Local Casimir Energies for a Thin Spherical Shell

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Abstract

The local Casimir energy density for a massless scalar field associated with step-function potentials in a $3+1$ dimensional spherical geometry is considered. The potential is chosen to be zero except in a shell of thickness δ, where it has height h, with the constraint $h\delta = 1$. In the limit of zero thickness, an ideal δ-function shell is recovered. The behavior of the energy density as the surface of the shell is approached is studied in both the strong and weak coupling regimes. The former case corresponds to the well-known Dirichlet shell limit. New results, which shed light on the nature of surface divergences and on the energy contained within the shell, are obtained in the weak coupling limit. In that case, the energy has a contribution not only from the local energy density, but from an energy term residing entirely on the surface. It is shown that the latter coincides with the integrated local energy density within the shell. We also study the dependence of local and global quantities on the conformal parameter. In particular new insight is provided on the reason for the divergence in the global Casimir energy in third order in the coupling.

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I. INTRODUCTION

The subject of local energy density associated with the confinement of quantum fields by surfaces has a rather long history. For example, Brown and Maclay [1] computed the vacuum expectation value of the electromagnetic energy-momentum tensor between two parallel perfectly conducting plates, which is twice that of a conformally coupled scalar field satisfying Dirichlet or Neumann boundary conditions on the plates, namely

$$\langle T^{\mu\nu}\rangle = \frac{\pi^2}{1440a^4}\mathrm{diag}\,(-1,-3,1,1), \tag{1.1}$$

which corresponds precisely to the attractive energy or pressure found by Casimir in the same situation [2]. If a nonconformal scalar stress tensor is used, a position-dependent term in the stress tensor appears, which does not contribute to either the total energy or the pressure on the plates [3, 4].

Local surface divergences were first discussed for arbitrary smooth boundaries by Deutsch and Candelas [5]. They found cubic divergences in the energy density as one approaches the surface; for example, outside a Dirichlet sphere (that is, for a conformally-coupled scalar field satisfying Dirichlet boundary conditions on the surface) the energy density diverges as

$$r \to a+: \quad u \sim \frac{1}{360\pi^2}\frac{1}{a(r-a)^3}, \tag{1.2}$$

where a is the radius of the sphere. This raises the question: How can it be that the total Casimir energy of a Dirichlet sphere (or a perfectly conducting sphere in electrodynamics) is finite? The electromagnetic case is the well-known one first calculated by Boyer [6]

$$E^{EM} = \frac{0.04618}{a}, \tag{1.3}$$

while the scalar case was first worked out by Bender and Milton [7],

$$E^S = \frac{0.002817}{a}. \tag{1.4}$$

Thus there has been a feeling since the time of Deutsch and Candelas that there was something suspect about the calculations of Casimir self energies of ideal closed boundaries. (We note that there is now a proof that any such smooth perfectly conducting boundary possesses a finite electromagnetic Casimir energy [8]. Whether such an idealized limit is physical is, of course, another question.) This suspicion has been recently intensified by a series of talks and papers by Jaffe's group [9]. The essential outcome of their analysis is that for a δ-function sphere, described by the following Lagrangian for a massless scalar field,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\frac{\lambda}{a^2}\delta(r-a)\phi^2, \tag{1.5}$$

a divergence occurs in third order in λ. (They claimed a divergence in second order, but that was spurious [3, 10, 11].) This divergence in fact was discovered much earlier by Bordag, Kirsten, and Vassilevich [12], and possible ways of dealing with it have been suggested [13, 14]. (Recently, the effects of the spherical δ-function potential at finite temperature have been considered by McKenzie-Smith and Naylor [15].) Objections complementary to those of Ref. [9] have also been voiced by Barton [16], all of which raise doubts concerning the physical relevance of results such as those in Eqs. (1.3) and (1.4).

