Multiple Scattering Methods in Casimir Calculations

Kimball A. Milton* and Jef Wagner†

Oklahoma Center for High Energy Physics and Homer L. Dodge Department of Physics and Astronomy, University of Oklahoma, Norman, OK 73019, USA

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Multiple scattering formulations have been recently rediscovered as a method of studying the quantum vacuum or Casimir interactions between distinct bodies. The methods are hardly new, but increased computing power and advances in understanding allow us to extract information efficiently. Here we review the method in the simple context of δ-function potentials, so-called semitransparent bodies. (In the limit of strong coupling, a semitransparent boundary becomes a Dirichlet one.) After applying the method to rederive the Casimir force between two semitransparent plates and the Casimir self-stress on a semitransparent sphere, we obtain expressions for the Casimir energies between disjoint parallel semitransparent cylinders and between disjoint semitransparent spheres. Simplifications occur for weak and strong coupling. In particular, after performing a power series expansion in the ratio of the radii of the objects to the separation between them, we are able to sum the weak-coupling expansions exactly to obtain explicit closed forms for the Casimir interaction energy. The same can be done for the interaction of a weak-coupling sphere or cylinder with a Dirichlet plane. We show that the proximity force approximation (PFA), which becomes the proximity force theorem when the objects are almost touching, is very poor for finite separations.

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I. INTRODUCTION

Recently, there has been a flurry of papers concerning "exact" methods of calculating Casimir energies or forces between arbitrary distinct bodies. Most notable is the recent paper by Emig, Graham, Jaffe, and Kardar [1]. (Details, applied to a scalar field, are supplied in Ref. [2]. See also Refs. [3, 4].) In fact it is clear that the methods are not so novel: Certainly the multiple scattering method was explicit in the famous papers of Balian and Duplantier [5–7]. Multipole expansion methods can be traced back at least as far as Sommerfeld [8]. Most explicitly, an earlier drafted paper by Kenneth and Klich [9] appeared which shows that the basis of the approach lies in the Lippmann-Schwinger formulation of scattering theory [10]. In fact, the most related precursors seem to be papers by Bulgac, Marierski, and Wirzba [11–13] and by Bordag [14, 15]. See also Reynaud et al. [16] and references therein.

In fact, Emig and earlier collaborators [17, 18] have published a series of papers, using closely related methods to calculate numerically forces between distinct bodies, starting from periodically deformed ones. They start from the change in the density of states,

$$E = \frac{\hbar c}{2}\int_0^\infty dq\, q\, \delta\rho(q), \tag{1.1a}$$

$$\delta\rho(q) = -\frac{1}{\pi}\frac{\partial}{\partial q}\mathrm{Tr}\ln \mathcal{M}\mathcal{M}_\infty^{-1}, \tag{1.1b}$$

where the matrix operator $\mathcal{M}$ is given by the Euclidean Green's function

$$G_0(\mathbf{x},\mathbf{x}',q) = \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}e^{-q|\mathbf{x}-\mathbf{x}'|} \tag{1.2}$$

evaluated on the (Dirichlet, for example) surfaces. $\mathcal{M}_\infty$ is defined at infinite surface separation. Emig first used this method to calculate the force between corrugated surfaces [19]. They later used the technique to calculate the exact force between a cylinder (radius a) and a plate (distance of closest approach d) [18]. The determinant is obtained by a truncation on partial waves; $l = 25$ is sufficient even for $d/a = 0.1$. Strong deviation from the proximity force

*Electronic address: milton@nhn.ou.edu; URL: http://www.nhn.ou.edu/%7Emilton
†Electronic address: wagner@nhn.ou.edu

approximation (PFA) is seen for $d/a \geq 1$. As Gies and Klingmüller note [20], 1% deviations from the PFA occur when the ratio of the distance between the cylinder and the plate to the radius of the cylinder exceeds 0.01. We will not discuss the worldline method of Gies and collaborators [21–23] further, as that method lies rather outside our discussion here. Bordag [14] has precisely quantified the first correction to the PFA both for a cylinder and a sphere near a plane.

Bulgac et al. [11–13] use a modified Krein formula [24] for the change in the density of states at energy ϵ due to the presence of N scatterers,

$$\delta g(\epsilon) = \frac{1}{2\pi i}\frac{d}{d\epsilon}\ln\det S_N(\epsilon), \tag{1.3}$$

where S_N is the scattering matrix for N point scatterers. This leads to

$$E = \frac{\hbar c}{2\pi}\int_0^\infty dk_4 \ln\det M(ik_4), \tag{1.4}$$

where M is the multiple-scattering matrix, an energy integral over the multiple-scattering phase shift. They obtain results for the interaction of two spheres, or a sphere and a plate (Dirichlet), stating that "The exact results ... are easy to calculate and definitely simpler to evaluate than in a path integral approach." Further, "Proximity formula and the semiclassical/orbit approaches are limited to small separations only, typically much smaller than the curvature radii of the two surfaces." [11]

Bordag [14, 15] rederives the representation for the Casimir energy found by Bulgac et al. [11], and by Emig et al. [17, 18], using a path integral approach: (τ is the time the configuration exists)

$$E = \frac{1}{2\tau}\mathrm{Tr}\ln K, \tag{1.5a}$$

$$K(z,z') = \int dx\,dy\,H(x,z)G_0(x,y)H(y,z'), \tag{1.5b}$$

where G_0 is the free propagator and H is the surface profile function,

$$H(x,z) = \delta(x - f(z)). \tag{1.5c}$$

He obtains an exact expression for the interaction between a cylinder and a plane, with thereby corrections to the PFA, for both TE and TM modes, agreeing with Emig et al. For a sphere and a plane he also obtains the large-separation limit, agreeing with Bulgac et al., as well as corrections to the PFA. He uses the same method to calculate the first correction to the PFA for a cylindrical graphene sheet in front of a flat graphene sheet or dielectric plate.

Dalvit et al. [25, 26] use the argument principle to calculate the interaction between conducting cylinders (of length L) with parallel axes,

$$E_{12} = \frac{\hbar c L}{4\pi}\int_0^\infty dy\,y\ln M(iy), \tag{1.6}$$

where M is a function which vanishes at the eigenvalues. This is essentially an exactly solvable model. They only consider a configuration in which one cylinder is contained within another.

Capasso et al. [27] calculate forces from stress tensors using the familiar construction of the stress tensor in terms of Green's dyadics [28, 29]. They use a numerical engineering method: Finite-difference frequency-domain methods are employed in two dimensions to obtain forces between metal squares and plates to 3% accuracy, "using reasonable computational resources." There may be problems with scalability of this procedure.

It is clear, then, with the exception of the last method, these approaches are fundamentally equivalent. We will refer to all of these methods as multiple scattering techniques. We will now proceed to state the formulation in a simple, straightforward way, and apply it to various situations, all characterized by δ-function potentials. (A preliminary version of some of our results has already appeared [30].)

II. FORMALISM

We begin by noting that, as others have observed, the derivation of the chief result of Emig et al. [1] is much more general than that given in their paper. In fact, it is a consequence of the general formula for Casimir energies (for simplicity here we restrict attention to a massless scalar field) [31]

$$E = \frac{i}{2\tau}\mathrm{Tr}\ln G, \tag{2.1}$$

where τ is the "infinite" time that the configuration exists, and G is the Green's function in the presence of a potential V satisfying (matrix notation)

$$(-\partial^2 + V)G = 1, \tag{2.2}$$

subject to some boundary conditions at infinity. (For example, we can use causal or Feynman boundary conditions, or, alternatively, retarded Green's functions.) In Appendix A we give a heuristic derivation of this fundamental formula.

The above formula for the Casimir energy is defined up to an infinite constant, which can be at least partially compensated by inserting a factor as do Kenneth and Klich [9]:

$$E = \frac{i}{2\tau}\operatorname{Tr}\ln GG_0^{-1}. \tag{2.3}$$

Here G_0 satisfies, with the same boundary conditions as G, the free equation

$$-\partial^2 G_0 = 1. \tag{2.4}$$

Now we define the T-matrix (note that our definition of T differs by a factor of 2 from that in Ref. [1])

$$T = S - 1 = V(1 + G_0 V)^{-1}. \tag{2.5}$$

We then follow standard scattering theory [10], as reviewed in Kenneth and Klich [9]. (Note that there seem to be some sign and ordering errors in that reference.) The Green's function can be alternatively written as

$$G = G_0 - G_0 T G_0 = \frac{1}{1 + G_0 V} G_0 = V^{-1} T G_0, \tag{2.6}$$

which results in two formulæ for the Casimir energy

$$E = \frac{i}{2\tau}\operatorname{Tr}\ln\frac{1}{1 + G_0 V} \tag{2.7a}$$

$$= \frac{i}{2\tau}\operatorname{Tr}\ln V^{-1} T. \tag{2.7b}$$

If the potential has two disjoint parts,

$$V = V_1 + V_2, \tag{2.8}$$

it is easy to show that

$$T = (V_1 + V_2)(1 - G_0 T_1)(1 - G_0 T_1 G_0 T_2)^{-1}(1 - G_0 T_2), \tag{2.9}$$

where

$$T_i = V_i(1 + G_0 V_i)^{-1}, \quad i = 1, 2. \tag{2.10}$$

Thus, we can write the general expression for the interaction between the two bodies (potentials) in two alternative forms:

$$E_{12} = -\frac{i}{2\tau}\operatorname{Tr}\ln(1 - G_0 T_1 G_0 T_2) \tag{2.11a}$$

$$= -\frac{i}{2\tau}\operatorname{Tr}\ln(1 - V_1 G_1 V_2 G_2), \tag{2.11b}$$

where

$$G_i = (1 + G_0 V_i)^{-1} G_0, \quad i = 1, 2. \tag{2.12}$$

The first form is exactly that given by Emig et al. [1], and by Kenneth and Klich [9], while the latter is actually easily used if we know the individual Green's functions. (The effort involved in calculating with either is identical.) In fact, the general form (2.11a) was recognized earlier and applied to planar geometries by Maia Neto, Lambrecht, and Reynaud [16, 32, 33].

