where

$$w(n,D)=\frac{(2n+D-2)\Gamma(n+D-2)}{n!},\qquad(2.20)$$

$x=ka$, and the generalized Ricatti-Bessel functions are

$$s_n(x)=x^{D/2-1}J_\nu(x),\quad e_n(x)=x^{D/2-1}H^{(1)}_\nu(x).\qquad(2.21)$$

It is a small check to observe that for $D=2$ we recover the known result [15]

$$f^{\rm TM}_{D=2}=-\frac{i}{8\pi^2a^3}\int_{-\infty}^\infty dx\,x\sum_{m=-\infty}^\infty\left(1-\frac{m^2}{x^2}\right)\left(\frac{J_m(x)}{J_m'(x)}+\frac{H_m^{(1)}(x)}{H_m^{(1)\prime}(x)}\right),\qquad(2.22)$$

where the half-weight at $n=0$ is a result of the *limit* $D\to2$. In two dimensions, the vector Casimir effect consists of only the TM mode contribution.

In general, we can combine the TE mode contribution, given in [8], and the TM mode contribution, found here, into the following simple formula[2]:

$$f^{\rm TM+TE}=\frac{i}{\pi^{(D+1)/2}2^Da^{D+1}}\sum_{n=0}^\infty\frac{w(n,D)}{\Gamma(\frac{D-1}2)}$$
$$\times\int_0^\infty dx\,x\left\{\frac{s_n'(x)}{s_n(x)}+\frac{e_n'(x)}{e_n(x)}+\frac{s_n''(x)}{s_n'(x)}+\frac{e_n''(x)}{e_n'(x)}\right\}.\qquad(2.23)$$

It will be noted that, for $D=3$, this result agrees with that found for the usual electrodynamic Casimir force/area, when the $n=0$ mode is properly excluded. [See (4.7) of [12] with the cutoff $\epsilon=0$.] Of course, this only coincides with electrodynamics in three dimensions. The number of electrodynamic modes changes discontinuously with dimension, there being

[2]We will not concern ourselves with a constant term in the integrand, which we will deal with in Sec. IV.

only one in $D=2$, the TM mode, and none in $D=1$, in general there being $D-1$ modes. Equation (2.23) is of interest in a mathematical sense, because significant cancellations do occur between TE and TM modes in general.

The integrals in (2.18) and (2.19) are oscillatory and therefore very difficult to evaluate numerically. Thus, it is advantageous to perform a rotation of 90 degrees in the complex-ω plane. The resulting expression for $f^{\rm TM}$ is

$$f^{\rm TM}=-\sum_{n=0}^{\infty}\frac{w(n,D)}{2^D\pi^{\frac{D+1}{2}}a^{D+1}\Gamma\left(\frac{D-1}{2}\right)}\int_0^\infty dx\,x\frac{d}{dx}\ln\left[x^{2(3-D)}\left(x^{D/2-1}K_\nu(x)\right)'\left(x^{D/2-1}I_\nu(x)\right)'\right]. \tag{2.24}$$

III. ENERGY DERIVATION

As a check of internal consistency, it would be reassuring to derive the same result by integrating the energy density due to the field fluctuations. The latter is computable from the vacuum expectation value of the stress tensor, which in turn is directly related to the Green's function, $\mathcal{G}_\omega$:

$$\langle T_{00}\rangle=\frac{i}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}(\omega^2+\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}')\mathcal{G}_\omega\Big|_{\mathbf{r}=\mathbf{r}'}. \tag{3.1}$$

Again, because we are going to integrate this over all space, we can integrate by parts, replacing, in effect,

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}'\to-\nabla^2\to\omega^2, \tag{3.2}$$

which uses the Green's function equation (2.3). [Point splitting is always implicitly assumed, so that delta functions may be omitted.] Then, using the area of a unit sphere in D dimensions,

$$A_D=\frac{2\pi^{D/2}}{\Gamma(D/2)}, \tag{3.3}$$

we find the Casimir energy to be given by

$$E=\frac{i2\pi^{D/2}}{\Gamma(D/2)}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\omega^2\int_0^\infty r^{D-1}dr\,\mathcal{G}_\omega(r,r). \tag{3.4}$$

