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Casimir Energy for a Spherical Cavity in a Dielectric: Applications to Sonoluminescence

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Abstract

In the final few years of his life, Julian Schwinger proposed that the "dynamical Casimir effect" might provide the driving force behind the puzzling phenomenon of sonoluminescence. Motivated by that exciting suggestion, we have computed the static Casimir energy of a spherical cavity in an otherwise uniform material. As expected the result is divergent; yet a plausible finite answer is extracted, in the leading uniform asymptotic approximation. This result agrees with that found using ζ-function regularization. Numerically, we find far too small an energy to account for the large burst of photons seen in sonoluminescence. If the divergent result is retained, it is of the wrong sign to drive the effect. Dispersion does not resolve this contradiction. In the static approximation, the Fresnel drag term is zero; on the other hand,

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electrostriction could be comparable to the Casimir term. It is argued that this adiabatic approximation to the dynamical Casimir effect should be quite accurate.

I. INTRODUCTION

In a series of papers in the last three years of his life, Julian Schwinger proposed [1] that the dynamical Casimir effect could provide the energy that drives the copious production of photons in the puzzling phenomenon of sonoluminescence [2–4]. In fact, however, he guessed an approximate (static) formula for the Casimir energy of a spherical bubble in water, based on a general, but incomplete, analysis [5]. He apparently was unaware that one of us had, in the late 1970's, completed the analysis of the Casimir force for a dielectric ball [6]. It is our purpose here to carry out the very straightforward calculation for the complementary situation, for a cavity in an infinite dielectric medium.[1] In fact, we will consider the general case of spherical region, of radius a, having permittivity ϵ' and permeability μ', surrounded by an infinite medium of permittivity ϵ and permeability μ.

Of course, this calculation is not directly relevant to sonoluminescence, which is anything but static. It is offered as only a preliminary step, but it should give an idea of the orders of magnitude of the energies involved. It is a significant improvement over the crude estimation used in [1]. Attempts at dynamical calculations exist [8–10]; but they are subject to possibly serious methodological objections, some of which will be discussed below. (Other theoretical models to explain sonoluminescence are given in [11].) In fact, we anticipate that because the relevant scale of the electromagnetic Casimir effect is in the optical region, with characteristic time scale $t \sim 10^{-15}$s, and the scale of the bubble collapse is of order $\tau \sim 10^{-6}$s (more relevant may be the duration of each flash, which is $\lesssim 10^{-11}$s), the adiabatic approximation of treating the bubble as static for calculating the Casimir energy should be very

[1] A preliminary version of this paper appeared in [7].

accurate. Sonoluminescence aside, this calculation is of interest for its own sake, as one of a relatively few nontrivial Casimir calculations with nonplanar boundaries [12–19]. It represents a significant generalization on the calculation of Brevik and Kolbenstvedt [20], who consider the same geometry with $\mu\epsilon = \mu'\epsilon' = 1$, a special case, possibly relevant to hadronic physics, in which the result is unambiguously finite. It is, as noted above, a straightforward generalization of the result in [6]; the most significant technical improvement is that here the energy is calculated directly. We also examine Fresnel drag and electrostriction; the latter may be numerically significant.

In the next section we review the Green's dyadic formalism we shall employ, and compute the Green's functions in this case for the TE and TM modes. Then, in Section 3, we compute the force on the cavity from the discontinuity of the stress tensor. The energy is computed similarly in Section 4, and the expected relation between stress and energy is found. Fresnel drag, in the static approximation, is considered in Section 5, and electrostriction in Section 6. Estimates in Section 7 show that the Casimir energy so constructed, even with physically required subtractions, and including both interior and exterior contributions, is divergent, but that if one supplies a plausible contact term, a finite result (at least in leading approximation) follows. This finite result agrees with that found using ζ-function regularization. (Physically, we expect that the divergence is regulated by including dispersion.) Numerical estimates of both the divergent and finite terms are given in the conclusion, and comparison is made with the calculations of Schwinger and others. A simple estimate is given which suggests that any macroscopic electromagnetic phenomenon such as the Casimir effect cannot possibly supply the energy required for sonoluminescence. There, and in the Appendix, where we discuss the form of the force on the surface due to the fluctuating electric and magnetic fields, a comparison with [9] is made.

II. GREEN'S DYADIC FORMULATION

We follow closely the formulation given in [14,6]. We start with Maxwell's equations in rationalized units, with a polarization source $\mathbf{P}$: (in the following we set $c = \hbar = 1$)

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{H} &= \frac{\partial}{\partial t}\mathbf{D} + \frac{\partial}{\partial t}\mathbf{P}, \qquad \boldsymbol{\nabla} \cdot \mathbf{D} = -\boldsymbol{\nabla} \cdot \mathbf{P}, \\ -\boldsymbol{\nabla} \times \mathbf{E} &= \frac{\partial}{\partial t}\mathbf{B}, \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0, \end{aligned} \tag{2.1}$$

where, for an homogeneous, isotropic, nondispersive medium

$$\mathbf{D} = \epsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H}. \tag{2.2}$$

We define a Green's dyadic $\boldsymbol{\Gamma}$ by

$$\mathbf{E}(\mathbf{r}, t) = \int (d\mathbf{r}')\, dt'\, \boldsymbol{\Gamma}(\mathbf{r}, t; \mathbf{r}', t') \cdot \mathbf{P}(\mathbf{r}', t') \tag{2.3}$$

and introduce a Fourier transform in time

$$\boldsymbol{\Gamma}(\mathbf{r}, t; \mathbf{r}', t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \boldsymbol{\Gamma}(\mathbf{r}, \mathbf{r}'; \omega), \tag{2.4}$$

where in the following the ω argument will be suppressed. Maxwell's equations then become (which define $\boldsymbol{\Phi}$)