III. CASIMIR INTERACTION BETWEEN δ-PLATES

We now use the second formula above (2.11b) to calculate the Casimir energy between two parallel semitransparent plates, with potential

$$V = \lambda_1 \delta(z - z_1) + \lambda_2 \delta(z - z_2), \tag{3.1}$$

where the dimension of λ_i is L^{-1}. The free reduced Green's function is (where we have performed the evident Fourier transforms in time and the transverse directions)

$$g_0(z, z') = \frac{1}{2\kappa} e^{-\kappa|z-z'|}, \quad \kappa^2 = \zeta^2 + k^2. \tag{3.2}$$

Here $\mathbf{k} = \mathbf{k}_\perp$ is the transverse momentum, and $\zeta = -i\omega$ is the Euclidean frequency. The Green's function associated with a single δ-function potential is

$$g_i(z, z') = \frac{1}{2\kappa} \left(e^{-\kappa|z-z'|} - \frac{\lambda_i}{\lambda_i + 2\kappa} e^{-\kappa|z-z_i|} e^{-\kappa|z'-z_i|} \right). \tag{3.3}$$

Then the energy/area is

$$\mathcal{E} = \frac{1}{16\pi^3} \int d\zeta \int d^2k \int dz \ln(1 - A)(z, z), \tag{3.4}$$

where, in virtue of the δ-function potentials ($a = |z_2 - z_1|$)

$$\begin{aligned} A(z, z') &= \frac{\lambda_1 \lambda_2}{4\kappa^2} \delta(z - z_1) \left(1 - \frac{\lambda_1}{\lambda_1 + 2\kappa}\right) e^{-\kappa|z_1 - z_2|} \left(1 - \frac{\lambda_2}{\lambda_2 + 2\kappa}\right) e^{-\kappa|z' - z_2|} \\ &= \frac{\lambda_1}{\lambda_1 + 2\kappa} \frac{\lambda_2}{\lambda_2 + 2\kappa} e^{-\kappa a} e^{-\kappa|z'-z_2|} \delta(z - z_1). \end{aligned} \tag{3.5}$$

We expand the logarithm according to

$$\ln(1 - A) = -\sum_{s=1}^{\infty} \frac{A^s}{s}. \tag{3.6}$$

For example, the leading term is easily seen to be

$$\mathcal{E}^{(2)} = -\frac{\lambda_1 \lambda_2}{16\pi^3} \int \frac{d\zeta\, d^2k}{4\kappa^2}\, e^{-2\kappa a} = -\frac{\lambda_1 \lambda_2}{32\pi^2 a}, \tag{3.7}$$

which uses the change to polar coordinates,

$$d\zeta\, d^2k = d\kappa\, \kappa^2\, d\Omega. \tag{3.8}$$

In general, it is easy to check that, because $A(z, z')$ factorizes here, $A(z, z') = B(z)C(z')$, $\operatorname{Tr} A^n = (\operatorname{Tr} A)^n$, or

$$\operatorname{Tr} \ln(1 - A) = \ln(1 - \operatorname{Tr} A), \tag{3.9}$$

so the Casimir interaction between the two semitransparent plates is

$$\mathcal{E} = \frac{1}{4\pi^2} \int_0^\infty d\kappa\, \kappa^2 \ln \left(1 - \frac{\lambda_1}{\lambda_1 + 2\kappa} e^{-\kappa a} \frac{\lambda_2}{\lambda_2 + 2\kappa} e^{-\kappa a}\right), \tag{3.10}$$

which is exactly the well-known result [34].

IV. CASIMIR SELF-ENERGY FOR A SINGLE SEMITRANSPARENT SPHERE

Before we embark on new calculations, let us also confirm the known result for the self-stress on a single sphere of radius a using this formalism. (This demonstrates, as did the rederivation of the Boyer result [35] by Balian and

Duplantier [6], that the multiple scattering method is equally applicable to the calculation of self-energies.) We start from the general formula (2.7a), where

$$V(\mathbf{r},\mathbf{r}') = \lambda\delta(r-a)\delta(\mathbf{r}-\mathbf{r}'). \tag{4.1}$$

We use the Fourier representation for the propagator in Euclidean space,

$$G_0(\mathbf{r},\mathbf{r}') = \frac{e^{-|\zeta||\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = \int \frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^2+\zeta^2}, \tag{4.2}$$

as well as the partial wave expansion of the plane wave

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{lm} 4\pi i^l j_l(kr) Y_{lm}(\hat{\mathbf{r}}) Y^*_{lm}(\hat{\mathbf{k}}). \tag{4.3}$$

Then, from the orthonormality of the spherical harmonics,

$$\int d\hat{\mathbf{k}} Y^*_{lm}(\hat{\mathbf{k}}) Y_{l'm'}(\hat{\mathbf{k}}) = \delta_{ll'}\delta_{mm'}, \tag{4.4}$$

we obtain the representation

$$G_0(\mathbf{r},\mathbf{r}') = \frac{2}{\pi}\sum_{lm}\int_0^\infty \frac{dk\, k^2}{k^2+\zeta^2} j_l(kr) j_l(kr') Y_{lm}(\hat{\mathbf{r}}) Y^*_{lm}(\hat{\mathbf{r}}'). \tag{4.5}$$

Now we combine the representation for the free Green's function with the spherical potential (4.1) to obtain

$$(G_0 V)(\mathbf{r},\mathbf{r}') = \frac{2\lambda}{\pi}\delta(r'-a)\sum_{lm}\int_0^\infty \frac{dk\, k^2}{k^2+\zeta^2} j_l(ka) j_l(kr) Y_{lm}(\hat{\mathbf{r}}) Y^*_{lm}(\hat{\mathbf{r}}'). \tag{4.6}$$

When this, or powers of this, is traced (that is, $\mathbf{r}$ and $\mathbf{r}'$ are set equal, and integrated over), we obtain a poorly defined expression; to regulate this, we assume $r \neq a$, for example, $r < a$. (This is a type of point-split regulation.) Then, because

$$j_l(ka) = \frac{1}{2}\left(h_l^{(1)}(ka) + h_l^{(2)}(ka)\right) = \frac{1}{2}\left(h_l^{(1)}(ka) + (-1)^l h_l^{(1)}(-ka)\right), \tag{4.7}$$

while $j_l(kr) = (-1)^l j_l(-kr)$, we see that the k integration in Eq. (4.6) can be evaluated as[1]

$$\int_0^\infty \frac{dk\, k^2}{k^2+\zeta^2} j_l(ka) j_l(kr) = \frac{\pi}{a} K_{l+1/2}(|\zeta|a) I_{l+1/2}(|\zeta|r), \quad r < a. \tag{4.8}$$

Thus, it is easily seen that an arbitrary power of G_0V has trace

$$\mathrm{Tr}\,(G_0V)^n = (\lambda a)^n \sum_{lm}\left(K_{l+1/2}(|\zeta|a) I_{l+1/2}(|\zeta|a)\right)^n, \tag{4.9}$$

and that therefore the total self-energy of the semitransparent sphere is given by the well-known expression [36, 37]

$$E = \frac{1}{2\pi a}\sum_{l=0}^\infty (2l+1)\int_0^\infty dx \ln\left(1+\lambda a I_{l+1/2}(x) K_{l+1/2}(x)\right), \quad x = |\zeta|a. \tag{4.10}$$

Actually, a slightly different form involving integration by parts was given in Refs. [38, 39], which results in the energy being finite though order λ^2.

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[1] Of course, this result is the immediate consequence of the usual partial wave expansion

$$G_0(\mathbf{r},\mathbf{r}') = ik\sum_{lm} j_l(kr_<) h_l^{(1)}(kr_>) Y_{lm}(\hat{\mathbf{r}}) Y^*_{lm}(\hat{\mathbf{r}}'), \quad k = |\omega|.$$

The point of our slightly more elaborate approach here is that it generalizes to the corresponding two-body case—see Eq. (5.7).