Clearly, then, there are issues here unresolved. The purpose of the present paper is to examine the local energy density for a scalar field in the presence of a spherically symmetric background, but as suggested by Ref. [9] not so singular as a delta function. So we consider a spherical shell, with a finite step potential. In the limit as the height goes to infinity and the width to zero we recover the δ-function sphere. This enables us to explore how the quantum vacuum energy of the shell itself contributes to the total energy of the configuration. In the next section, we will construct the Green's function for such a sphere, and in the following section the local energy density (for an arbitrarily coupled scalar) inside and outside the region of the shell. In Sec. IV we will consider the thin shell limit and examine how the energy density diverges as the surface is approached. As expected, the divergence in the energy density as the surface is approached is weakest for the conformally coupled case. In Sec. V we will compute the energy of each region, and thereby determine the total energy of the spherical geometry. There we will see that the latter involves not just the local energy density, which diverges as the surface is approached, but also an energy lying entirely on the surfaces separating the shell and the interior and the exterior regions, except for a special, nonconformal value of the conformal parameter. The net effective surface energy for the interior and exterior regions of a thin shell is identified as the integrated energy density within the shell.

II. GREEN'S FUNCTION FOR λ SPHERE

As discussed in Refs. [10, 11] for parallel planes, we consider the potential

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{2a^2}\phi^2\sigma(r), \tag{2.1a}$$

where

$$\sigma(r) = \begin{cases} 0, & r < a_-, \\ h, & a_- < r < a_+, \\ 0, & a_+ < r. \end{cases} \tag{2.1b}$$

Here $a_\pm = a \pm \delta/2$, and we set $h\delta = 1$. Compared to Refs. [10, 11], we have changed the dimensions of λ so that the total energy of interaction does not explicitly refer to the radius a. In the limit as $\delta \to 0$ (or $h \to \infty$) we recover the δ-function sphere considered first in Ref. [12].

A straightforward solution of the Green's function equation, for a massless particle, with $\kappa^2 = -\omega^2$,

$$\left(-\nabla^2 + \kappa^2 + \frac{\lambda}{a^2}\sigma\right)\mathcal{G}(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \tag{2.2}$$

in terms of the reduced Green's function,

$$\mathcal{G}(\mathbf{r},\mathbf{r}') = \sum_{lm} g_l(r,r')Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi'), \tag{2.3}$$

and the modified Ricatti-Bessel functions,

$$s_l(x) = \sqrt{\frac{\pi x}{2}}I_{l+1/2}(x), \quad e_l(x) = \sqrt{\frac{2x}{\pi}}K_{l+1/2}(x), \tag{2.4}$$

is as follows, outside of the shell,

$$r, r' < a_- : \quad g_l = \frac{1}{\kappa r r'}\left[s_l(\kappa r_<)e_l(\kappa r_>) - \frac{\tilde{\Xi}}{\Xi}s_l(\kappa r)s_l(\kappa r')\right], \tag{2.5a}$$

$$r, r' > a_+ : \quad g_l = \frac{1}{\kappa r r'}\left[s_l(\kappa r_<)e_l(\kappa r_>) - \frac{\hat{\Xi}}{\Xi}e_l(\kappa r)e_l(\kappa r')\right], \tag{2.5b}$$

where the denominator is

$$\begin{aligned}\Xi &= [\kappa s_l'(\kappa a_-)e_l(\kappa' a_-) - \kappa' s_l(\kappa a_-)e_l'(\kappa' a_-)][\kappa' e_l(\kappa a_+)s_l'(\kappa' a_+) - \kappa e_l'(\kappa a_+)s_l(\kappa' a_+)]\\ &\quad - [\kappa s_l'(\kappa a_-)s_l(\kappa' a_-) - \kappa' s_l(\kappa a_-)s_l'(\kappa' a_-)][\kappa' e_l(\kappa a_+)e_l'(\kappa' a_+) - \kappa e_l'(\kappa a_+)e_l(\kappa' a_+)].\end{aligned} \tag{2.6}$$

The numerator in (2.5a), $\tilde{\Xi}$, is obtained from Ξ by replacing $s_l(\kappa a_-) \to e_l(\kappa a_-)$:

$$\begin{aligned}\tilde{\Xi} &= [\kappa e_l'(\kappa a_-)e_l(\kappa' a_-) - \kappa' e_l(\kappa a_-)e_l'(\kappa' a_-)][\kappa' e_l(\kappa a_+)s_l'(\kappa' a_+) - \kappa e_l'(\kappa a_+)s_l(\kappa' a_+)]\\ &\quad - [\kappa e_l'(\kappa a_-)s_l(\kappa' a_-) - \kappa' e_l(\kappa a_-)s_l'(\kappa' a_-)][\kappa' e_l(\kappa a_+)e_l'(\kappa' a_+) - \kappa e_l'(\kappa a_+)e_l(\kappa' a_+)],\end{aligned} \tag{2.7}$$

while that in (2.5b), $\hat{\Xi}$, is obtained from Ξ by replacing $e_l(\kappa a_+) \to s_l(\kappa a_+)$.