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{\Gamma} &= i\omega \boldsymbol{\Phi}, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{\Phi} = 0, \\ \frac{1}{\mu} \boldsymbol{\nabla} \times \boldsymbol{\Phi} &= -i\omega\epsilon \boldsymbol{\Gamma}', \qquad \boldsymbol{\nabla} \cdot \boldsymbol{\Gamma}' = 0, \end{aligned} \tag{2.5}$$

in which $\boldsymbol{\Gamma}' = \boldsymbol{\Gamma} + \mathbf{1}/\epsilon$, where $\mathbf{1}$ includes a spatial delta function. The two solenoidal Green's dyadics given here satisfy the following second-order equations:

$$(\nabla^2 + \omega^2 \epsilon\mu)\boldsymbol{\Gamma}' = -\frac{1}{\epsilon} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{1}), \tag{2.6a}$$

$$(\nabla^2 + \omega^2 \epsilon\mu)\boldsymbol{\Phi} = i\omega\mu \boldsymbol{\nabla} \times \mathbf{1}. \tag{2.6b}$$

They can be expanded in vector spherical harmonics [21,22] defined by

$$\mathbf{X}_{lm} = \frac{1}{\sqrt{l(l+1)}} \mathbf{L} Y_{lm}, \tag{2.7}$$

as follows:

$$\mathbf{\Gamma}'(\mathbf{r},\mathbf{r}') = \sum_{lm}\left(f_l(r,\mathbf{r}')\mathbf{X}_{lm}(\Omega) + \frac{i}{\omega\epsilon\mu}\boldsymbol{\nabla}\times g_l(r,\mathbf{r}')\mathbf{X}_{lm}(\Omega)\right), \tag{2.8a}$$

$$\mathbf{\Phi}(\mathbf{r},\mathbf{r}') = \sum_{lm}\left(\tilde{g}_l(r,\mathbf{r}')\mathbf{X}_{lm}(\Omega) - \frac{i}{\omega}\boldsymbol{\nabla}\times \tilde{f}_l(r,\mathbf{r}')\mathbf{X}_{lm}(\Omega)\right). \tag{2.8b}$$

When these are substituted in Maxwell's equations (2.5) we obtain, first,

$$g_l = \tilde{g}_l, \qquad f_l = \tilde{f}_l + \frac{1}{\epsilon}\frac{1}{r^2}\delta(r-r')\mathbf{X}^*_{lm}(\Omega'), \tag{2.9}$$

and then the second-order equations

$$(D_l + \omega^2\mu\epsilon)g_l(r,\mathbf{r}') = i\omega\mu\int d\Omega''\,\mathbf{X}^*_{lm}(\Omega'')\cdot\boldsymbol{\nabla}''\times\mathbf{1}, \tag{2.10a}$$

$$\begin{aligned}(D_l + \omega^2\mu\epsilon)f_l(r,\mathbf{r}') &= -\frac{1}{\epsilon}\int d\Omega''\,\mathbf{X}^*_{lm}(\Omega'')\cdot\boldsymbol{\nabla}''\times(\boldsymbol{\nabla}''\times\mathbf{1})\\ &= \frac{1}{\epsilon}D_l\frac{1}{r^2}\delta(r-r')\mathbf{X}^*_{lm}(\Omega'),\end{aligned} \tag{2.10b}$$

where

$$D_l = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}. \tag{2.11}$$

These equations can be solved in terms of Green's functions satisfying

$$(D_l + \omega^2\epsilon\mu)F_l(r,r') = -\frac{1}{r^2}\delta(r-r'), \tag{2.12}$$

which have the form

$$F_l(r,r') = \begin{cases} ik'j_l(k'r_<)[h_l(k'r_>) - Aj_l(k'r_>)], & r,r' < a,\\ ikh_l(kr_>)[j_l(kr_<) - Bh_l(kr_<)], & r,r' > a,\end{cases} \tag{2.13}$$

where

$$k = |\omega|\sqrt{\mu\epsilon}, \qquad k' = |\omega|\sqrt{\mu'\epsilon'}, \tag{2.14}$$

and $h_l = h_l^{(1)}$ is the spherical Hankel function of the first kind. Specifically, we have

$$\tilde{f}_l(r,\mathbf{r}') = \omega^2\mu F_l(r,r')\mathbf{X}^*_{lm}(\Omega'), \tag{2.15a}$$

$$g_l(r,\mathbf{r}') = -i\omega\mu\boldsymbol{\nabla}'\times G_l(r,r')\mathbf{X}^*_{lm}(\Omega'), \tag{2.15b}$$

where F_l and G_l are Green's functions of the form (2.13) with the constants A and B determined by the boundary conditions given below. Given F_l, G_l, the fundamental Green's dyadic is given by

$$\begin{aligned}\boldsymbol{\Gamma}'(\mathbf{r},\mathbf{r}') = \sum_{lm}\Big\{&\omega^2\mu F_l(r,r')\mathbf{X}_{lm}(\Omega)\mathbf{X}^*_{lm}(\Omega')\\ &-\frac{1}{\epsilon}\boldsymbol{\nabla}\times G_l(r,r')\mathbf{X}_{lm}(\Omega)\mathbf{X}^*_{lm}(\Omega')\times\overleftarrow{\boldsymbol{\nabla}}'\\ &+\frac{1}{\epsilon}\frac{1}{r^2}\delta(r-r')\mathbf{X}_{lm}(\Omega)\mathbf{X}^*_{lm}(\Omega')\Big\}.\end{aligned} \tag{2.16}$$