So, from the form for the Green's function given in (2.9a) and (2.10a), we see that we need to evaluate integrals such as

$$\int_0^a r\,dr\,J_\nu(kr)J_{-\nu}(kr), \tag{3.5}$$

which are given in terms of the indefinite integral

$$\int dx\,xZ_\nu(x)\mathcal{Z}_\nu(x)=\frac{x^2}{2}\left(\left(1-\frac{\nu^2}{x^2}\right)Z_\nu(x)\mathcal{Z}_\nu(x)+Z'_\nu(x)\mathcal{Z}'_\nu(x)\right), \tag{3.6}$$

valid for any two Bessel functions Z_ν, $\mathcal{Z}_\nu$ of order ν. Thus we find for the Casimir energy of the TM modes the formula

$$\begin{aligned}E^{\rm TM}=&-\frac{i}{2\pi\Gamma(D-1)a}\sum_{n=0}^\infty w(n,D)\\&\times\int_0^\infty\frac{dx}{x}\left[(x^2-n(n+D-2))\left(\frac{s_n(x)}{s'_n(x)}+\frac{e_n(x)}{e'_n(x)}\right)+(2-D)x\right].\end{aligned} \tag{3.7}$$

We obtain the stress on the spherical shell by differentiating this expression with respect to a (which agrees with (2.18) and (2.19), apart from the constant in the integrand), followed again doing the complex frequency rotation, which yields

$$F^{\rm TM}=\frac{1}{2\pi a^2\Gamma(D-1)}\sum_{n=0}^\infty w(n,D)Q_n, \tag{3.8}$$

where the integrals are

$$Q_n=-\int_0^\infty dx\,x\frac{d}{dx}\ln q(x), \tag{3.9}$$

where

$$\begin{aligned}q(x)=&\left[\left(\frac{D}{2}-1\right)I_\nu(x)+\frac{x}{2}(I_{\nu+1}(x)+I_{\nu-1}(x))\right]\\&\times\left[\left(\frac{D}{2}-1\right)K_\nu(x)-\frac{x}{2}(K_{\nu+1}(x)+K_{\nu-1}(x))\right].\end{aligned} \tag{3.10}$$

This agrees with the form found directly from the force density, (2.24), again, apart from a constant in the x integrand.

IV. NUMERICAL EVALUATION OF THE STRESS

We now need to evaluate the formal expression (3.8) for arbitrary dimension D. We implicitly assumed in its derivation that $D > 2$ and that D was not an even integer, but we will argue that (3.8) can be continued to all D.

A. Convergent reformulation of (3.8)

First of all, it is apparent that as it sits, the integral Q_n in (3.9) does not exist. (The form in (2.24) does exist for the special case of $D = 3$.) As in the scalar case [8], we argue that since

$$\sum_{n=0}^{\infty} w(n, D) = 0 \quad \text{for} \quad D < 1, \tag{4.1}$$

we can add an arbitrary term, independent of n, to Q_n in (3.8) without effect as long as $D < 1$. In effect then, we can multiply the quantity in the logarithm in (3.9) by an arbitrary power of x without changing the value for the force for $D < 1$. We choose that multiplicative factor to be $-2/x$ because then a simple asymptotic analysis shows that the integrals converge. Then, we analytically continue the resulting expression to all D. The constant -2 is, of course, without effect in (3.9), but allows us to integrate by parts, ignoring the boundary terms. The result of this process is that the expression for the Casimir force is still given by (3.8), but with Q_n replaced by

$$Q_n = \int_0^\infty dx \ln\left[-\frac{2}{x}q(x)\right], \tag{4.2}$$

$q(x)$ being given by (3.10).

Now the individual integrals in (3.8) converge, but the sum still does not. We can see this by making the uniform asymptotic approximations for the Bessel functions in (3.10) [19], which leads to

$$Q_n \sim \frac{\pi\nu}{2}\left(1 + \frac{-101 + 80D - 16D^2}{64\nu^2} + \frac{-5861 + 11152D - 7680D^2 + 2304D^3 - 256D^4}{16384\nu^4} + \dots\right) \quad (n \to \infty). \tag{4.3}$$

(Note that the coefficients in this expansion depend on the dimension D, unlike the scalar case, given in (3.17) of [8].) Because of this behavior, it is apparent that the series diverges for all positive D, except for $D = 1$, where the series truncates.