$$\begin{aligned}\hat{\Xi} &= [\kappa s_l'(\kappa a_-)e_l(\kappa' a_-) - \kappa' s_l(\kappa a_-)e_l'(\kappa' a_-)][\kappa' s_l(\kappa a_+)s_l'(\kappa' a_+) - \kappa s_l'(\kappa a_+)s_l(\kappa' a_+)]\\ &\quad - [\kappa s_l'(\kappa a_-)s_l(\kappa' a_-) - \kappa' s_l(\kappa a_-)s_l'(\kappa' a_-)][\kappa' s_l(\kappa a_+)e_l'(\kappa' a_+) - \kappa s_l'(\kappa a_+)e_l(\kappa' a_+)].\end{aligned} \tag{2.8}$$

Here $\kappa' = \sqrt{\kappa^2 + \lambda h/a^2}$. It might be noticed that under the interchange

$$s_l(\kappa a_\mp) \leftrightarrow e_l(\kappa a_\pm), \tag{2.9}$$

and similarly for functions depending on κ', Ξ is unchanged, while $\hat{\Xi} \leftrightarrow \tilde{\Xi}$.

The Green's function within the shell has a somewhat more complicated form. For $a - \frac{\delta}{2} < r, r' < a + \frac{\delta}{2}$ we find

$$\begin{aligned}g_l &= \frac{1}{\kappa' r r'}\Bigg\{s_l(\kappa' r_<)e_l(\kappa' r_>) - \frac{1}{\Xi}\Bigg[[s_l(\kappa' r)e_l(\kappa' r') + s_l(\kappa' r')e_l(\kappa' r)]\\ &\qquad \times[\kappa e_l'(\kappa a_+)e_l(\kappa' a_+) - \kappa' e_l(\kappa a_+)e_l'(\kappa' a_+)][\kappa s_l'(\kappa a_-)s_l(\kappa' a_-) - \kappa' s_l(\kappa a_-)s_l'(\kappa' a_-)]\\ &\quad - s_l(\kappa' r')s_l(\kappa' r)[\kappa e_l'(\kappa a_+)e_l(\kappa' a_+) - \kappa' e_l(\kappa a_+)e_l'(\kappa' a_+)]\\ &\qquad \times[\kappa s_l'(\kappa a_-)e_l(\kappa' a_-) - \kappa' s_l(\kappa a_-)e_l'(\kappa' a_-)]\\ &\quad - e_l(\kappa' r')e_l(\kappa' r)[\kappa e_l'(\kappa a_+)s_l(\kappa' a_+) - \kappa' e_l(\kappa a_+)s_l'(\kappa' a_+)]\\ &\qquad \times[\kappa s_l'(\kappa a_-)s_l(\kappa' a_-) - \kappa' s_l(\kappa a_-)s_l'(\kappa' a_-)]\Bigg]\Bigg\}.\end{aligned} \tag{2.10}$$

We will now use the Green's function to compute the energy density.

III. ENERGY DENSITY

We can calculate the local energy density from the stress tensor:

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\mathcal{L} - \xi(\partial^\mu\partial^\nu - g^{\mu\nu}\partial^2)\phi^2, \tag{3.1}$$

from which the energy density follows:

$$T^{00} = \frac{1}{2}\left[\partial^0\phi\partial^0\phi + \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi + \frac{\lambda}{a^2}\sigma\phi^2\right] - \xi\nabla^2\phi^2, \tag{3.2}$$

where the conformal value is given by $\xi = 1/6$. To obtain the vacuum expectation values at one loop, we use the identification with the classical Green's function

$$\langle\phi(x)\phi(x')\rangle = \frac{1}{i}G(x,x'). \tag{3.3}$$

The energy density thus is, within or outside the shell,

$$\langle T^{00}\rangle = \frac{1}{2i}\left(\partial^0\partial'^0 + \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}' + \left\{\begin{array}{c}\lambda h/a^2\\ 0\end{array}\right\}\right)G(x,x')\bigg|_{x'=x} - \frac{\xi}{i}\nabla^2 G(x,x). \tag{3.4}$$