Now we consider a sphere of radius a centered at the origin, with properties ϵ', μ' in the interior and ϵ, μ outside. Because of the boundary conditions that

$$\mathbf{E}_\perp, \quad \epsilon E_r, \quad B_r, \quad \frac{1}{\mu}\mathbf{B}_\perp \tag{2.17}$$

be continuous at $r=a$, we find for the constants A and B in the two Green's functions in (2.16)

$$A_F = \frac{\sqrt{\epsilon\mu'}\tilde{e}_l(x')\tilde{e}'_l(x) - \sqrt{\epsilon'\mu}\tilde{e}_l(x)\tilde{e}'_l(x')}{\Delta_l}, \tag{2.18a}$$

$$B_F = \frac{\sqrt{\epsilon\mu'}\tilde{s}_l(x')\tilde{s}'_l(x) - \sqrt{\epsilon'\mu}\tilde{s}_l(x)\tilde{s}'_l(x')}{\Delta_l}, \tag{2.18b}$$

$$A_G = \frac{\sqrt{\epsilon'\mu}\tilde{e}_l(x')\tilde{e}'_l(x) - \sqrt{\epsilon\mu'}\tilde{e}_l(x)\tilde{e}'_l(x')}{\tilde{\Delta}_l}, \tag{2.18c}$$

$$B_G = \frac{\sqrt{\epsilon'\mu}\tilde{s}_l(x')\tilde{s}'_l(x) - \sqrt{\epsilon\mu'}\tilde{s}_l(x)\tilde{s}'_l(x')}{\tilde{\Delta}_l}. \tag{2.18d}$$

Here we have introduced $x=ka$, $x'=k'a$, the Riccati-Bessel functions

$$\tilde{e}_l(x) = xh_l(x), \qquad \tilde{s}_l(x) = xj_l(x), \tag{2.19}$$

and the denominators

$$\begin{aligned}\Delta_l &= \sqrt{\epsilon\mu'}\tilde{s}_l(x')\tilde{e}'_l(x) - \sqrt{\epsilon'\mu}\tilde{s}'_l(x')\tilde{e}_l(x),\\ \tilde{\Delta}_l &= \sqrt{\epsilon'\mu}\tilde{s}_l(x')\tilde{e}'_l(x) - \sqrt{\epsilon\mu'}\tilde{s}'_l(x')\tilde{e}_l(x),\end{aligned} \tag{2.20}$$

and have denoted differentiation with respect to the argument by a prime.

III. STRESS ON THE SPHERE

We can calculate the stress (force per unit area) on the sphere by computing the discontinuity of the radial-radial component of the stress tensor (see the Appendix):

$$\mathcal{F} = T_{rr}(a-) - T_{rr}(a+), \tag{3.1}$$

where

$$T_{rr} = \frac{1}{2}\langle[\epsilon(E_\perp^2 - E_r^2) + \mu(H_\perp^2 - H_r^2)]\rangle. \tag{3.2}$$

The vacuum expectation values of the product of field strengths are given directly by the Green's dyadics computed in Section 2:

$$i\langle \mathbf{E}(\mathbf{r})\mathbf{E}(\mathbf{r}')\rangle = \boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}'), \tag{3.3a}$$

$$i\langle \mathbf{B}(\mathbf{r})\mathbf{B}(\mathbf{r}')\rangle = -\frac{1}{\omega^2}\boldsymbol{\nabla}\times\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}')\times\overleftarrow{\boldsymbol{\nabla}}', \tag{3.3b}$$

where here and in the following we ignore δ functions because we are interested in the *limit* as $\mathbf{r}'\to\mathbf{r}$. It is then rather immediate to find for the stress on the sphere (the *limit* $t'\to t$ is assumed)

$$\begin{aligned}\mathcal{F} = {}& \frac{1}{2ia^2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\sum_{l=1}^{\infty}\frac{2l+1}{4\pi} \\ & \times\Bigg\{(\epsilon'-\epsilon)\left[\frac{k^2}{\epsilon}a^2F_l(a+,a+) + \left(\frac{l(l+1)}{\epsilon'} + \frac{1}{\epsilon}\frac{\partial}{\partial r}r\frac{\partial}{\partial r'}r'\right)G_l(r,r')\Big|_{r=r'=a+}\right] \\ & + (\mu'-\mu)\left[\frac{k^2}{\mu}a^2G_l(a+,a+) + \left(\frac{l(l+1)}{\mu'} + \frac{1}{\mu}\frac{\partial}{\partial r}r\frac{\partial}{\partial r'}r'\right)F_l(r,r')\Big|_{r=r'=a+}\right]\Bigg\} \qquad (3.4\text{a}) \\ = {}& \frac{i}{2a^4}\int_{-\infty}^{\infty}\frac{dy}{2\pi}e^{-iy\delta}\sum_{l=1}^{\infty}\frac{2l+1}{4\pi}x\frac{d}{dx}\ln\Delta_l\tilde{\Delta}_l, \qquad (3.4\text{b})\end{aligned}$$

where $y=\omega a$, $\delta=(t-t')/a$, and

$$\ln\Delta_l\tilde{\Delta}_l = \ln\left[(\tilde{s}_l(x')\tilde{e}_l'(x) - \tilde{s}_l'(x')\tilde{e}_l(x))^2 - \xi^2(\tilde{s}_l(x')\tilde{e}_l'(x) + \tilde{s}_l'(x')\tilde{e}_l(x))^2\right] + \text{constant}. \tag{3.5}$$

Here the parameter ξ is

$$\xi = \frac{\sqrt{\frac{\epsilon'}{\epsilon}\frac{\mu}{\mu'}}-1}{\sqrt{\frac{\epsilon'}{\epsilon}\frac{\mu}{\mu'}}+1}. \tag{3.6}$$

This is not yet the answer. We must remove the term which would be present if either medium filled all space (the same was done in the case of parallel dielectrics [23]). The corresponding Green's function is

$$F_l^{(0)} = \begin{cases} ik' j_l(k' r_<) h_l(k' r_>), & r, r' < a \\ ik j_l(k r_<) h_l(k r_>), & r, r' > a \end{cases} \tag{3.7}$$

The resulting stress is

$$\mathcal{F}^{(0)} = \frac{1}{a^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} \sum_{l=1}^{\infty} \frac{2l+1}{4\pi} \Big\{ x' [\tilde{s}_l'(x') \tilde{e}_l'(x') - \tilde{e}_l(x') \tilde{s}_l''(x')]$$
$$- x [\tilde{s}_l'(x) \tilde{e}_l'(x) - \tilde{e}_l(x) \tilde{s}_l''(x)] \Big\}. \tag{3.8}$$