V. $2+1$ SPATIAL GEOMETRIES

We now proceed to apply this method to the interaction between bodies, which leads, for example, as Emig et al. [1, 2] point out, to a multipole expansion. In this section we illustrate this idea with a $2+1$ dimensional version, which allows us to describe, for example, cylinders with parallel axes. We seek an expansion of the free Green's function for $\mathbf{R} = \mathbf{R}_\perp$ entirely in the x-y plane,

$$G_0(\mathbf{R}+\mathbf{r}'-\mathbf{r}) = \frac{e^{i|\omega||\mathbf{r}-\mathbf{R}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{R}-\mathbf{r}'|} = \int \frac{dk_z}{2\pi} e^{ik_z(z-z')} g_0(\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp), \tag{5.1}$$

where the reduced Green's function is

$$g_0(\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp) = \int \frac{(d^2k_\perp)}{(2\pi)^2} \frac{e^{-i\mathbf{k}_\perp\cdot\mathbf{R}_\perp} e^{i\mathbf{k}_\perp\cdot(\mathbf{r}_\perp - \mathbf{r}'_\perp)}}{k_\perp^2 + k_z^2 + \zeta^2}. \tag{5.2}$$

As long as the two potentials do not overlap, so that we have $\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp \neq 0$, we can write an expansion in terms of modified Bessel functions:

$$g_0(\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp) = \sum_{m,m'} I_m(\kappa r) e^{im\phi} I_{m'}(\kappa r') e^{-im'\phi'} \tilde{g}^0_{m,m'}(\kappa R), \quad \kappa^2 = k_z^2 + \zeta^2. \tag{5.3}$$

By Fourier transforming, and using the definition of the Bessel function

$$i^m J_m(kr) = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-im\phi} e^{ikr\cos\phi}, \tag{5.4}$$

we easily find

$$\tilde{g}^0_{m,m'}(\kappa R) = \frac{1}{2\pi} \int_0^\infty \frac{dk\,k}{k^2+\kappa^2} J_{m-m'}(kR) \frac{J_m(kr) J_{m'}(kr')}{I_m(\kappa r) I_{m'}(\kappa r')}, \tag{5.5}$$

which is in fact independent of r, r'.

As in the previous section, the k integral here can actually be evaluated as a contour integral, as Bordag noted [14]. No point-splitting is required here, because the bodies are non-overlapping, so $r/R, r'/R < 1$. We write the dominant Bessel function in terms of Hankel functions,

$$J_{m-m'}(x) = \frac{1}{2}\left[H^{(1)}_{m-m'}(x) + H^{(2)}_{m-m'}(x)\right] = \frac{1}{2}\left[H^{(1)}_{m-m'}(x) + (-1)^{m-m'+1} H^{(1)}_{m-m'}(-x)\right], \tag{5.6}$$

and then we can carry out the integral over k by closing the contour in the upper half plane. We are left with

$$\int_0^\infty \frac{dx\,x}{x^2+y^2} J_{m-m'}(x) J_m(xr/R) J_{m'}(xr'/R) = (-1)^{m'} K_{m-m'}(y) I_m(yr/R) I_{m'}(yr'/R), \tag{5.7}$$

and therefore the reduced Green's function has the simple form

$$\tilde{g}^0_{m,m'}(\kappa R) = \frac{(-1)^{m'}}{2\pi} K_{m-m'}(\kappa R). \tag{5.8}$$

Thus we can derive an expression for the interaction energy per unit length between two bodies, in terms of discrete matrices,

$$\mathfrak{E} \equiv \frac{E_{\text{int}}}{L} = \frac{1}{8\pi^2} \int d\zeta\, dk_z \ln\det\left(1 - \tilde{g}^0 t_1 \tilde{g}^{0\top} t_2\right), \tag{5.9}$$

where $\top$ denotes transpose, and where the T matrix elements are given by

$$t_{mm'} = \int dr\, r\, d\phi \int dr'\, r'\, d\phi' I_m(\kappa r) e^{-im\phi} I_{m'}(\kappa r') e^{im'\phi'} T(r,\phi; r',\phi'). \tag{5.10}$$

A. Interaction between semitransparent cylinders

Consider, as an example, two parallel semitransparent cylinders, of radii a and b, respectively, lying outside each other, described by the potentials

$$V_1 = \lambda_1 \delta(r-a), \quad V_2 = \lambda_2 \delta(r'-b), \tag{5.11}$$

with the separation between the centers R satisfying $R > a+b$. It is easy to work out the scattering matrix in this situation,

$$T_1 = V_1 - V_1 G_0 V_1 + V_1 G_0 V_1 G_0 V_1 - \dots, \tag{5.12}$$

so the matrix element is easily seen to be

$$(t_1)_{mm'} = 2\pi\lambda_1 a \delta_{mm'} \frac{I_m^2(\kappa a)}{1+\lambda_1 a I_m(\kappa a) K_m(\kappa a)}. \tag{5.13}$$

Again, we used here the regularized integral[2]

$$\int_0^\infty \frac{dk\,k}{k^2+\kappa^2} J_m(kr) J_m(kr') = K_m(\kappa r') I_m(\kappa r), \quad r < r'. \tag{5.14}$$

Thus the Casimir energy per unit length is

$$\mathfrak{E} = \frac{1}{4\pi}\int_0^\infty d\kappa\,\kappa\,\mathrm{tr}\ln(1-A), \tag{5.15}$$

where

$$A = B(a)B(b), \tag{5.16}$$

in terms of the matrices

$$B_{mm'}(a) = K_{m+m'}(\kappa R)\frac{\lambda_1 a I_{m'}^2(\kappa a)}{1+\lambda_1 a I_{m'}(\kappa a) K_{m'}(\kappa a)}. \tag{5.17}$$

B. Interaction between cylinder and plane

As a check, let us rederive the result derived by Bordag [14] for a cylinder in front of a Dirichlet plane perpendicular to the x axis. We start from the interaction (2.11a) written in terms of $\bar{G}_2$, the deviation from the free Green's function induced by a single potential,

$$\bar{G}_2 = G_2 - G_0 = -G_0 T_2 G_0, \tag{5.18}$$

so the interaction energy has the form

$$E = -\frac{i}{2\tau}\mathrm{Tr}\ln(1+T_1\bar{G}_2). \tag{5.19}$$

When the second body is a Dirichlet plane, $\bar{G}$ may be found by the method of images, with the origin taken at the center of the cylinder,

$$\bar{G}(\mathbf{r},\mathbf{r}') = -G_0(\mathbf{r},\bar{\mathbf{r}}'), \quad \bar{\mathbf{r}}' = (R-x', y', z'), \tag{5.20}$$

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[2] Again, this is equivalent to the use of the two-dimensional Green's function

$$H_0(kP) = \sum_{m=-\infty}^{\infty} i^m e^{-im\phi'} J_m(k\rho') e^{im\phi} H_m^{(1)}(k\rho), \quad \rho' < \rho,$$

where $P = \sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}$.

where R is the distance between the center of the cylinder and its image at $\mathbf{R}_\perp$, that is, $R/2$ is the distance between the center of the cylinder and the plane. (We keep R here, rather than $R/2 = D$, because of the close connection to the two cylinder case.) Now we encounter the 2-dimensional Green's function

$$g(\mathbf{r}_\perp + \mathbf{r}'_\perp - \mathbf{R}_\perp) = \sum_{mm'} I_m(\kappa r) I_{m'}(\kappa r') e^{im\phi} e^{im'\phi'} g_{mm'}(\kappa R), \tag{5.21}$$

(because the cylinder has $y \to -y$ reflection symmetry) where the argument given above yields

$$g_{mm'}(\kappa R) = \frac{1}{2\pi} K_{m+m'}(\kappa R). \tag{5.22}$$

Thus the interaction between the semitransparent cylinder and a Dirichlet plane is

$$\mathfrak{E} = \frac{1}{4\pi} \int_0^\infty \kappa\, d\kappa\, \mathrm{tr}\, \ln(1 - B(a)), \tag{5.23}$$

where $B(a)$ is given by Eq. (5.17). In the strong-coupling limit this result agrees with that given by Bordag, because

$$\mathrm{tr}\, B^s = \mathrm{tr}\, \tilde{B}^s, \quad \tilde{B}_{mm'} = \frac{1}{K_m(\kappa a)} K_{m+m'}(\kappa R) I_{m'}(\kappa a). \tag{5.24}$$

C. Weak-coupling

In weak coupling, the formula (5.15) for the interaction energy between two cylinders is

$$\mathfrak{E} = -\frac{\lambda_1 \lambda_2 ab}{4\pi R^2} \sum_{m,m'=-\infty}^{\infty} \int_0^\infty dx\, x\, K^2_{m+m'}(x) I^2_m(xa/R) I^2_{m'}(xb/R). \tag{5.25}$$

Similarly, the energy of interaction between a weakly-coupled cylinder and a Dirichlet plane is from Eq. (5.23)

$$\mathfrak{E} = -\frac{\lambda a}{4\pi R^2} \sum_{m=-\infty}^{\infty} \int_0^\infty dx\, x\ K_{2m}(x) I^2_m(xa/R). \tag{5.26}$$

D. Power series expansion

It is straightforward to develop a power series expansion for the interaction between weakly-coupled semitransparent cylinders. One merely exploits the small argument expansion for the modified Bessel functions $I_m(xa/R)$ and $I_{m'}(xb/R)$:

$$I^2_m(x) = \left(\frac{x}{2}\right)^{2|m|} \sum_{n=0}^{\infty} Z_{|m|,n} \left(\frac{x}{2}\right)^{2n}, \tag{5.27}$$

where the coefficients $Z_{m,n}$ are

$$\begin{aligned} Z_{m,n} &= \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!\,\Gamma(k+m+1)\,\Gamma(n-k+m+1)} \\ &= \frac{2^{2(m+n)}\,\Gamma\left(m+n+\frac{1}{2}\right)}{\sqrt{\pi}\, n!\,(2m+n)!\,\Gamma(m+n+1)}. \end{aligned} \tag{5.28}$$