When we insert the partial wave decomposition of the Green's function (2.3), we encounter, in terms of the angle γ between the two directions (θ,φ), (θ',φ'),

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi'), \tag{3.5}$$

the evaluation

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}' P_l(\cos\gamma)\bigg|_{\theta=\theta',\phi=\phi'} = \frac{1}{r^2}2P_l'(1) = \frac{l(l+1)}{r^2}. \tag{3.6}$$

Therefore, the expression for the energy density is immediately reduced to (inside or outside the shell, but not within it)

$$\langle T^{00}\rangle = \int_0^\infty \frac{d\kappa}{2\pi}\sum_{l=0}^\infty \frac{2l+1}{4\pi}\left\{\left[-\kappa^2 + \partial_r\partial_r' + \frac{l(l+1)}{r^2}\right]g_l(r,r')\bigg|_{r'=r} - 2\xi\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}g_l(r,r)\right\}. \tag{3.7}$$

Note that there has been no need of some dubious argument (such as appears in Ref. [11], Eq. (4.21)) concerning partial integration in the angular coordinates.

We insert the Green's function in the exterior region, but delete the free part,

$$g_l^0 = \frac{1}{\kappa r r'}s_l(\kappa r_<)e_l(\kappa r_>), \tag{3.8}$$

which corresponds to the *bulk energy* which would be present if either medium filled all of space, leaving us with for $r > a_+$

$$\begin{aligned} u(r) = -(1-4\xi)\int_0^\infty \frac{d\kappa}{2\pi}\sum_{l=0}^\infty \frac{2l+1}{4\pi}\frac{\hat{\Xi}}{\Xi}\bigg\{&\frac{e_l^2(\kappa r)}{\kappa r^2}\left[-\kappa^2\frac{1+4\xi}{1-4\xi} + \frac{l(l+1)}{r^2} + \frac{1}{r^2}\right] \\ &- \frac{2}{r^3}e_l(\kappa r)e_l'(\kappa r) + \frac{\kappa}{r^2}e_l'^2(\kappa r)\bigg\}. \end{aligned} \tag{3.9}$$

Inside the shell, $r < a_-$, the energy is given by a similar expression obtained from Eq. (3.9) by replacing e_l by s_l and $\hat{\Xi}$ by $\tilde{\Xi}$. We will discuss the energy within the shell below.

IV. SURFACE DIVERGENCES

We want to examine the singularity structure as $r \to a_+$. For this purpose we use the leading uniform asymptotic expansion, $l \to \infty$,

$$\begin{aligned} e_l(x) &\sim \sqrt{zt}\, e^{-\nu\eta}, \quad s_l(x) \sim \frac{1}{2}\sqrt{zt}\, e^{\nu\eta}, \\ e_l'(x) &\sim -\frac{1}{\sqrt{zt}}\, e^{-\nu\eta}, \quad s_l'(x) \sim \frac{1}{2}\frac{1}{\sqrt{zt}}\, e^{\nu\eta}, \end{aligned} \tag{4.1}$$

where ($\nu = l + 1/2$)

$$x = \nu z, \quad t = (1+z^2)^{-1/2}, \quad \frac{d\eta}{dz} = \frac{1}{zt}. \tag{4.2}$$

Let us consider the thin shell limit, $\delta \to 0$, $h\delta = 1$, where it is easy to check that

$$\hat{\Xi} \to \frac{\frac{\lambda}{\kappa a^2} s_l^2(\kappa a)}{1 + \frac{\lambda}{\kappa a^2} e_l(\kappa a) s_l(\kappa a)}, \tag{4.3}$$

which is exactly the coefficient occurring in the δ-function potential (1.5). There are two simple limits of this, strong and weak coupling:

$$\frac{\lambda}{a} \to \infty: \quad \hat{\Xi} \to \frac{s_l(\kappa a)}{e_l(\kappa a)}, \tag{4.4a}$$

$$\frac{\lambda}{a} \to 0: \quad \hat{\Xi} \to \frac{\lambda}{\kappa a^2} s_l^2(\kappa a), \tag{4.4b}$$

if we assume that the relevant scale of κ is $1/a$, since we expect that the significant values of κ are determined by the argument of the Bessel functions in Eq. (3.9).