The final formula for the stress is obtained by subtracting (3.8) from (3.4b):

$$\mathcal{F} = -\frac{1}{2a^4} \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{iy\delta} \sum_{l=1}^{\infty} \frac{2l+1}{4\pi} \Big\{ x \frac{d}{dx} \ln \Delta_l \tilde{\Delta}_l$$
$$+ 2x' [s_l'(x') e_l'(x') - e_l(x') s_l''(x')] - 2x [s_l'(x) e_l'(x) - e_l(x) s_l''(x)] \Big\}, \tag{3.9}$$

where we have now performed a Euclidean rotation,

$$y \to iy, \quad x \to ix, \quad \tau = t - t' \to i(x_4 - x_4') \quad [\delta = (x_4 - x_4')/a],$$
$$\tilde{s}_l(x) \to s_l(x) = \sqrt{\frac{\pi x}{2}} I_{l+1/2}(x), \quad \tilde{e}_l(x) \to e_l(x) = \frac{2}{\pi} \sqrt{\frac{\pi x}{2}} K_{l+1/2}(x). \tag{3.10}$$

IV. TOTAL ENERGY

In a similar way we can directly calculate the Casimir energy of the configuration, starting from the energy density

$$U = \frac{\epsilon E^2 + \mu H^2}{2}. \tag{4.1}$$

In terms of the Green's dyadic, the total energy is

$$E = \int (d\mathbf{r})\, U$$

$$= \frac{1}{2i}\int r^2 dr\, d\Omega \left[\epsilon \mathrm{Tr}\, \mathbf{\Gamma}(\mathbf{r},\mathbf{r}) - \frac{1}{\omega^2\mu}\mathrm{Tr}\, \mathbf{\nabla}\times\mathbf{\Gamma}(\mathbf{r},\mathbf{r})\times \overleftarrow{\mathbf{\nabla}}\right] \tag{4.2a}$$

$$= \frac{1}{2i}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\sum_{l=1}^{\infty}(2l+1)\int_0^\infty r^2\, dr$$
$$\times\left\{2k^2[F_l(r,r)+G_l(r,r)] + \frac{1}{r^2}\frac{\partial}{\partial r}r\frac{\partial}{\partial r'}r'[F_l+G_l](r,r')|_{r'=r}\right\}, \tag{4.2b}$$

where there is no explicit appearance of ϵ or μ. (However, the value of k depends on which medium we are in.) As in [14] we can easily show that the total derivative term integrates to zero. We are left with

$$E = \frac{1}{2i}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}\sum_{l=1}^{\infty}(2l+1)\int_0^\infty r^2\, dr\, 2k^2[F_l(r,r)+G_l(r,r)]. \tag{4.3}$$

However, again we should subtract off that contribution which the formalism would give if either medium filled all space. That means we should replace F_l and G_l by

$$\tilde{F}_l, \tilde{G}_l = \begin{cases} -ik'A_{F,G}j_l(k'r)j_l(k'r'), & r,r'<a \\ -ikB_{F,G}h_l(kr)h_l(kr'), & r,r'>a \end{cases} \tag{4.4}$$

so then (4.3) says

$$E = -\sum_{l=1}^{\infty}(2l+1)\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}\left\{\int_0^a r^2 dr\, k'^3(A_F+A_G)j_l^2(k'r)\right.$$
$$\left. + \int_a^\infty r^2 dr\, k^3(B_F+B_G)h_l^2(kr)\right\}. \tag{4.5}$$

The radial integrals may be done by using the following indefinite integral for any spherical Bessel function j_l:

$$\int dx\, x^2 j_l^2(x) = \frac{x}{2}[((xj_l)')^2 - j_l(xj_l)' - xj_l(xj_l)'']. \tag{4.6}$$

But we must remember to add the contribution of the total derivative term in (4.2b) which no longer vanishes when the replacement (4.4) is made. The result is precisely that expected from the stress (3.9),

$$E = 4\pi a^3\mathcal{F}, \qquad \mathcal{F} = \frac{1}{4\pi a^2}\left(-\frac{\partial}{\partial a}\right)E, \tag{4.7}$$

where the derivative is the naive one, that is, the cutoff δ has no effect on the derivative.

It is useful here to make contact with the formalism introduced by Schwinger [5]. In terms of an imaginary frequency ζ and a parameter w, he derives the following simple formula for the energy from the proper-time formalism

$$E = -\frac{1}{2\pi}\int_0^\infty d\zeta \int_0^\infty dw\, \mathrm{Tr}_s G, \tag{4.8}$$

where the trace refers to space, the Green's function is

$$G = \frac{1}{w+H} \tag{4.9}$$

and the Hamiltonian appropriate to the two modes is (for a nonmagnetic material)

$$H = \begin{cases} \mathrm{TE}: & \partial_0\epsilon\partial_0 - \nabla^2 \\ \mathrm{TM}: & \partial_0^2 - \boldsymbol{\nabla}\cdot(1/\epsilon)\boldsymbol{\nabla} \end{cases} \tag{4.10}$$

Consider the TE part (the TM part is similar, but not explicitly considered by Schwinger). In terms of Green's function satisfying (2.12), we have

$$E = \frac{1}{2\pi}\int_0^\infty d\zeta \int_0^\infty dw \sum_{l=1}^\infty (2l+1)\int_0^\infty dr\, r^2 F_l(r,r;\zeta^2\epsilon + w), \tag{4.11}$$

where the third argument of the Green's function reflects the substitution in (2.12) of $\omega^2\epsilon \to -\zeta^2\epsilon - w$. We now introduce polar coordinates by writing

$$\zeta^2\epsilon + w = \rho^2, \quad d\zeta\, dw = \frac{1}{\sqrt{\epsilon}}2\rho^2\cos\theta\, d\rho\, d\theta, \tag{4.12}$$

and integrate over θ from 0 to 2π. The result coincides with the first term in (4.3).