The Casimir energy per unit length (5.25) is now given as

$$\mathfrak{E} = -\frac{\lambda_1 a \lambda_2 b}{4\pi R^2} \int_0^\infty \mathrm{d}x\ x \sum_{m=-\infty}^{\infty} \sum_{m'=-\infty}^{\infty} \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \left(\frac{xa}{2R}\right)^{2|m|} Z_{|m|,n} \left(\frac{xa}{2R}\right)^{2n} \left(\frac{xb}{2R}\right)^{2|m'|} Z_{|m'|,n'} \left(\frac{xb}{2R}\right)^{2n'} K^2_{m+m'}(x). \tag{5.29}$$

Reordering terms gives a more compact formula

$$\mathfrak{E} = -\frac{\lambda_1 a \lambda_2 b}{4\pi R^2} \sum_{m=-\infty}^{\infty} \sum_{m'=-\infty}^{\infty} \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} Z_{|m|,n} \left(\frac{a}{R}\right)^{2(|m|+n)} Z_{|m'|,n'} \left(\frac{b}{R}\right)^{2(|m'|+n')} J_{|m|+|m'|+n+n',m+m'}, \tag{5.30}$$

where the two index symbol $J_{p,q}$ represents the integral over x, which evaluates to

$$J_{p,q} = 2\int_0^\infty dx \left(\frac{x}{2}\right)^{2p+1} K_q^2(x) = \frac{\sqrt{\pi}\, p!\, \Gamma(p+q+1)\Gamma(p-q+1)}{2^{2p+2}\Gamma\left(p+\frac{3}{2}\right)}. \tag{5.31}$$

In order to simplify the power series expansion in terms of $\frac{a}{R}$ and $\frac{b}{R}$ we need to reorder the m sums so that only non-negative values of m appear. There are several ways to break up the m sums; one of them is to decompose the sum into the $m = m' = 0$ term, the m, m' same sign terms, and the m, m' different sign terms, giving

$$\begin{aligned}\mathfrak{E} = -\frac{\lambda_1 a \lambda_2 b}{4\pi R^2} \Bigg[&\sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} Z_{0,n} \left(\frac{a}{R}\right)^{2n} Z_{0,n'} \left(\frac{b}{R}\right)^{2n'} J_{n+n',0} \\ &+2 \sum_{m=1}^{\infty} \sum_{m'=0}^{\infty} \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} Z_{m,n} \left(\frac{a}{R}\right)^{2(m+n)} Z_{m',n'} \left(\frac{b}{R}\right)^{2(m'+n')} J_{m+m'+n+n',m+m'} \\ &+2 \sum_{m=0}^{\infty} \sum_{m'=1}^{\infty} \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} Z_{m,n} \left(\frac{a}{R}\right)^{2(m+n)} Z_{m',n'} \left(\frac{b}{R}\right)^{2(m'+n')} J_{m+m'+n+n',m-m'} \Bigg]. \end{aligned} \tag{5.32}$$

It is now possible to combine the multiple infinite power series into a single infinite power series, where each term is given by (possible multiple) finite sum(s). In this case we get an amazingly simple result

$$\mathfrak{E} = -\frac{\lambda_1 a \lambda_2 b}{4\pi R^2} \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{a}{R}\right)^{2n} P_n(\mu), \tag{5.33}$$

where $\mu = b/a$, and where by inspection we identify the binomial coefficients

$$P_n(\mu) = \sum_{k=0}^{n} \binom{n}{k}^2 \mu^{2k}. \tag{5.34}$$

Remarkably, it is possible to perform the sums [40], so we obtain the following closed form for the interaction between two weakly-coupled cylinders:

$$\mathfrak{E} = -\frac{\lambda_1 a \lambda_2 b}{8\pi R^2} \left[\left(1 - \left(\frac{a+b}{R}\right)^2\right) \left(1 - \left(\frac{a-b}{R}\right)^2\right) \right]^{-1/2}. \tag{5.35}$$

We note that in the limit $R - a - b = d \to 0$, d being the distance between the closest points on the two cylinders, we recover the proximity force theorem in this case (B4),

$$U(d) = -\frac{\lambda_1 \lambda_2}{32\pi} \sqrt{\frac{2ab}{R}} \frac{1}{d^{1/2}}, \quad d \ll a, b. \tag{5.36}$$

In Figs. 1–2 we compare the exact energy (5.35) with the proximity force approximation (5.36). Evidently, the former approaches the latter when the sum of the radii $a+b$ of the cylinders approaches the distance R between their centers. The rate of approach is linear (with slope $3/2$) for the equal radius case, but with slope $b^2/4a^2$ when $a \ll b$. More precisely, the ratio of the exact energy to the PFA is

$$\frac{\mathfrak{E}}{U} \approx 1 - \frac{1+\mu+\mu^2}{4\mu} \frac{d}{R} \approx 1 - \frac{R^2 - aR + a^2}{4a(R-a)} \frac{d}{R}. \tag{5.37}$$

This correction to the PFA is derived by another method in Appendix C. The reader should note that the the PFA is actually only defined in the limit $d \to 0$, so the functional form away from that point is ambiguous. Corrections to the PFA depend upon the specific form assumed for $U(d)$.

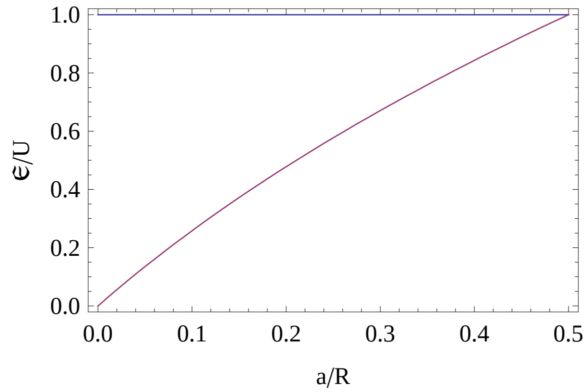

FIG. 1: Plotted is the ratio of the exact interaction energy (5.35) of two weakly-coupled cylinders to the proximity force approximation (5.36) as a function of the cylinder radius a for $a = b$.

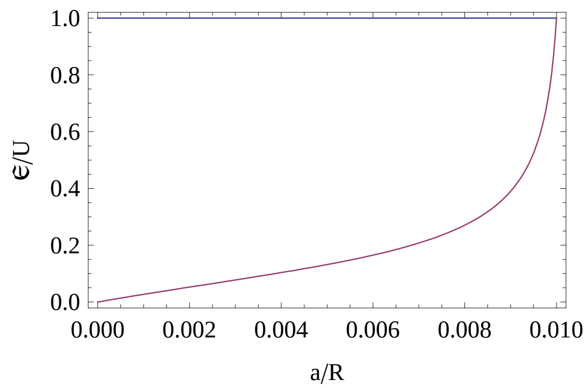

FIG. 2: Plotted is the ratio of the exact interaction energy (5.35) of two weakly-coupled cylinders to the proximity force approximation (5.36) as a function of the cylinder radius a for $b/a = 99$.

E. Exact result for interaction between plane and cylinder

In exactly the same way, starting from Eq. (5.26), we can obtain a closed-form result for the interaction energy between a Dirichlet plane and a weakly-coupled cylinder of radius a separated by a distance $R/2$. The result is again quite simple:

$$\mathfrak{E} = -\frac{\lambda a}{4\pi R^2}\left[1-\left(\frac{2a}{R}\right)^2\right]^{-3/2}. \tag{5.38}$$

In the limit as $d \to 0$, this agrees with the PFA:

$$U(d) = -\frac{\lambda}{64\pi}\frac{\sqrt{2a}}{d^{3/2}}. \tag{5.39}$$

Note again that this form is ambiguous: the proximity force theorem is equally well satisfied if we replace a by $R/2$, for example, in $U(d)$. The comparison between this PFA and the exact result (5.38) is given in Fig. 3.

F. Strong coupling (Dirichlet) limit

The interaction between Dirichlet cylinders is given by Eq. (5.15) in the limit $\lambda_1 = \lambda_2 \to \infty$, that is

$$\mathfrak{E} = \frac{1}{4\pi R^2}\int_0^\infty dx\, x\, \mathrm{tr}\ln(1-A), \tag{5.40a}$$

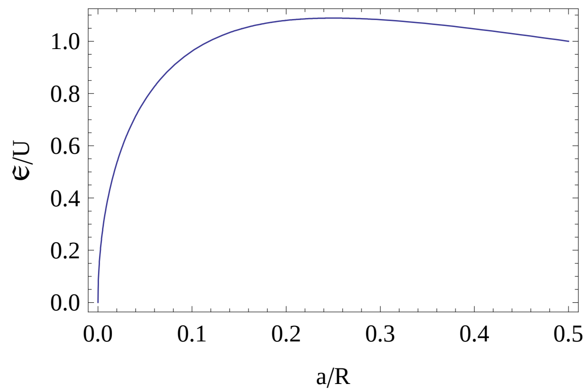

FIG. 3: Plotted is the ratio of the exact interaction energy (5.38) of a weakly-coupled cylinder above a Dirichlet plane to the proximity force approximation (5.39) as a function of the cylinder radius a.

where

$$A_{mm'} = \sum_{m''} K_{m+m''}(x) K_{m''+m'}(x) \frac{I_{m''}(xa/R)}{K_{m''}(xa/R)} \frac{I_{m'}(xb/R)}{K_{m'}(xb/R)}. \tag{5.40b}$$

Here the trace of the logarithm can be interpreted as in Eq. (3.6).