In either case, we carry out the asymptotic sum over angular momentum using Eq. (4.1) and

$$\sum_{l=0}^{\infty} e^{-\nu\chi} = \frac{1}{2\sinh\frac{\chi}{2}}. \tag{4.5}$$

Here

$$\chi = 2\left[\eta(z) - \eta\left(z\frac{a}{r}\right)\right] \approx 2z\eta'(z)\frac{r-a}{r} = \frac{2}{t}\frac{r-a}{r}. \tag{4.6}$$

The remaining integrals over z are elementary, and in this way we find that the leading divergences in Eq. (3.9) are as $r \to a+$,

$$\frac{\lambda}{a} \to \infty: \quad u \sim -\frac{1}{16\pi^2}\frac{1-6\xi}{(r-a)^4}, \tag{4.7a}$$

$$\frac{\lambda}{a} \to 0: \quad u^{(n)} \sim \left(-\frac{\lambda}{a}\right)^n \frac{\Gamma(4-n)}{96\pi^2 a^4}(1-6\xi)\left(\frac{a}{r-a}\right)^{4-n}, \quad n < 4, \tag{4.7b}$$

the latter being the leading divergence in order n, which results clearly seem to demonstrate the virtue of the conformal value of $\xi = 1/6$; but see below. (The value for the Dirichlet sphere (4.7a) first appeared in Ref. [5]; it recently was rederived in Ref. [17], where, however, the subdominant term, the leading term if $\xi = 1/6$, namely (4.12), was not calculated.)

Thus, for $\xi = 1/6$ we must keep subleading terms. This includes keeping the subdominant term in χ,

$$\chi \approx \frac{2}{t}\frac{r-a}{r} - t\left(\frac{r-a}{r}\right)^2, \tag{4.8}$$

the distinction between $t(z)$ and $\tilde{t} = t(\tilde{z} = za/r)$,

$$\tilde{z}\tilde{t} \approx zt - t^3 z\frac{r-a}{r}, \tag{4.9}$$

as well as the next term in the uniform asymptotic expansion of the Bessel functions,

$$\begin{aligned}
s_l(x) &\sim \frac{1}{2}\sqrt{zt}\, e^{\nu\eta}\left(1 + u_1(t) + \dots\right), && \text{(4.10a)}\\
e_l(x) &\sim \sqrt{zt}\, e^{-\nu\eta}\left(1 - u_1(t) + \dots\right), && \text{(4.10b)}\\
s_l'(x) &\sim \frac{1}{2}\frac{1}{\sqrt{zt}}\, e^{\nu\eta}\left(1 + v_1(t) + \dots\right), && \text{(4.10c)}\\
e_l'(x) &\sim -\frac{1}{\sqrt{zt}}\, e^{-\nu\eta}\left(1 - v_1(t) + \dots\right), && \text{(4.10d)}\\
& && \text{(4.10e)}
\end{aligned}$$

where

$$u_1(t) = \frac{3t - 5t^3}{24}, \quad v_1(t) = \frac{3t + 7t^3}{24}. \tag{4.11}$$

Including all this, it is straightforward to recover the well-known result (1.2) [5] for strong coupling (Dirichlet boundary conditions):

$$\frac{\lambda}{a} \to \infty: \quad u \sim \frac{1}{360\pi^2}\frac{1}{a(r-a)^3}, \tag{4.12}$$

Following the same process for weak coupling, we find that the leading divergence in order n, $1 \le n < 3$, is ($r \to a\pm$)

$$\lambda \to 0: \quad u^{(n)} \sim \left(\frac{\lambda}{a^2}\right)^n \frac{1}{1440\pi^2}\frac{1}{a(a-r)^{3-n}}(n-1)(n+2)\Gamma(3-n). \tag{4.13}$$

Note that the subleading $O(\lambda)$ term again vanishes. Both Eqs. (4.12) and (4.13) apply for the conformal value $\xi = 1/6$.

V. TOTAL ENERGY

A. Cancellation of divergences

The above results for the conformally coupled scalar show that the inverse linear divergences in the local energy density which occur in either order λ or λ^2 cancel between inside and outside, when one computes the total energy, while the divergence encountered at $n = 3$:

$$u^{(3)} \sim \frac{\lambda^3}{a^7}\frac{1}{144\pi^2}\Gamma(0), \tag{5.1}$$

is correctly given by

$$u^{(3)} \sim -\frac{\lambda^3}{144\pi^2 a^7}\ln\frac{r-a}{a}, \tag{5.2}$$

as shown by explicit calculation. The integral of this, however, is finite, so this does not signal any difficulty with the total energy in order λ^3.