V. FRESNEL DRAG

As may easily inferred from Pauli's book [24], the nonrelativistic effect of material motion of the dielectric, $\boldsymbol{\beta}(\mathbf{r})$, is given by the so-called Fresnel drag term,

$$E' = \int (d\mathbf{r})\frac{\epsilon\mu - 1}{\epsilon}\boldsymbol{\beta}\cdot(\mathbf{D}\times\mathbf{H}) = \int(d\mathbf{r})(\epsilon\mu - 1)\boldsymbol{\beta}\cdot(\mathbf{E}\times\mathbf{H}). \tag{5.1}$$

To preserve spherical symmetry (of course, this is likely not to be realistic) we consider purely radial velocities,

$$\boldsymbol{\beta} = \beta\hat{\mathbf{r}}. \tag{5.2}$$

Then, what we seek is the asymmetrical structure

$$\begin{aligned}\hat{\mathbf{r}}\cdot\langle\mathbf{E}(\mathbf{r})\times\mathbf{H}(\mathbf{r}')\rangle &= -\hat{\mathbf{r}}\cdot\langle\mathbf{H}(\mathbf{r}')\times\mathbf{E}(\mathbf{r})\rangle = -\frac{1}{i\mu}\epsilon_{ijk}\hat{r}_i\cdot\Phi_{jk}(\mathbf{r}',\mathbf{r})\\ &= \omega\hat{\mathbf{r}}\cdot\sum_{lm}\Big\{\mathbf{X}_{lm}(\Omega')\times[\boldsymbol{\nabla}\times G_l(r',r)\mathbf{X}^*_{lm}(\Omega)]\\ &\quad+[\boldsymbol{\nabla}'\times F_l(r',r)\mathbf{X}_{lm}(\Omega')]\times\mathbf{X}^*_{lm}(\Omega)\Big\}.\end{aligned} \tag{5.3}$$

This is easily seen to reduce to

$$\begin{aligned}\hat{\mathbf{r}}\cdot\langle\mathbf{E}\times\mathbf{H}\rangle &= \omega\frac{1}{r}\frac{\partial}{\partial r}r\sum_{lm}G_l(r',r)\mathbf{X}_{lm}(\Omega')\cdot\mathbf{X}^*_{lm}(\Omega)\\ &\quad-\omega\frac{1}{r'}\frac{\partial}{\partial r'}r'\sum_{lm}F_l(r',r)\mathbf{X}_{lm}(\Omega')\cdot\mathbf{X}^*_{lm}(\Omega),\end{aligned} \tag{5.4}$$

so when Ω and Ω' are identified, and the angular integral is carried out, we obtain the corresponding energy for a *slow, adiabatic, radially symmetric motion*,

$$E' = \beta\int_0^\infty r^2dr\,(\epsilon\mu-1)\int_{-\infty}^\infty\frac{d\omega}{2\pi}\omega\,e^{-i\omega\tau}\sum_{l=1}^\infty(2l+1)\left\{\frac{1}{r}\frac{\partial}{\partial r}rG_l(r',r)-\frac{1}{r'}\frac{\partial}{\partial r'}r'F_l(r',r)\right\}\bigg|_{r'=r}. \tag{5.5}$$

It is clear, immediately, that if the cutoff τ is set equal to zero, this vanishes because the integrand is odd in ω; compare to (4.3). Since the sign of τ is certainly irrelevant, we therefore claim that in this quasistatic approximation Fresnel drag is absent.

Related is the Abraham value of the field momentum [25,26],

$$\mathbf{G} = \mathbf{E}\times\mathbf{H}, \tag{5.6}$$

which gives then an extra contribution to the force density,

$$\mathbf{f}' = (\epsilon-1)\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{H}). \tag{5.7}$$

However, as Brevik noted [26], this is also zero, because, in the Fourier transform, successive action of the time derivative brings down ω and $-\omega$. So the continuing controversy about which field momentum to use is without consequence here. This should be already obvious, because the energy is well-defined, and we have already seen that the force is related to the energy by (4.7). See also the Appendix.

VI. ELECTROSTRICTION

When a dielectric medium is deformed, there is an additional contribution to the force density, that of electrostriction [22,25],

$$\mathbf{f}_{\mathrm{ES}} = \frac{1}{2}\boldsymbol{\nabla}\left(E^2\rho\frac{\partial\epsilon}{\partial\rho}\right), \tag{6.1}$$

where ρ is the density of the medium. This term is without effect for computation of the *force* on the dielectric, because it is a total derivative, yet here, where we are calculating the *stress* on the surface, it can be significant. The simplest model for describing the density dependence of the dielectric constant is that given by the Clausius-Mossotti equation,

$$\frac{\epsilon-1}{\epsilon+2} = K\rho, \tag{6.2}$$

where K is a constant. Consequently the logarithmic derivative appearing in (6.1) is

$$\rho\frac{\partial\epsilon}{\partial\rho} = \frac{1}{3}(\epsilon-1)(\epsilon+2). \tag{6.3}$$

The calculation of the electrostrictive Casimir effect for a dielectric ball is given by Brevik [26]. We have confirmed his result, and generalized it to the situation at hand. Again, the contribution if either medium fills all space has been subtracted. The result for the integrated stress on the spherical cavity, after the Euclidean transformation is performed, is

$$\begin{aligned}
F_{\mathrm{ES}} = &-\frac{1}{12a^2}\sum_{l=1}^\infty(2l+1)\int_{-\infty}^\infty dy\, e^{iy\delta}\\
&\cdot\Bigg\{\frac{(\epsilon'-1)(\epsilon'+2)}{\epsilon'}\Bigg[\frac{A_G}{2x'}\left(x'^2 I^2_{l+1/2}(x')\right)'\\
&-x'(A_F+A_G)\int_0^{x'}d\xi\, I^2_{l+1/2}(\xi) + x'A_G\int_0^{x'}\frac{d\xi}{\xi}I^2_{l+1/2}(\xi)\Bigg]\\
&+\frac{(\epsilon-1)(\epsilon+2)}{\epsilon}\left(\frac{2}{\pi}\right)^2\Bigg[-\frac{B_G}{2x}\left(x^2K^2_{l+1/2}(x)\right)'\\
&-x(B_F+B_G)\int_x^\infty d\xi\, K^2_{l+1/2}(\xi) + xB_G\int_x^\infty\frac{d\xi}{\xi}K^2_{l+1/2}(\xi)\Bigg]\Bigg\}.
\end{aligned} \tag{6.4}$$

VII. ASYMPTOTIC ANALYSIS AND NUMERICAL RESULTS

The result for the stress (3.9) is an immediate generalization of that given in [6], and therefore, the asymptotic analysis given there can be applied nearly unchanged. The result for the energy is new, and seems not to have been recognized earlier.