Because it no longer appears possible to obtain a closed-form solution, we want to verify analytically that as the surfaces of the two cylinders nearly touch, we recover the result of the proximity force theorem. We use a variation of the scheme explained by Bordag for a cylinder next to a plane [14]. [The analysis is a bit simpler in the weak-coupling case, which leads to Eq. (5.36). See Appendix C.] First we replace the products of Bessel functions in A by their leading uniform asymptotic approximants for all m's large:

$$B_{mm''}(a)B_{m''m'}(b) \sim \frac{1}{2\pi} \frac{1}{\sqrt{m+m''}} \frac{1}{\sqrt{m'+m''}} \left(1 + \left(\frac{x}{m+m''}\right)^2\right)^{-1/4} \left(1 + \left(\frac{x}{m'+m''}\right)^2\right)^{-1/4} e^{-\chi}, \tag{5.41}$$

where the exponent is

$$\chi = (m+m'')\eta\left(\frac{x}{m+m''}\right) + (m'+m'')\eta\left(\frac{x}{m'+m''}\right) - 2m''\eta\left(\frac{xa}{m''R}\right) - 2m'\eta\left(\frac{xb}{m'R}\right), \tag{5.42}$$

in terms of

$$\eta(z) = t^{-1} + \ln\frac{z}{1+t^{-1}}, \quad \eta'(z) = \frac{1}{zt}, \quad \eta''(z) = -\frac{t}{z^2}, \tag{5.43}$$

and

$$t = (1+z^2)^{-1/2}. \tag{5.44}$$

We write the trace of the sth power of A as (summed on repeated indices)

$$(A^s)_{m_1m_1} = B_{m_1m_1'}(a)B_{m_1'm_2}(b)B_{m_2m_2'}(a)B_{m_2'm_3}(b)\cdots B_{m_sm_s'}(a)B_{m_s'm_1}(b). \tag{5.45}$$

We rescale variables in terms of a large variable M and relatively small variables:

$$m_i' = M\alpha_i, \quad m_i = M\beta_i, \tag{5.46}$$

where without loss of generality we take only $2s-1$ of the α's and β's as independent:

$$\sum_{i=1}^{s}(\alpha_i + \beta_i) = s. \tag{5.47}$$

This normalization is chosen so at the critical point where $\chi = 0$ for $a+b=R$,

$$\alpha_i = \frac{a}{R} \quad \beta_i = 1 - \frac{a}{R}, \quad \forall i. \tag{5.48}$$

Away from this point, we consider fluctuations,

$$\alpha_i = \frac{a}{R} + \hat\alpha_i, \quad \beta_i = 1 - \frac{a}{R} + \hat\beta_i, \tag{5.49}$$

with the constraint

$$\sum_{i=1}^s (\hat\alpha_i + \hat\beta_i) = 0. \tag{5.50}$$

The Jacobian of this transformation is sM^{2s-1}.

Now, we expand the exponent in $\operatorname{tr} A^s$, to first order in $d = R - a - b$, and to second order in $\hat\alpha_i$, $\hat\beta_i$. The result is

$$\chi = \frac{2Msd}{tR} + Mt\left(\frac{R}{a} - 1\right)\sum_{i=1}^s \left[\hat\alpha_i - \frac{1}{2}\frac{a}{R-a}(\hat\beta_i + \hat\beta_{i+1})\right]^2 + \frac{Mt}{4}\frac{a}{R-a}\sum_{i=1}^s (\hat\beta_i - \hat\beta_{i+1})^2. \tag{5.51}$$

The $\hat\alpha_i$ terms lead to trivial Gaussian integrals. The difficulty with the quadratic $\hat\beta_i$ terms is that only $s-1$ of the differences are independent. But, in view of the constraint (5.50) there are only $s-1$ independent β_i variables. In fact, it is easy to check that

$$\sum_{i=1}^s (\hat\beta_i - \hat\beta_{i+1})^2 = \sum_{i=1}^{s-1} \frac{i+1}{i}\left[\hat\beta_i - \hat\beta_{i+1} + \frac{1}{i+1}\sum_{j=i+1}^{s-1}(\hat\beta_j - \hat\beta_{j+1})\right]^2, \tag{5.52}$$

which now enables us to perform each successive $\hat\beta_i - \hat\beta_{i+1}$ integration. The Jacobian of the transformation to the difference variables $u_i = \hat\beta_i - \hat\beta_{i+1}$, $i = 1, \dots, s-1$, is $1/s$. Thus, we can immediately write down

$$\begin{aligned}
\mathfrak{E} &\sim -\frac{1}{4\pi R^2}\int_0^\infty dz\, z \sum_{s=1}^\infty \frac{t^s}{s}\int_0^\infty dM \frac{M^{2s+1}}{(2\pi M)^s} e^{-2Msd/tR}\left[\int_{-\infty}^\infty d\alpha_i e^{-Mt(R-a)\alpha_i^2/a}\right]^s \prod_{i=i}^{s-1}\int_{-\infty}^\infty du_i e^{-Mat\sum_{i=1}^{s-1}\frac{i+1}{i}u_i^2/4(R-a)} \\
&= -\frac{1}{4\pi R^2}\int_0^\infty dz\, z \sum_{s=1}^\infty \frac{1}{s}\int_0^\infty dM\, M^{2s+1}\frac{t^s}{(2\pi M)^s} e^{-2Msd/tR}\left[\frac{\pi a}{(R-a)Mt}\right]^{s/2}\left[\frac{4\pi(R-a)}{Mta}\right]^{(s-1)/2} s^{-1/2} \\
&= -\frac{\sqrt{2a(R-a)}\pi^3}{3840 R^3}\left(\frac{R}{d}\right)^{5/2},
\end{aligned} \tag{5.53}$$

which is exactly the result expected from the proximity force theorem, according to Eq. (B5).

We will forego further discussion of strong coupling, and presentation of numerical results, for these have been extensively discussed in several recent papers, especially in Ref. [2].

VI. 3-DIMENSIONAL FORMALISM

The three-dimensional formalism is very similar. In this case, the free Green's function has the representation

$$G_0(\mathbf{R} + \mathbf{r}' - \mathbf{r}) = \sum_{lm,l'm'} j_l(i|\zeta|r) j_{l'}(i|\zeta|r') Y^*_{lm}(\hat{\mathbf{r}}) Y_{l'm'}(\hat{\mathbf{r}}') g_{lm,l'm'}(\mathbf{R}). \tag{6.1}$$

The reduced Green's function can be written in the form

$$g^0_{lm,l'm'}(\mathbf{R}) = (4\pi)^2 i^{l'-l}\int \frac{(d\mathbf{k})}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot\mathbf{R}}}{k^2+\zeta^2}\frac{j_l(kr)j_{l'}(kr')}{j_l(i|\zeta|r)j_{l'}(i|\zeta|r')} Y_{lm}(\hat{\mathbf{k}}) Y^*_{l'm'}(\hat{\mathbf{k}}). \tag{6.2}$$

Now we use the plane-wave expansion (4.3) once again, this time for $e^{i\mathbf{k}\cdot\mathbf{R}}$, so now we encounter something new, an integral over three spherical harmonics,

$$\int d\hat{\mathbf{k}} Y_{lm}(\hat{\mathbf{k}}) Y^*_{l'm'}(\hat{\mathbf{k}}) Y^*_{l''m''}(\hat{\mathbf{k}}) = C_{lm,l'm',l''m''}, \tag{6.3}$$

where

$$C_{lm,l'm',l''m''} = (-1)^{m'+m''}\sqrt{\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}}\begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix}. \tag{6.4}$$

The three-j symbols (Wigner coefficients) here vanish unless $l+l'+l''$ is even. This fact is crucial, since because of it we can follow the previous method of writing $j_{l''}(kR)$ in terms of Hankel functions of the first and second kind, using the reflection property of the latter,

$$h^{(2)}_{l''}(kR) = (-1)^{l''} h^{(1)}_{l''}(-kR), \tag{6.5}$$

and then extending the k integral over the entire real axis to a contour integral closed in the upper half plane. The residue theorem then supplies the result for the reduced Green's function[3]

$$g^0_{lm,l'm'}(\mathbf{R}) = 4\pi i^{l'-l}\sqrt{\frac{2|\zeta|}{\pi R}}\sum_{l''m''} C_{lm,l'm',l''m''} K_{l''+1/2}(|\zeta|R) Y_{l''m''}(\hat{\mathbf{R}}). \tag{6.6}$$

A. Casimir interaction between semitransparent spheres

For the case of two semitransparent spheres that are totally outside each other,

$$V_1(r) = \lambda_1\delta(r-a), \quad V_2(r') = \lambda_2(r'-b), \tag{6.7}$$

in terms of spherical coordinates centered on each sphere, it is again very easy to calculate the scattering matrices,

$$T_1(\mathbf{r},\mathbf{r}') = \frac{\lambda_1}{a^2}\delta(r-a)\delta(r'-a)\sum_{lm}\frac{Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')}{1+\lambda_1 a K_{l+1/2}(|\zeta|a) I_{l+1/2}(|\zeta|a)}, \tag{6.8}$$

and then the harmonic transform is very similar to that seen in Eq. (5.13), ($k = i|\zeta|$)

$$\begin{aligned}(t_1)_{lm,l'm'} &= \int (d\mathbf{r})(d\mathbf{r}') j_l(kr) Y^*_{lm}(\hat{\mathbf{r}}) j_{l'}(kr') Y_{l'm'}(\hat{\mathbf{r}}') T_1(\mathbf{r},\mathbf{r}') \\ &= \delta_{ll'}\delta_{mm'}(-1)^l \frac{\lambda_1 a\pi}{2|\zeta|}\frac{I^2_{l+1/2}(|\zeta|a)}{1+\lambda_1 a K_{l+1/2}(|\zeta|a) I_{l+1/2}(|\zeta|a)}. \end{aligned} \tag{6.9}$$