B. Surface energy

However, as discussed first by Dowker, Kennedy and Critchley [18, 19], and later elaborated by Saharian and Romeo [20], and put in a broader context by Fulling [21], for situations when other than Neumann or Dirichlet boundary conditions apply, an additional term must be supplied in calculating the energy, a term which resides entirely on the surface. For the case of the general stress tensor (3.1), that extra term is [11]

$$E_S = -\frac{1-4\xi}{2i}\int_S d\mathbf{S}\cdot\boldsymbol{\nabla}G(x,x')\bigg|_{x'\to x}, \tag{5.3}$$

where the direction of the normal is out of the region in question, which arises from the T^{0i} component of the stress tensor, and from $\partial_\mu T^{\mu\nu}=0$. The total energy in a given region V bounded by a surface S is not, therefore, just the integral of the local energy density, but has this extra contribution [11]:

$$\mathcal{E} = \int_V (d\mathbf{r})\langle T^{00}\rangle + E_S = \frac{1}{2i}\int_V (d\mathbf{r})\int\frac{d\omega}{2\pi}2\omega^2\mathcal{G}(\mathbf{r},\mathbf{r})e^{-i\omega\tau}, \tag{5.4}$$

which is independent of ξ. The latter expression has a rather evident interpretation in terms of summing zero-point energies. We have inserted a temporal point-splitting regulator τ as first introduced in Ref. [22]. The limit $\tau\to 0$ must be taken at the end of the calculation. Of course, the total energy in all space is just the sum of the integrated local energy densities in each region, because the sum of the inside and outside surface energy for each surface is zero, owing to the continuity of the Green's function and its normal derivative across each surface for the nonsingular potential (2.1). This is not the case for the singular potential (1.5).

C. Shell energy

In the limit of $h\to\infty$ for the region in the shell, $a_- < r, r' < a_+$, Eq. (2.10) becomes

$$g_l \to \frac{1}{2\kappa r r'}\frac{e_l(\kappa a)s_l(\kappa a)}{1+\frac{\lambda}{\kappa a^2}e_l(\kappa a)s_l(\kappa a)}\left[\cosh\frac{\sqrt{\lambda h}}{a}(r-r') + \cosh\frac{\sqrt{\lambda h}}{a}(r+r'-a_+-a_-)\right]. \tag{5.5}$$

In the thin shell limit this leads to an energy density in the shell nearly independent of r ($\delta\to 0$), leading to the energy ($y=|x|$, $\epsilon=\tau/a$, τ now being the Euclidean time)

$$E_s = \frac{\lambda}{2\pi a^2}(1-4\xi)\sum_{l=0}^\infty(2l+1)\frac{1}{2}\int_{-\infty}^\infty dx\frac{I_\nu(y)K_\nu(y)}{1+\frac{\lambda}{a}I_\nu(y)K_\nu(y)}e^{ix\epsilon}. \tag{5.6}$$

However, we have to include the surface term in the shell,

$$E_{Ss} = -\frac{1-4\xi}{2i}\int d\mathbf{S}\cdot\boldsymbol{\nabla}G(\mathbf{r},t;\mathbf{r}',t-\tau)\Big|_{\mathbf{r}'=\mathbf{r},r=a_-}^{\mathbf{r}'=\mathbf{r},r=a_+}, \tag{5.7}$$

which exactly cancels this: $E_s + E_{Ss} = 0$, because the total energy of the shell is given by (5.4)

$$E_{s,\text{total}} = \frac{1}{2i}\int_{\text{shell}}(d\mathbf{r})\int\frac{d\omega}{2\pi}2\omega^2\mathcal{G}(\mathbf{r},\mathbf{r})e^{-i\omega\tau}, \tag{5.8}$$

which clearly vanishes as the thickness of the shell $\delta\to 0$. However, we shall see shortly that E_s in (5.6) plays a special role. [The reader should note that if we set $\tau = 0$ expressions such as Eq. (5.6) fail to converge. As we shall see shortly, if this point coincident limit is expanded in powers of λ, divergences in the total energy occur in orders λ and λ^3.]