We first remark on the special case $\sqrt{\epsilon\mu} = \sqrt{\epsilon'\mu'}$. Then $x = x'$ and the energy reduces to

$$E = -\frac{1}{4\pi a}\int_{-\infty}^{\infty} dy\, e^{iy\delta}\sum_{l=1}^{\infty}(2l+1)x\frac{d}{dx}\ln[1-\xi^2((s_l e_l)')^2], \tag{7.1}$$

where

$$\xi = \frac{\mu-\mu'}{\mu+\mu'}. \tag{7.2}$$

If $\xi = 1$ we recover the case of a perfectly conducting spherical shell, treated in [14], for which E is finite. In fact (7.1) is finite for all ξ, and if we use the leading uniform asymptotic approximation for the Bessel functions we obtain

$$E \sim \frac{3}{64a}\xi^2. \tag{7.3}$$

Further analysis of this special case is given by Brevik and Kolbenstvedt [20].

In general, using the uniform asymptotic behavior, with $x = \nu z$, $\nu = l + 1/2$, and, for simplicity looking at the large z behavior, we have

$$E \sim -\frac{1}{2\pi a}\frac{1}{\sqrt{\epsilon\mu}}\sum_l \nu^2\int_{-\infty}^{\infty} dz\, e^{iz\nu\delta/\sqrt{\epsilon\mu}} z\frac{d}{dz}\ln\left[1+\frac{1}{16z^4}\left(\frac{\epsilon\mu}{\epsilon'\mu'}-1\right)^2(1-\xi^2)\right], \tag{7.4}$$

which exhibits a cubic divergence as $\delta \to 0$. To be more explicit, let us content ourselves with with the case when $\epsilon - 1$, $\epsilon' - 1$ are both small and $\mu = \mu' = 1$. Then, the leading ν term is

$$\begin{aligned} E &\sim -\frac{(\epsilon'-\epsilon)^2}{16\pi a}\sum_{l=1}^{\infty}\nu^2\frac{1}{2}\int_{-\infty}^{\infty} dz\, e^{i\nu z\delta} z\frac{d}{dz}\frac{1}{(1+z^2)^2} \\ &= -\frac{(\epsilon'-\epsilon)^2}{64a}\left(\frac{16}{\delta^3}+\frac{1}{4}\right) \to -\frac{(\epsilon'-\epsilon)^2}{256a}. \end{aligned} \tag{7.5}$$

Here, the last arguable step is made plausible by noting that since $\delta = \tau/a$ the divergent term represents a contribution to the surface tension on the bubble, which should be cancelled by a suitably chosen counter term (contact term). This argument is given somewhat more weight by the discussion in [27]. In essence, justification is provided there for the use of zeta-function regularization, which directly gives the finite part here:

$$E \sim \frac{(\epsilon' - \epsilon)^2}{32\pi a} \sum_{l=1}^{\infty} \nu^2 \frac{\pi}{2} = \frac{(\epsilon' - \epsilon)^2}{64a} \left(-\frac{1}{4} \right), \tag{7.6}$$

because $\sum_{l=0}^{\infty} \nu^s = (2^{-s} - 1)\zeta(-s)$ vanishes at $s = 2$.

Alternatively, one could argue that dispersion should be included [28–30], crudely modelled by

$$\epsilon(\omega) - 1 = \frac{\epsilon_0 - 1}{1 - \omega^2/\omega_0^2}. \tag{7.7}$$

If this rendered the expression for the stress finite [we consider the stress, not the energy, for it is not necessary to consider the dispersive factor $d(\omega\epsilon(\omega))/d\omega$ there], we could drop the cutoff δ and the sign of the force would be positive: (at last, we set $\epsilon' = 1$)

$$\mathcal{F} \sim +\frac{(\epsilon_0 - 1)^2}{128\pi^2 a^4} \sum_{l=1}^{\infty} \nu^2 \int_{-\infty}^{\infty} dz \, \frac{1}{(1 + z^2)^2} \frac{1}{(1 + z^2/z_0^2)^2}, \tag{7.8}$$

where $z_0 = \omega_0 a/\nu$. As $\nu \to \infty$, $z_0 \to 0$, and the integral here approaches $\pi z_0/2$, and so

$$\mathcal{F} \sim \frac{(\epsilon_0 - 1)^2}{256\pi a^3} \omega_0 \sum_{l=1}^{\nu_c} \nu \sim \frac{(\epsilon_0 - 1)^2}{512\pi a} \omega_0^3, \tag{7.9}$$

if we take as the cutoff[2] of the angular momentum sum $\nu_c \sim \omega_0 a$. The corresponding energy is obtained by integrating $-4\pi a^2 \mathcal{F}$,

$$E \sim -\frac{(\epsilon_0 - 1)^2}{256} \omega_0^3 a^2, \tag{7.10}$$

which is of the form of (7.5) with $1/\delta \to \omega_0 a/4$.

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[2]Inconsistently, for then $z_0 \sim 1$. If $z_0 = 1$ in (7.8), however, the same angular momentum cutoff gives 5/12 of the value in (7.9).