There were actually two procedures which were used to turn the corresponding sum in the scalar case into a convergent series, and to extract numerical results, although only one of those procedures was described in the paper [8]. In that procedure, we subtract from the summand the leading terms in the $1/n$ expansion, derived from (4.3), identifying those summed subtractions with Riemann zeta functions:

$$\begin{aligned} F^{\mathrm{TM}} \approx \frac{1}{2\pi a}\Bigg\{ Q_0 &+ \frac{1}{\Gamma(D-1)}\sum_{n=1}^{N}\left[w(n,D)Q_n - \pi n^{D-1}\left(1+\sum_{k=1}^{K}\frac{b_k}{n^k}\right)\right] \\ &+ \frac{\pi}{\Gamma(D-1)}\left[\zeta(1-D)+\sum_{k=1}^{K+1}b_k\zeta(k+1-D) - b_{K+1}\sum_{n=1}^{N} n^{D-K-2}\right]\Bigg\}. \end{aligned} \tag{4.4}$$

Here b_k are the coefficients in the asymptotic expansion of the summand in (3.8), of which the first two are

$$b_1 = \frac{(D-2)(D-1)}{2}, \tag{4.5a}$$

$$b_2 = \frac{81 - 448D + 456D^2 - 176D^3 + 24D^4}{192}. \tag{4.5b}$$

In (4.4) we keep K terms in the asymptotic expansion, and, after N terms in the sum, we approximate the subtracted integrand by the next term in the large n expansion. The series converges for $D < K + 1$, so more and more terms in the asymptotic expansion are required as D increases.

There is a second method which gives identical results, and is, in fact, more convergent. The results given in [8] were, in fact, first computed by this procedure, which is based on analytic continuation in dimension. Here, we simply subtract from Q_n the first two terms in the asymptotic expansion (4.3), and then argue, as a generalization of (4.1), that

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+D-2)}{n!} = 0 \quad \text{for} \quad D < 2, \qquad \sum_{n=0}^{\infty}\frac{\Gamma(n+D-2)}{n!}\nu^2 = 0 \quad \text{for} \quad D < 0. \tag{4.6}$$

Therefore, by continuing from negative dimension, we argue that we can make the subtraction without introducing any additional terms. Thus, if we define

$$\hat{Q}_n = Q_n - \frac{\pi\nu}{2}\left(1 + \frac{-101 + 80D - 16D^2}{64\nu^2}\right), \tag{4.7}$$

we have

$$\begin{aligned} F^{\mathrm{TM}} &= \frac{1}{2\pi a^2 \Gamma(D-1)} \sum_{n=0}^{\infty} w(n,D)\hat{Q}_n \\ &\approx \frac{1}{2\pi a^2 \Gamma(D-1)} \left(\sum_{n=0}^{N} w(n,D)\hat{Q}_n + \pi g(D) \sum_{n=N+1}^{\infty} \frac{\Gamma(n+D-2)}{n!\,\nu^2} \right), \end{aligned} \tag{4.8}$$

where $g(D)$ is the coefficient of ν^{-4} in (4.3). The last sum in (4.8) can be evaluated according to

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\,(n+\alpha/2)^2} = \frac{\pi^2}{2} \frac{\Gamma(\alpha/2)}{\Gamma(1-\alpha/2)} \frac{1}{\sin^2 \pi\alpha/2}. \tag{4.9}$$

The approximation given in (4.8) converges for $D < 4$.

B. Casimir stress for integer $D \leq 1$

The case of integers ≤ 1 is of special note, because, for those cases, the series truncates. For example, for $D = 0$ only the $n = 0, 2$ terms appear, where the integrals cancel by virtue of the symmetry of Bessel functions,

$$K_\nu(x) = K_{-\nu}(x), \quad I_n(x) = I_{-n}(x), \tag{4.10}$$

for n an integer. However, using the first procedure (4.4), we have a residual zeta function contribution:

$$F^{\mathrm{TM}}_{D=0} = \frac{1}{2\pi a^2}(Q_0 - Q_2 + \pi) = \frac{1}{2a^2}, \tag{4.11}$$

because both $\zeta(1-D)$ and $\Gamma(D-1)$ have simple poles, with residue -1, at $D = 0$. This result for $D = 0$ is the negative of the result found in the scalar case, (3.22) of [8], which is a direct consequence of the fact that the n^{D-2} term in the asymptotic expansion cancels when the TE and TM modes are combined [compare (4.5a) with the corresponding term in (3.23) of [8].] The continuation in D method gives the same result, because then

$$F^{\rm TM}_{D=0} = \frac{1}{2\pi a^2}(\hat{Q}_0 + D\hat{Q}_1 - \hat{Q}_2) = \frac{1}{2a^2}\left(1 - \frac{101}{64} + \frac{101}{64}\right) = \frac{1}{2a^2}, \tag{4.12}$$

where a limiting procedure, $D \to 0$, is employed to deal with the singularity which occurs for $n = 1$, where $\nu \to 0$.