D. Total energy of λ sphere

Likewise, if one integrates the interior and exterior energy density in the thin shell limit, one gets, for arbitrary ξ,

$$E_{\text{in}} + E_{\text{out}} = \mathcal{E} + E', \tag{5.9}$$

where the total energy is that found in Ref. [11, 12], ($y = |x|$)

$$\mathcal{E} = -\frac{1}{2\pi a}\sum_{l=0}^{\infty}(2l+1)\frac{1}{2}\int_{-\infty}^{\infty}dx\,y\frac{d}{dy}\ln\left[1+\frac{\lambda}{a}I_\nu(y)K_\nu(y)\right]e^{ix\epsilon}, \tag{5.10}$$

where the extra term precisely cancels the sum of the surface terms for the inside and outside regions,

$$E' + E_S = 0, \tag{5.11}$$

and again $\mathcal{E}$ is exactly that obtained from the integral of the Green's function, as in Eq. (5.4).

The above can be verified by computing the pressure. This is obtained by evaluating the radial-radial component of the stress tensor (3.1), or

$$T_{rr} = \frac{1}{2}\left[(\partial_r\phi)^2 + (\partial_0\phi)^2 - (\boldsymbol{\nabla}_\perp\phi)^2\right] - \xi(\partial_0^2 - \nabla_\perp^2)\phi^2, \tag{5.12}$$

or more precisely, the discontinuity of the vacuum expectation value of this across the surface:

$$\begin{aligned} P &= \langle T_{rr}\rangle_{\text{in}} - \langle T_{rr}\rangle_{\text{out}} \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\zeta}{2\pi}\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\left[\partial_r\partial_{r'} - \kappa^2 - \frac{l(l+1)}{r^2}\right]g(r,r')\Big|_{r'=r=a+}^{r'=r=a-}e^{i\zeta\tau}, \end{aligned} \tag{5.13}$$

with $\kappa = |\zeta|$. We note that the ξ dependence drops out immediately, and the terms not involving radial derivatives of Bessel functions cancel between inside and outside, leaving us with, in the thin-shell limit,

$$P = -\frac{\lambda}{8\pi^2 a^5}\frac{1}{2}\int_{-\infty}^{\infty}dx\sum_{l=0}^{\infty}(2l+1)\frac{y[I_\nu(y)K_\nu(y)]' - I_\nu(y)K_\nu(y)}{1+\frac{\lambda}{a}I_\nu(y)K_\nu(y)}e^{ix\epsilon}, \tag{5.14}$$

which, when multiplied by the area of the sphere to give the stress on the surface, is exactly that obtained by differentiating the total energy (5.10) with respect to the radius (with λ fixed),

$$-\frac{\partial \mathcal{E}}{\partial a} = 4\pi a^2 P. \tag{5.15}$$

(The scaling law for λ was incorrectly stated in Refs. [10, 11], as was the sign of the pressure.)

However, there is more to say here. As noted above, the integral of the local energy, inside and outside the sphere, is finite perturbatively, because of cancellations between inside and outside, for the conformally coupled scalar. But it is well known that divergences occur in the total energy at order λ^3 [9, 10, 12]. These evidently must arise from the surface term (5.3). So let us consider the latter, which is given in the outside region by

$$E_S = a^2(1-4\xi)\sum_{l=0}^\infty 2\nu \frac{1}{2}\int_{-\infty}^\infty \frac{d\zeta}{2\pi}\frac{\partial}{\partial r} g_l(r,r')\bigg|_{r=r'=a+} e^{i\zeta\tau}. \tag{5.16}$$

In the strong coupling limit, there is, of course, no surface term. This is because then

$$r, r' > a: \quad g_l(r,r') = \frac{1}{\kappa r r'}\left[s_l(\kappa r_<)e_l(\kappa r_>) - \frac{s_l(\kappa a)}{e_l(\kappa a)} e_l(\kappa r) e_l(\kappa r')\right], \tag{5.17}$$

which vanishes on the surface, and has a derivative proportional to the Wronskian.