It is rather more difficult to extract numerical results from the formula for electrostriction, (6.4). Indeed, Brevik [25] only considers two special cases, $\epsilon \gg 1$, appropriate to a perfect conductor, and $\epsilon - 1 \ll 1$. In fact, in the latter case, corresponding to (7.5), he was able to consider only the single $l = 1$ term in the sum. This is highly unreliable, as such a term may be completely unrepresentative (such as having the wrong sign [14]). Because this electrostrictive stress presents new and somewhat difficult to understand divergences, we will defer its consideration to a later publication, and only remark that it is highly likely to contribute a term comparable to the finite Casimir estimate.

VIII. CONCLUSIONS

So finally, what can we say about sonoluminescence? To calibrate our remarks, let us recall (a simplified version of) the argument of Schwinger [1]. On the basis of a provocative but incomplete analysis he argued that a bubble ($\epsilon' = 1$) in water ($\epsilon \approx (4/3)^2$) possessed a positive Casimir energy[3]

$$E_c \sim \frac{4\pi a^3}{3} \int \frac{(d\mathbf{k})}{(2\pi)^3} \frac{1}{2} k \left(1 - \frac{1}{\sqrt{\epsilon}}\right) \sim \frac{a^3 K^4}{12\pi} \left(1 - \frac{1}{\sqrt{\epsilon}}\right), \tag{8.1}$$

where K is a wavenumber cutoff. Putting in his estimate, $a \sim 4 \times 10^{-3}$cm, $K \sim 2 \times 10^5$cm^{-1} (in the UV), we find a large Casimir energy, $E_c \sim 13$ MeV, and something like 3 million photons would be liberated if the bubble collapsed.

What does our full (albeit static) calculation say? If we believe the subtracted result, the last form in (7.5), and say that the bubble collapses from an initial radius $a_i = 4 \times 10^{-3}$cm to a final radius $a_f = 4 \times 10^{-4}$cm, as suggested by experiments [3], we find that the change in the Casimir energy is $\Delta E \sim +10^{-4}$eV. This is far to small to account for the observed emission.

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[3]Note, for small $\epsilon - 1$, Schwinger's result goes like $(\epsilon - 1)$, rather than $(\epsilon - 1)^2$, indicating that he had not removed the "vacuum" contribution corresponding to (3.8). This is the essential physical reason for the discrepancy between his results and ours.

On the other hand, perhaps we should retain the divergent result, and put in reasonable cutoffs. If we do so, we have

$$E = -\frac{(\epsilon-1)^2}{4} a^2 K^3 \sim -4\times 10^5 \mathrm{eV}, \tag{8.2}$$

perhaps of acceptable magnitude, but of the *wrong* sign. (The emission occurs *during* the collapse.) The same conclusion follows if one uses dispersion, as (7.10) shows.

So we are unable to see how the Casimir effect could possibly supply energy relevant to the copious emission of light seen in sonoluminescence. Of course, dynamical effects could change this conclusion, but elementary arguments suggest that this is impossible unless ultrarelativistic velocities are achieved. For example, consider the Larmor formula, appropriate to dipole radiation; it gives the power radiated:

$$P = \frac{2}{3}\frac{(\ddot{\mathbf{d}})^2}{c^3}, \tag{8.3}$$

where $\mathbf{d}$ is the dipole moment. If our bubble, with N atoms, coherently emits radiation, we expect

$$|\ddot{\mathbf{d}}| \sim \frac{N d_a}{\tau^2}, \tag{8.4}$$

where d_a is an atomic dipole moment, and τ a characteristic collapse time for the bubble. Thus the energy emitted during one collapse of a bubble in water is

$$E \sim 4\pi\alpha\hbar c \left(10^{23}\left(\frac{a}{\mathrm{cm}}\right)^3\right)^2 \frac{(d_a/e)^2}{(c\tau)^3}. \tag{8.5}$$

So with $a \sim 10^{-3}$ cm, $\tau \sim 10^{-5}$ s (suggested by experiments [3]), and $d_a \sim 10^{-8} e$-cm, we get an energy of only $E \sim 10^{-10}$ eV. This is in spite of the assumption of coherent radiation. Note that in (8.5), τ would have to be $\sim 10^{-11}$ s (which is the upper bound to the observed flash duration) to yield a total energy of 10 MeV; this would correspond to a velocity across the bubble of 10^8 cm/s, well in excess of the speed of sound, thus precluding the presumed coherent radiation process.

We therefore believe that in Eberlein's calculation [9] there is an implicit assumption of superluminal velocities.[4] We note that the short wavelength result of Eberlein, (4.7) of the first reference in [9] or (10) of the second, can be cast in the dipole form (8.3) by integrating by parts. Up to factors nearly equal to one,

$$\left(\frac{d}{e}\right)_E \approx a\frac{\dot{a}}{c}, \tag{8.6}$$

where $a(t)$ is the bubble radius. Because $\dot{a}/c < 1$, we find that emission energy of 10 MeV requires a time scale of $\tau_E \sim 10^{-17}$ s. This seems to us an implausibly short scale unless remarkable relativistic phenomena are involved. [Incidentally, the magnitude of our cutoff estimate, (8.2), also agrees with (8.3) if $K \sim 1/\tau$. This demonstrates that there is nothing classical about the estimate (8.5)].

The only plausible origin of such short time scales lies in the formation of a shock. Such a model was sketched by Greenspan and Nadim in [11]. But this picture has nothing to do with quantum radiation.

Recent experimental results have made it even more difficult to accommodate any explanation based on macroscopic considerations. In particular, Hiller and Putterman [4] have found a remarkably strong isotope effect when water (H_2O) is replaced by heavy water (D_2O), where the dielectric properties change by no more than 10%. This, together with the already known strong temperature dependence, and strong dependence on gas concentration and the gas mixture, may rule out Casimir effect explanations entirely. Yet the subject of vacuum energy is sufficiently subtle that surprises could be in store.

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[4]Indeed, if one follows Eberlein and uses $\gamma \sim 1$ fs (though the experimental value seems to be closer to 10 ps) in her model profile, one finds the maximum speed of the bubble surface to exceed the speed of light by almost two orders of magnitude. Actually, even with such a small γ, we find her result yields an energy output of only 10^{-3} MeV, insufficient to explain sonoluminescence. Eberlein agrees with these points, and no longer stands by this model [31].