For the negative even integers we achieve a similar cancellation between pairs of integers, with no zeta function residual because the ζ functions no longer have poles there. For example, for $D = -2$ we have

$$F^{\rm TM}_{D=-2} = \frac{1}{2\pi a^2}(Q_0 - 2Q_1 + 2Q_3 - Q_4) = 0 \tag{4.13}$$

because $Q_0 = Q_4$ and $Q_1 = Q_3$. Again, the other method of regularization gives the same result when a careful limit is taken.

For odd integer ≤ 1, trucation occurs without cancellation, because $I_\nu \neq I_{-\nu}$. For example, for $D = 1$,

$$F^{\rm TM}_{D=1} = \frac{1}{2\pi}(Q_0 + Q_1) = -0.2621 + 0.6032i. \tag{4.14}$$

C. Numerical results

We have used both methods described above to extract numerical results for the stress on a sphere due to TM fluctuations in the interior and exterior. Results are plotted in Fig. 1. Salient features are the following:

- As in the scalar case, poles occur for positive even dimension.
- The integrals become complex for $D < 2$ because the function $q(x)$, (3.10), occurring in the logarithm develops zeros. (This phenomenon started at $D = 0$ for the scalar case.) Correspondingly, there are logarithmic singularities, and cusps, occurring at 2, 1, 0, -1, -2, ..., rather than just at the nonpositive even integers.
- The sign of the Casimir force changes dramatically with dimension. Here this is even more striking than in the scalar case, where the sign was constant between the poles

for $D > 0$. For the TM modes, the Casimir force vanishes for $D = 2.60$, being repulsive for $2 < D < 2.60$ and attractive for $2.60 < D < 4$.

- Also in Fig. 1, the results found here are compared with those found in the scalar or TE case, [8]. The correspondence is quite remarkable. In particular, for $D < 2$ the qualitative structure of the curves are very similar when the scale of the dimensions in the TE case is reduced by a factor of 2; that is, the interval $0 < D < 2$ in the TE case corresponds to the interval $1 < D < 2$ in the TM, $-2 < D < 0$ for TE corresponds to $0 < D < 1$ for TM, etc.

- Physically, the most interesting result is at $D = 3$. The TM mode calculated here has the value $F^{\rm TM}_{D=3} = -0.02204$. However, if we wish to compare this to the electrodynamic result [12], we must subtract off the $n = 0$ mode, which is given in terms of the integral $Q_0 = 0.411233 = \pi^2/24$, which displays the accuracy of our numerical integration. Similarly removing the $n = 0$ mode ($= -\pi/24$) from the result quoted in (3.24) of [8], $F^{\rm TM}_{D=3} = 0.0028168$, gives agreement with the familiar result [10,13,14,12,9]:

$$F^{\rm TM+TE}_{n>0}\Big|_{D=3} = \frac{0.0462}{a^2}, \tag{4.15}$$

To conclude, this paper adds one more example to our collection of known results concerning the dimensional and boundary dependence of the Casimir effect. Unfortunately, we are no closer to understanding intuitively the sign of the phenomenon.

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APPENDIX: TOWARD A FINITE $D = 2$ CASIMIR EFFECT

The truly disturbing aspect of our results here and in [8] are the pole in even dimensions. In particular many very interesting condensed matter systems are well-approximated by being two dimensional. Are we to conclude that the Casimir effect does not exist in two dimensions?

One trivial way to extract a finite answer from our expressions, which have simple poles at $D = 2$ (I will set aside the logarithmic singularity there in the TM mode, because that only occurs in one integral, Q_0), is to average over the singularity. If we do so for the scalar result in [8], we obtain

$$F^{\rm TE}_{D=2} = -\frac{0.01304}{a^2}, \tag{A1}$$

while for the TM result here, we find

$$F^{\rm TM}_{D=2} = -\frac{0.340}{a^2}, \tag{A2}$$

which numbers, incidentally, are remarkably close to the leading Q_0 term, as stated[3] in [15], which are -0.0140 and -0.254. But, there seems to be no reason to have any belief in these numbers.