Let us suppose that the two spheres lie along the z-axis, that is, $\mathbf{R} = R\hat{\mathbf{z}}$. Then we can simplify the expression for the energy somewhat by using $Y_{lm}(\theta = 0) = \delta_{m0}\sqrt{(2l+1)/4\pi}$. The formula for the energy of interaction becomes

$$E = \frac{1}{2\pi}\int_0^\infty d\zeta\, \mathrm{tr}\ln(1-A), \tag{6.10}$$

where the matrix

$$A_{lm,l'm'} = \delta_{m,m'}\sum_{l''} B_{ll''m}(a) B_{l''l'm}(b) \tag{6.11}$$

is given in terms of the quantities

$$B_{ll'm}(a) = \frac{\sqrt{\pi}}{\sqrt{2\zeta R}} i^{-l+l'}\sqrt{(2l+1)(2l'+1)}\sum_{l''}(2l''+1)\begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix}\frac{K_{l''+1/2}(\zeta R)\lambda_1 a I^2_{l'+1/2}(\zeta a)}{1+\lambda_1 a I_{l'+1/2}(\zeta a) K_{l'+1/2}(\zeta a)}. \tag{6.12}$$

Note that the phase always cancels in the trace in Eq. (6.10). For strong coupling, this result reduces to that found by Bulgac, Wirzba et al. [11, 13] for Dirichlet spheres, and recently generalized by Emig et al. [2] for Robin boundary conditions. (See also Ref. [41].)

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[3] This differs by a (conventional) factor of $|\zeta|$ from the quantity $U_{lml'm'}$ defined by Emig et al. [2].

B. Weak coupling

For weak coupling, a major simplification results because of the orthogonality property,

$$\sum_{m=-l}^{l} \begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix} \begin{pmatrix} l & l' & l''' \\ m & -m & 0 \end{pmatrix} = \delta_{l''l'''} \frac{1}{2l''+1}, \quad l \leq l'. \tag{6.13}$$

Then the formula for the energy of interaction between the two spheres is

$$E = -\frac{\lambda_1 a \lambda_2 b}{4R} \int_0^\infty \frac{dx}{x} \sum_{ll'l''} (2l+1)(2l'+1)(2l''+1) \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix}^2 K^2_{l''+1/2}(x) I^2_{l+1/2}(xa/R) I^2_{l'+1/2}(xb/R). \tag{6.14}$$

There is no infrared divergence because for small x the product of Bessel functions goes like $x^{2(l+l'-l'')+1}$, and $l'' \leq l+l'$.

As with the cylinders, we expand the modified Bessel functions of the first kind in power series in $a/R, b/R < 1$. This expansion yields the infinite series

$$E = -\frac{\lambda_1 a \lambda_2 b}{4\pi R} \frac{ab}{R^2} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{m=0}^{n} D_{n,m} \left(\frac{a}{R}\right)^{2(n-m)} \left(\frac{b}{R}\right)^{2m}, \tag{6.15}$$

where by inspection of the first several $D_{n,m}$ coefficients we can identify them as

$$D_{n,m} = \frac{1}{2} \binom{2n+2}{2m+1}, \tag{6.16}$$

and now we can immediately sum the expression (6.15) for the Casimir interaction energy to give the closed form

$$E = \frac{\lambda_1 a \lambda_2 b}{16\pi R} \ln \left(\frac{1 - \left(\frac{a+b}{R}\right)^2}{1 - \left(\frac{a-b}{R}\right)^2} \right). \tag{6.17}$$

Again, when $d = R - a - b \ll a, b$, the proximity force theorem (B9) is reproduced:

$$U(d) \sim \frac{\lambda_1 \lambda_2 ab}{16\pi R} \ln(d/R), \quad d \ll a, b. \tag{6.18}$$

However, as Figs. 4, 5 demonstrate, the approach is not very smooth, even for equal-sized spheres. The ratio of the energy to the PFA is

$$\frac{E}{U} = 1 + \frac{\ln[(1+\mu)^2/2\mu]}{\ln d/R}, \quad d \ll a, b, \tag{6.19}$$

for $b/a = \mu$. Truncating the power series (6.15) at $n = 100$ would only begin to show the approach to the proximity force theorem limit. The error in using the PFA between spheres can be very substantial.

Again we will forego discussion of the strong-coupling (Dirichlet) limit here because of the extensive discussion already in the literature [2, 11, 13].

C. Exact result for interaction between plane and sphere

In just the way indicated above, we can obtain a closed-form result for the interaction energy between a weakly-coupled sphere and a Dirichlet plane. Using the simplification that

$$\sum_{m=-l}^{l} (-1)^m \begin{pmatrix} l & l & l' \\ m & -m & 0 \end{pmatrix} \begin{pmatrix} l & l & l' \\ 0 & 0 & 0 \end{pmatrix} = \delta_{l'0}, \tag{6.20}$$

we can write the interaction energy in the form

$$E = -\frac{\lambda a}{2\pi R} \int_0^\infty dx \sum_{l=0}^{\infty} \sqrt{\frac{\pi}{2x}} (2l+1) K_{1/2}(x) I^2_{l+1/2}(x(a/R)). \tag{6.21}$$

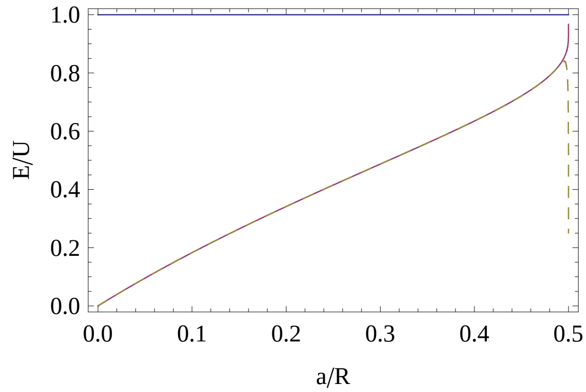

FIG. 4: Plotted is the ratio of the exact interaction energy (6.17) of two weakly-coupled spheres to the proximity force approximation (6.18) as a function of the sphere radius a for $a = b$. Shown also by a dashed line is the power series expansion (6.15), truncated at $n = 100$, indicating that it is necessary to include very high powers to capture the proximity force limit.

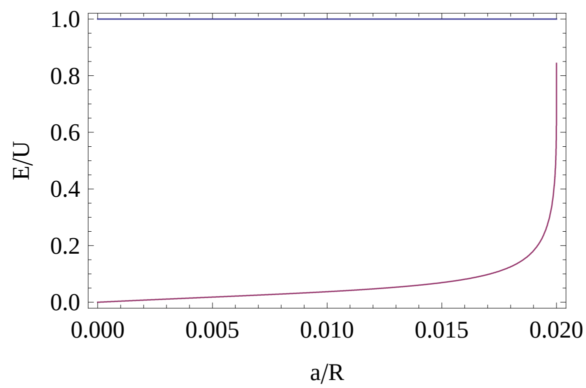

FIG. 5: Plotted is the ratio of the exact interaction energy (6.17) of two weakly-coupled spheres to the proximity force approximation (6.18) as a function of the sphere radius a for $b/a = 49$.

Then in terms of $R/2$ as the distance between the center of the sphere and the plane, the exact interaction energy is

$$E = -\frac{\lambda}{2\pi}\left(\frac{a}{R}\right)^2 \frac{1}{1-(2a/R)^2}, \tag{6.22}$$

which as $a \to R/2$ reproduces the proximity force limit, contained in the (ambiguously defined) PFA formula

$$U = -\frac{\lambda}{8\pi}\frac{a}{d}. \tag{6.23}$$

The exact energy and this PFA approximation are compared in Fig. 6.