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APPENDIX: DISCUSSION OF FORM OF FORCE ON SURFACE

There seems, in the literature, to be some confusion about the correct form for the stress on a surface due to electromagnetic fields (here, of course, we are interested in vacuum expectation values of those fields). The definitive discussion seems to be given in Stratton [22]. We also refer here to the manuscript in process by one of us [32].

In the text, we computed the force on the surface by considering the discontinuity of the stress tensor,

$$T_{nn} = \frac{1}{2}\epsilon(E_\perp^2 - E_n^2) + \frac{1}{2}\mu(H_\perp^2 - H_n^2), \tag{A1}$$

across the surface, where $\mathbf{n}$ denotes the direction normal to the surface, and $\perp$ directions tangential to the surface. This follows directly from a consideration of the interpretation to T_{nn} as the flow of nth component of momentum in the direction $\mathbf{n}$. Because of the boundary conditions that

$$\mathbf{E}_\perp, \quad D_n, \quad \mathbf{H}_\perp, \quad B_n \tag{A2}$$

be continuous, the stress on the surface is

$$\mathcal{F} = T_{nn}(-) - T_{nn}(+) = \frac{1}{2}\left[(\epsilon' - \epsilon)E_\perp^2 - \left(\frac{1}{\epsilon'} - \frac{1}{\epsilon}\right) D_n^2 + (\mu' - \mu)H_\perp^2 - \left(\frac{1}{\mu'} - \frac{1}{\mu}\right) B_n^2\right], \tag{A3}$$

in terms of fields on the surface, and where a prime denotes quantities on the "$-$" side of the surface. This is obviously equivalent to the following form for the force density,

$$\mathbf{f} = -\frac{1}{2}\left(E_\perp^2 \boldsymbol{\nabla}\epsilon - D_n^2 \boldsymbol{\nabla}\frac{1}{\epsilon} + H_\perp^2 \boldsymbol{\nabla}\mu - B_n^2 \boldsymbol{\nabla}\frac{1}{\mu}\right) = -\frac{1}{2}(E^2 \boldsymbol{\nabla}\epsilon + H^2 \boldsymbol{\nabla}\mu), \tag{A4}$$

which is just what is obtained from the stationary principle for the energy [32].

The controversy seems to center around the additional "Abraham" term

$$\mathbf{f}' = (\epsilon - 1)\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{H}). \tag{A5}$$

(Henceforward we restrict ourselves to nonmagnetic material, $\mu = 1$.) As noted in Section 5, this makes no contribution to Casimir effect, because the vacuum expectation value is stationary. Furthermore, the existence of such a term is dependent upon the (essentially arbitrary) split between field and particle momentum. The Minkowski choice for field momentum

$$\mathbf{G}_M = \mathbf{D} \times \mathbf{H} \tag{A6}$$

would not imply this additional force density. The analysis of experimental data given by Brevik [25], however, seems to favor the Abraham value.

If we were calculating a net *force* on the surface, (A5) would indeed give a further contribution to the force beyond that given by (A3). Through use of Maxwell's equations, we easily find

$$f_n' = (\epsilon - 1)\left[-\frac{1}{2\epsilon}\nabla_n B^2 - \frac{1}{2}\nabla_n E^2 + \frac{1}{\epsilon}\boldsymbol{\nabla}\cdot(\mathbf{B}B_n) + \boldsymbol{\nabla}\cdot(\mathbf{E}E_n)\right]. \tag{A7}$$

If $\mathbf{n}$ were a fixed direction, the volume integral of this force density would turn into a surface integral, and the result given by Eberlein follows,

$$F_n' = \frac{1}{2}\int dS\,\left[\left(\frac{1}{\epsilon} - \frac{1}{\epsilon'}\right)(B_n^2 - B_\perp^2) + (\epsilon - \epsilon')E_\perp^2 - \left(\frac{1}{\epsilon} - \frac{1}{\epsilon'}\right)\left(1 - \frac{1}{\epsilon} - \frac{1}{\epsilon'}\right)D_n^2\right]. \tag{A8}$$

But this result cannot be used to compute the stress. Thus, the formula (C5) given in Appendix C of the first paper in [9] is wrong, and, accordingly, so is (3.18) there.[5]

[5]The first derivation there is based on the formula for the force given in the following paragraph,

Finally, we note there is yet another formula for the force on a dielctric given in terms of polarization charges and currents,

$$\mathbf{F} = \int (d\mathbf{r}) \left[\rho_{\rm pol}\mathbf{E} + \frac{1}{c}\mathbf{j}_{\rm pol} \times \mathbf{B} \right], \tag{A9}$$

where

$$\rho_{\rm pol} = -\boldsymbol{\nabla} \cdot \mathbf{P}, \quad \mathbf{j}_{\rm pol} = \frac{\partial}{\partial t}\mathbf{P} + c\boldsymbol{\nabla} \times \mathbf{M}, \tag{A10}$$

with the polarization and magnetization fields given by

$$\mathbf{P} = \mathbf{D} - \mathbf{E} = \left(1 - \frac{1}{\epsilon}\right)\mathbf{D}, \quad \mathbf{M} = \mathbf{B} - \mathbf{H} = (\mu - 1)\mathbf{H}. \tag{A11}$$

Again, it is easy to show that if one is calculating the force in a fixed direction, so one can freely integrate by parts, for a nonmagnetic medium, we recover the expected force including the Abraham term:

$$\mathbf{F} = \int (d\mathbf{r}) \left[-\frac{E^2}{2}\boldsymbol{\nabla}\epsilon + \frac{1}{c}\frac{\partial}{\partial t}(\epsilon - 1)\mathbf{E} \times \mathbf{B} \right]. \tag{A12}$$

But the integrand in (A9) is not interpretable as a force density from which the stress may be computed. In effect, it is that interpretation that [9] uses.

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while the second is based on an obviously incorrect extrapolation from the vacuum stress tensor, which of course gives vanishing stress.

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