However, something remarkable does happen in the scalar case. If we use the first procedure, (4.4), we note that the poles can arise both from the integrals and from the explicit zeta functions. For the latter, let the dependence on D be given by $r(D)/(D-2)$ which has a pole at $D = 2$. When we average over the pole, we obtain

$$\lim_{\epsilon\to 0}\frac{1}{2}\left(\frac{r(2+\epsilon)}{\epsilon} - \frac{r(2-\epsilon)}{\epsilon}\right) = r'(2), \tag{A3}$$

where the prime denotes differentiation. For the scalar modes it is easy to verify that $r'(2) = 0$. Thus, there is no contribution from those subtracted terms. In other words, they

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[3]The integral in (A12) of [15] was not evaluated very accurately there. The value, good to 6 figures, should be in our notation, $-Q_0^{\rm scalar} = 0.0880137$. Similarly $-Q_0^{\rm TM} = 1.5929$.

might just as well be omitted, which is what we would do if we inserted a cutoff and simply dropped the divergent terms. (This procedure does give the correct $D = 3$ results.) This provides some evidence for the validity of the procedure which yields (A1).

Unfortunately, the same effect does not occur for the TM modes, $r'(2) \neq 0$. Nor does it occur for higher dimensions, $D = 4, 6, \ldots$, even for scalars. And, even for scalars, it is not clear how the divergences of a massive (2+1) theory can be removed. So we are no closer to solving the divergence problem in even dimensions.[4] It is clear there is much more work to do on Casimir phenomena.

[4] For a discussion of the inadequacies of the dubious procedure of attempting to extract a finite result in [15] see [9].

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[19] See Chap. 9 of [18].

FIGURES

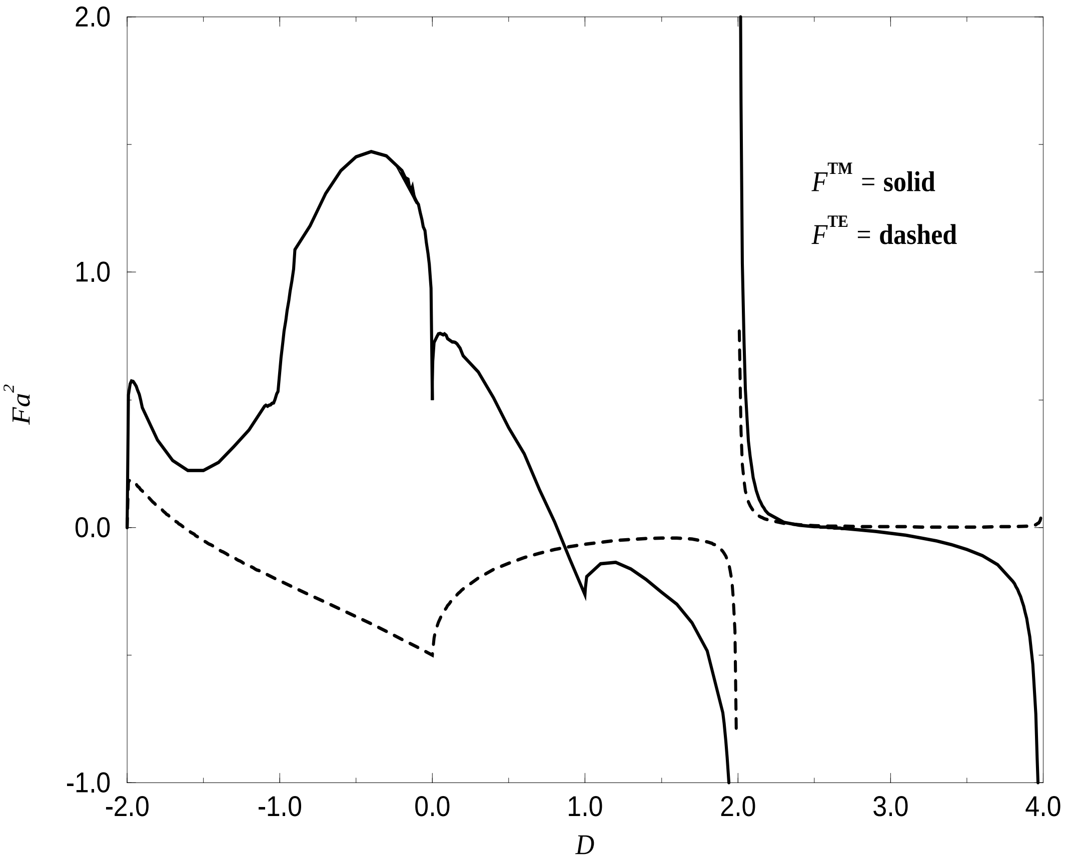

FIG. 1. A plot of the TM Casimir stress $F^{\rm TM}$ for $-2 < D < 4$ on a spherical shell, compared with $F^{\rm TE}$, taken from [8]. For $D < 2$ ($D < 0$) the stress $F^{\rm TM}$ ($F^{\rm TE}$) is complex and we have plotted $\mathrm{Re}\, F$.