VII. COMMENTS AND PROGNOSIS

The methods proposed recently are in fact not particularly novel, being utilized in this context in the 1970s [6]. What is new is the ability, largely due to enhancement in computing power and flexibility, to evaluate continuum determinants (or infinitely dimensional discrete ones) accurately numerically. This is making it possible to compute Casimir forces for geometries previously inaccessible. Some remarkable new results have been obtained. Here we have given a perhaps simpler and more transparent derivation of the procedure than in Refs. [1, 2]. For example, because we have approached the problem from a general field theoretic viewpoint, we see that the "translation matrix" introduced there is nothing other than the free Green's function. Our approach yields the general form first, and the multipole expansion as a derived consequence, not the other way around. We apply this multiple scattering method to obtain new results for the interaction between semitransparent cylinders and spheres, and we have analytically demonstrated the approach to the proximity force theorem. Most remarkably, we have derived explicit, very simple,

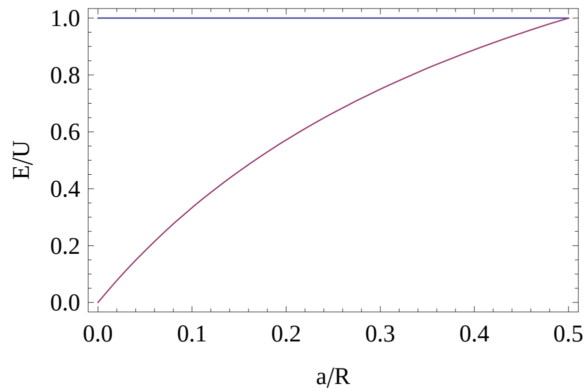

FIG. 6: Plotted is the ratio of the exact interaction energy (6.22) of a weakly-coupled sphere above a Dirichlet plane to the proximity force approximation (6.23) as a function of the sphere radius a.

closed-form expressions for the interaction between weakly coupled cylinders and between weakly coupled spheres, as well as between weakly-coupled cylinders or spheres and Dirichlet planes. These explicit results demonstrate the profound limitation of the proximity force approximation, which has been under serious attack for some time [42, 43]. We hope that these developments will lead to improved conceptual understanding, and to better comparison with experiment, when they are extended to realistic materials.

Acknowledgments

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APPENDIX A: DERIVATION OF VACUUM ENERGY FORMULA

Following Schwinger [31] we start from the vacuum amplitude in terms of sources,

$$\langle 0_+|0_-\rangle^K = e^{iW[K]}, \quad W[K] = \frac{1}{2}\int (dx)(dx')K(x)G(x,x')K(x'). \tag{A1}$$

Here G is the Green's function in the presence of some background potential. From this the effective field is

$$\phi(x) = \int (dx')G(x,x')K(x'). \tag{A2}$$

If the geometry of the region is altered slightly, as through moving one of the bounding surfaces, the vacuum amplitude is altered:

$$\delta W[K] = \frac{1}{2}\int (dx)(dx')K(x)\delta G(x,x')K(x') = -\frac{1}{2}\int (dx)(dx')\phi(x)\delta G^{-1}(x,x')\phi(x'), \tag{A3}$$

which uses the fact that

$$GG^{-1} = 1. \tag{A4}$$

Upon comparison with the two particle emission term in

$$e^{iW[K]} = e^{i\int(dx)K(x)\phi(x)+i\int(dx)\mathcal{L}} = \cdots + \frac{1}{2}\left[i\int (dx)K(x)\phi(x)\right]^2, \tag{A5}$$

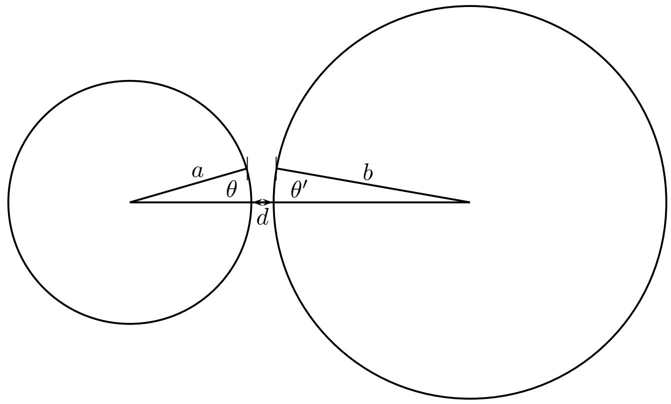

FIG. 7: Geometry of two cylinders (or two spheres) with radii a and b, respectively, and distances between their centers of $R > a + b$. The proximity force approximation applies when the distance of closest approach $d = R - a - b \ll a, b$. The approximation consists in assuming that the interaction is dominated by the interaction of adjacent surface elements, as shown.

we deduce that the effective two-particle source due to a geometry modification is

$$iK(x)K(x')\Big|_{\text{eff}} = -\delta G^{-1}(x, x'). \tag{A6}$$

Thus the change in the generating functional is

$$\delta W = \frac{i}{2}\int (dx)(dx')G(x, x')\delta G^{-1}(x', x) = -\frac{i}{2}\int (dx)(dx')\delta G(x, x')G^{-1}(x', x). \tag{A7}$$

From this, in matrix notation

$$\delta W = -\frac{i}{2}\delta \mathrm{Tr}\ln G \Rightarrow E = \frac{i}{2\tau}\mathrm{Tr}\ln G, \tag{A8}$$

because for a static configuration $W = -E\tau$, which is our starting point, Eq. (2.1).

APPENDIX B: PROXIMITY FORCE APPROXIMATION

In this Appendix we derive the proximity force approximation (PFA) for the energy of interaction between two semitransparent cylinders, or two semitransparent spheres, either in the strong or weak coupling regimes. This approximation, relating the force between nonplanar surfaces in terms of the forces between parallel plane surfaces, was first introduced in 1934 by Derjaguin [44]. This approximation is only valid when the separation between the bodies is very small compared to their sizes. It is now well established that the approximation cannot be extended beyond that limit, and that 1% errors occur if the PFA is applied when the ratio of the separation to the radius of curvature of the bodies is of order 1%. Fortunately, current experiments have not exceeded this limit. That should change in the near future, which is one reason the new numerical calculations are of importance. In fact, we have found that in general the errors in using the PFA may be much larger than indicated above. We concur with Bordag that while the proximity force theorem is exact at zero separation, any approximation based on extrapolation away from that point is subject to uncontrollable errors.

Consider first two parallel cylinders, of radius a and b, with their centers separated by a distance $R > a + b$. The distance of closest approach of the cylinders is $d = R - a - b$. The PFA consists of assuming that the energy between the two bodies is the sum of the energies between small parallel plane elements at the same height along the surfaces, that is, in polar coordinates at θ relative to the center of cylinder a and at θ' relative to the center of cylinder b, where as seen in Fig. 7,

$$a\sin\theta = b\sin\theta'. \tag{B1}$$

Because d is much smaller than either a or b, only small values of θ actually contribute, and the energy of interaction $U(d)$ between the surface may be expressed in terms of the energy per area $\mathcal{E}(h)$ for the corresponding parallel plate problem, with separation distance h:

$$U(d) = \int a\, d\theta\, \mathcal{E}[d + a(1-\cos\theta) + b(1-\cos\theta')]. \tag{B2}$$

Here, for weak coupling [see Eq. (3.7)],

$$\mathcal{E}(h) = -\frac{\lambda_1\lambda_2}{32\pi^2 h}. \tag{B3}$$

Because θ is small, the PFA energy per length is

$$U(d) = -\frac{\lambda_1\lambda_2}{32\pi^2}\frac{a}{d}\int_{-\pi}^{\pi} d\theta \left[1 + \frac{a}{d}\left(1+\frac{a}{b}\right)\frac{\theta^2}{2}\right]^{-1} = -\frac{\lambda_1\lambda_2}{32\pi}\sqrt{\frac{2ab}{R}}\frac{1}{d^{1/2}}. \tag{B4}$$

To obtain the corresponding result for strong coupling, we merely replace $\mathcal{E}(h) = -\pi^2/(1440h^3)$, and a similar calculation yields

$$U(d) = -\frac{\pi^3}{3840}\sqrt{\frac{2ab}{R}}\frac{1}{d^{5/2}}, \quad d \ll a, b. \tag{B5}$$

It is easy to reproduce the result given by Bordag [14] for a cylinder in front of a plane. For the strong coupling (Dirichlet) case we simply take the result (B5) and regard b as much larger than a, and obtain

$$U(d) = -\frac{\pi^3}{1920\sqrt{2}}\frac{a^{1/2}}{d^{5/2}}, \quad d \ll a. \tag{B6}$$

For a weakly coupled cylinder in front of a Dirichlet plane, we start from the corresponding interaction between two such planes, $\mathcal{E}(h) = -\lambda/(32\pi^2 h^2)$, which leads to

$$U(d) = -\frac{\lambda}{64\pi}\frac{(2a)^{1/2}}{d^{3/2}}. \tag{B7}$$

For nearly touching spheres the calculation goes just the same way. The result, for strong coupling (Dirichlet boundary conditions), for the PFA energy is

$$U(d) = -\frac{\pi^3}{1440}\frac{ab}{R}\frac{1}{d^2}, \quad d \ll a, b, \tag{B8}$$

while in the weak-coupling limit there is sensitivity to large θ signifying a logarithmic divergence,

$$U(d) \sim \frac{\lambda_1\lambda_2 ab}{16\pi R}\ln(d/R), \quad d \ll a, b. \tag{B9}$$

For a weakly-coupled sphere in front of a Dirichlet plane, a PFA approximation is

$$U(d) = -\frac{\lambda}{16\pi}\frac{a}{d}. \tag{B10}$$

APPENDIX C: SHORT DISTANCE LIMIT

1. Cylinders

In this appendix we want to discuss the short distance limit, for the case of weakly-coupled cylinders, where the closest distance between the cylinders is $R-a-b = d \ll a, b$, which should reduce to the proximity force approximation derived in Appendix B. We will calculate the first correction to the PFA, and compare to the exact result found in Sec. V D. In this limit, we replace the modified Bessel functions by their uniform asymptotic approximants, which in leading form yield

$$K_{m+m'}^2(x) I_m^2(xa/R) I_{m'}^2(xb/R) \sim \frac{1}{8\pi}\frac{1}{mm'(m+m')} t t_a t_b e^{-\chi}, \tag{C1}$$

where

$$t=(1+z^2)^{-1/2},\quad t_a=(1+z_a^2)^{-1/2},\quad t_b=(1+z_b^2)^{-1/2},\tag{C2}$$

and

$$z=\frac{x}{m+m'},\quad z_a=\frac{xa/R}{m},\quad z_b=\frac{xb/R}{m'}.\tag{C3}$$

The exponent here is

$$\chi=2(m+m')\eta(z)-2m\eta(z_a)-2m'\eta(z_b),\tag{C4}$$

where η is defined by Eq. (5.43). The reason that the force diverges as $a+b\to R$ is that χ vanishes there, for suitable values of m and m'. To make this systematic, let us rescale variables,

$$m=M\alpha,\quad m'=M\beta,\tag{C5}$$

and then when $b=R-a$, $\chi=0$ when $\beta a=\alpha b$.