In general, in the thin-shell limit, the sum of the inside and outside surface terms is given by

$$E_S = \frac{\lambda}{2\pi a^2}(1-4\xi)\frac{1}{2}\int_{-\infty}^\infty dx \sum_{l=0}^\infty (2l+1)\frac{I_\nu(y)K_\nu(y)}{1+\frac{\lambda}{a}I_\nu(y)K_\nu(y)} e^{ix\epsilon}, \tag{5.18}$$

because the free term in the Green's function cancels between inside and out. Perhaps not remarkably, this is precisely the same as the integrated local shell energy (5.6). Thus the surface energies within and outside the shell regions cancel. (As noted above, this is a general result, because the Green's function and its normal derivative are continuous across each surface at a_+, a_-. We have explicitly verified this using (2.5), (2.10), and the Wronskian for the Bessel functions.) For weak coupling, we expand this in powers of λ. Perhaps the easiest way to isolate the asymptotic behavior is to use the uniform asymptotic expansion,

$$\nu \to \infty: \quad I_\nu(x)K_\nu(x) \sim \frac{t}{2\nu}. \tag{5.19}$$

This yields the following expression for the nth term in the total surface energy, if we set $\epsilon = 0$ and regard n as a continuous variable, analytically continuing from $\mathrm{Re}\, n > 3$,

$$E_S^{(n)} \sim -\frac{(-1)^n}{2\sqrt{\pi}a}(1-4\xi)\left(\frac{\lambda}{2a}\right)^n \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}(2^{n-2}-1)\zeta(n-2). \tag{5.20}$$

Note that this expression vanishes for $n=2$; we must keep subleading corrections to see the order λ^2 term in the energy arising from the surface energy. However, for $n=3$ we obtain for the conformal value, $\xi = 1/6$,

$$E_S^{(3)} \sim \frac{\lambda^3}{24\pi a^4}\zeta(1), \tag{5.21}$$

precisely the divergent term in the energy given in Ref. [11], first found by the heat kernel calculation of Bordag, Kirsten, and Vassilevich [12]. Alternatively if we keep the point-split regulator, we find divergences in both $n = 2$ and $n = 3$:

$$E_S^{(2)} \sim -\frac{\lambda^2}{24\epsilon a^3}, \quad E_S^{(3)} \sim \frac{\lambda^3}{24\pi a^4}\Gamma(0). \tag{5.22}$$

The former might be regarded as a removable contact term since $\epsilon = \tau/a$ and λ/a is a dimensionless coupling, while the latter is divergent even in the presence of the regulator. In any case, as shown in Ref. [11], the total energy term $\mathcal{E}^{(\lambda^2)}$ from Eq. (5.10) is unambiguously finite:

$$\mathcal{E}^{(\lambda^2)} = \frac{\lambda^2}{32\pi a}. \tag{5.23}$$

E. Other values of ξ

We have shown, as expected, that for the global energy any value of ξ can be used. Fulling [21] has suggested that a value different from the conformal value may have advantages. Thus, because the surface (or shell) energy is proportional to $1-4\xi$, it vanishes for $\xi = 1/4$, as do most of the terms in the energy density (3.9). In that case, however, the surface divergences in the local energy density are intensified, which could possibly account for the divergence in the total energy in order λ^3 [although the inverse linear divergence seen in Eq. (4.7b) would cancel between the inside and the outside], but apparently leads to a divergence in λ^2. It is probably not surprising that the situation is most satisfactory only in the conformally coupled case, because it is well known that any other choice leads to more singular quantum corrections [23].

VI. CONCLUSIONS

We have derived the Green's function in general for a massless scalar field for a spherically symmetric step-function potential. By taking the limit as the width of the step function goes to zero, we recover a δ-function shell potential, for which we consider both the weak and strong coupling limits. The latter corresponds to a Dirichlet shell, for which we recover the well-known results for the energy density and total energy. For weak coupling, we derive for the first time the behavior of the energy density as the surface is approached. We also examine the energy content of the shell itself. The inner and outer surfaces of the shell contribute a vanishing net surface energy, but there is a net effective surface energy in the thin shell limit, to be added to the integrated local energy density for the inside and outside regions, which is exactly the integrated local energy density of the shell. This shell energy, for the conformally coupled theory, is finite in second order in the coupling (in at least two plausible regularization schemes), but diverges in third order. We show that the latter precisely corresponds to the known divergence of the total energy in this order. Thus we have established the suspected correspondence between surface divergences and divergences in the total energy, which has nothing to do with divergences in the local energy density as the surface is approached. This precise correspondence should enable us to absorb such divergences in a renormalization of the surface energy, and should lead to further advances of our understanding of quantum vacuum effects.

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