When $b=R-a-d$, with d small compared to the radius of either cylinder, we assume that the main contribution comes from the neighborhood of these values. So we define

$$\alpha=\frac{a}{R}+\hat\alpha,\quad \beta=1-\frac{a}{R}+\hat\beta,\tag{C6}$$

and we expand the exponent to first order in d and to second order in $\hat\alpha$ and $\hat\beta=-\hat\alpha$. (The latter constraint ensures that $\alpha+\beta=1$.) The result is

$$\chi=\frac{2Md}{tR}+\frac{MtR^2\hat\alpha^2}{a(R-a)}+O(\hat\alpha^3,d^2).\tag{C7}$$

Then

$$\begin{aligned}\mathfrak{E}&\sim-\frac{\lambda_1\lambda_2}{16\pi^2}\int_0^\infty dz\,z\,t^3\int_0^\infty dM\,e^{-2Md/tR}\int_{-\infty}^\infty d\hat\alpha\,e^{-Mt\hat\alpha^2R^2/[a(R-a)]}\\&=-\frac{\lambda_1\lambda_2}{32\pi}\sqrt{\frac{2}{d}}\sqrt{\frac{a(R-a)}{R}}=U,\end{aligned}\tag{C8}$$

which is exactly the result given by the proximity force theorem in Appendix B, Eq. (B4).

Now we calculate the correction to the PFA. We do this by keeping subleading terms in the uniform asymptotic approximation for the product of six Bessel function

$$\begin{aligned}K^2_{m+m'}(x)I^2_m(xa/R)I^2_{m'}(xb/R)\sim{}&\frac{1}{8\pi mm'}\frac{tt_at_b}{m+m'}\\&\times\left(1-\frac{u_1(t)}{m+m'}\right)^2\left(1+\frac{u_1(t_a)}{m}\right)^2\left(1+\frac{u_1(t_b)}{m'}\right)^2e^{-\chi},\end{aligned}\tag{C9}$$

where $t=t(z)$ with $z=x/(m+m')$, $z_a=xa/m$, $z_b=xb/m'$,

$$u_1(t)=\frac{3t-5t^3}{24},\tag{C10}$$

and χ is given by Eq. (C4). Now when we expand χ we must go out to order $\hat\alpha^4$, d^2, and $\hat\alpha^2 d$. The result is

$$\begin{aligned}e^{-\chi}\sim e^{-2Md/tR}e^{-M\hat\alpha^2tR^2/(a(R-a))}\bigg[&1-\frac{d^2Mt}{R(R-a)}+\frac{2\hat\alpha dMt}{R-a}\\&-\frac{\hat\alpha^2dMt(1-t^2)R}{(R-a)^2}+\frac{\hat\alpha^3Mt^3(R-2a)R^3}{3a^2(R-a)^2}+\frac{\hat\alpha^4Mt^3(1-3t^2)(R^2-3aR+3a^2)R^4}{12a^3(R-a)^3}\\&+\frac{2M^2\hat\alpha^2d^2t^2}{(R-a)^2}+\frac{M^2\hat\alpha^6t^6(R-2a)^2R^6}{18a^4(R-a)^4}+\frac{2}{3}\frac{M^2\hat\alpha^4t^4dR^3}{a^2(R-a)^3}(R-2a)\bigg].\end{aligned}\tag{C11}$$

As above, we replace

$$m = M\frac{a}{R}\left(1+\hat{\alpha}\frac{R}{a}\right), \quad m' = M\left(1-\frac{a}{R}\right)\left(1-\hat{\alpha}\frac{R}{R-a}\right). \tag{C12}$$

We expand t_a and t_b in the prefactor using

$$\frac{dt}{dz} = -zt^3, \quad \frac{d^2t}{dz^2} = 2t^3 - 3t^5. \tag{C13}$$

The PFA is obtained by using the integrals

$$\int_{-\infty}^{\infty} d\hat{\alpha}\, e^{-\hat{\alpha}^2\gamma} = \sqrt{\frac{\pi}{\gamma}}, \quad \gamma = MtR^2/a(R-a), \tag{C14a}$$

$$\int_0^\infty \frac{dM}{\sqrt{M}} e^{-2Md/t} = \Gamma\left(\frac{1}{2}\right)\left(\frac{2d}{t}\right)^{-1/2}, \tag{C14b}$$

and so, from the expansion we can obtain the result of the integrals over $\hat{\alpha}$ and M by the algebraic substitutions

$$\frac{1}{M} \to -\frac{4d}{Rt}, \quad M \to \frac{tR}{4d}, \tag{C15a}$$

$$\hat{\alpha}^2 \to -\frac{2a(R-a)d}{R^3t^2}, \quad M\hat{\alpha}^2 \to \frac{1}{2}\frac{a(R-a)}{R^2t}, \quad M\hat{\alpha}^4 \to -\frac{3a^2(R-a)^2d}{R^5t^3}, \tag{C15b}$$

$$M^2\hat{\alpha}^2 \to \frac{a(R-a)}{8Rd}, \quad M^2\hat{\alpha}^4 \to \frac{3}{4}\frac{a^2(r-a)^2}{R^4t^2}, \quad M^2\hat{\alpha}^6 \to -\frac{15}{2}\frac{a^3(R-a)^3d}{R^7t^4}. \tag{C15c}$$

The result is the following correction factor to the PFA in the form given in Eq. (C8):

$$\frac{\mathfrak{E}}{U} = 1 - \frac{R^2+aR+a^2}{4a(R-a)}\frac{d}{R}. \tag{C16}$$

Although this looks slightly different from Eq. (5.37), it agrees with the latter when the PFA formula (5.36) is expressed in terms of the form given in Eq. (C8), that is, writing $d = R-a-b$.

2. Spheres

Here we see how the proximity force limit is achieved for weakly-coupled spheres. Again, the strategy is to replace the modified Bessel functions by their leading uniform asymptotic approximants. The only new element is the appearance of the 3-j symbol. Because now only $m=0$ appears, there is a very simple approximant for the latter [45–47]:

$$\begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \sim \sqrt{\frac{\pi}{2}}\frac{\cos\frac{\pi}{2}(l+l'+l'')}{[(l+l'+l'')(l+l'-l'')(l-l'+l'')(-l+l'+l'')]^{1/4}}. \tag{C17}$$

This result is quite accurate, being within 1% of the true value of the Wigner coefficient for l's of order 100 (except very near the boundaries of the triangular region, where the approximant diverges weakly). Otherwise, the procedure is rather routine. Letting $\nu = l+1/2$, and similarly for the primed quantities, we expand the exponent resulting from the uniform asymptotic expansion about the critical point, with

$$\nu = N(a+\hat{\alpha}), \quad \nu' = N(1-a+\hat{\alpha}'), \quad \nu'' = N(1+\alpha''), \tag{C18}$$

with the constraint $\hat{\alpha}+\hat{\alpha}'+\hat{\alpha}'' = 0$. Replacing the sums over angular momenta by integrals, and changing variables:

$$\int dv\, d\nu'\, d\nu'' = 2\int_0^\infty dN\, N^2 \int_0^\infty d(\hat{\alpha}+\hat{\alpha}') \int_{-\infty}^{\infty} \frac{d(\hat{\alpha}-\hat{\alpha}')}{2}, \tag{C19}$$

which reflects the restriction emerging from the triangular relation of the Wigner coefficients, $\hat{\alpha}+\hat{\alpha}' > 0$, we find for the approximant to the energy when the two spheres are nearly touching:

$$\begin{aligned} E &\sim -\frac{\lambda_1 a\lambda_2 b}{4R}\frac{2}{\pi}\int_0^\infty \frac{dx}{x}\int_0^\infty dN\, N^2t^3e^{-2Nd/Rt}\frac{1}{4\pi N^2}\left[\frac{R^2}{a(1-a)}\right]^{1/2}\int_0^\infty \frac{d(\hat{\alpha}+\hat{\alpha}')}{(\hat{\alpha}+\hat{\alpha}')^{1/2}}e^{-4N(\hat{\alpha}+\hat{\alpha}')/t} \\ &\quad\times\int_{-\infty}^{\infty} d\left(\frac{\hat{\alpha}-\hat{\alpha}'}{2}\right)e^{-NtR^2(\hat{\alpha}-\hat{\alpha}')^2/4a(R-a)} \\ &\sim \frac{\lambda_1\lambda_2 ab}{16\pi R}\ln d, \quad d = R-a-b \ll a, b, \end{aligned} \tag{C20}$$

which is exactly the PFA result (B9).

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