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ENTROPY BOUNDS IN SPHERICAL SPACE

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Exact calculations are given for the Casimir energy for various fields in $R \times S^3$ geometry. The Green's function method naturally gives a result in a form convenient in the high-temperature limit, while the statistical mechanical approach gives a form appropriate for low temperatures. The equivalence of these two representations is demonstrated. Some discrepancies with previous work are noted. In no case, even for $\mathcal{N} = 4$ SUSY, is the ratio of entropy to energy found to be bounded.

1 Introduction

The remarkable appearance of the holographic principle has fostered the understanding that some hitherto distant branches of theoretical physics may have a much deeper common origin that was expected. One significant example of this sort is the relation, suggested by Verlinde [1] between the Cardy entropy formula [2] and the Friedmann equation for the evolution of the scale factor of the universe. Moreover, the proposal that in the early universe there exists a holographic bound on the cosmological entropy associated with Casimir energy suggests that there should be a deeper relation between Friedmann cosmology and the Casimir effect [3]. Specifically, there has been much interest in studying the entropy and energy arising from quantum and thermal fluctuations in conformal field theories [4, 5]. Whether the Verlinde bound for the ratio of the entropy to the thermal energy can be realized in realistic situations is a matter for specific calculations. Previous computations have been limited to the regime of high temperatures, so they are unable to provide definitive results. Here we obtain exact results for various fields in the $R \times S^3$ geometry, so the issues may be more decisively addressed [6].

2 Conformally Coupled Scalar

2.1 Green's Function Method

We can start from the formalism given in Kantowski and Milton [7]. The energy is given by the imaginary part of the Green's function,

$$U = V_3 \partial^0 \partial'^0 \Im G(x, y; x', y')|_{x=x', y=y'}, \tag{1}$$

where the "external" coordinates x consist only of the time. We introduce a Fourier transform there

$$G(t, y; t', y') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} g(y, y'; \omega), \quad g(y, y'; \omega) = \sum_{lm} \frac{Y_l^m(y) Y_l^{m*}(y')}{M_l^2/a^2 - \omega^2}, \tag{2}$$

with $M_l^2 = (l+1)^2$ for conformal coupling on S^3, so the Casimir energy is

$$U = -\frac{i}{4\pi} \int_c d\omega\, \omega^2 \sum_l \frac{D_l}{M_l^2/a^2 - \omega^2}, \quad D_l = (l+1)^2, \tag{3}$$

where the contour c encircles the poles on the positive axis in a negative sense, and those on the negative axis in a positive sense.

Temperature dependence is incorporated by the replacement

$$\int_c \frac{d\omega}{2\pi} \to \frac{4i}{\beta} \sum_{n=0}^{\infty}{}', \quad \omega^2 \to -\left(\frac{2\pi n}{\beta}\right)^2. \tag{4}$$

The prime means that the $n = 0$ term is counted with half weight.

We carry out the sum on l in Eq. (3) by using the general representation

$$\sum_{m=0}^{\infty} \frac{1}{m^2 - \alpha^2} = -\frac{\pi}{2\alpha} \cot \pi\alpha - \frac{1}{2\alpha^2}. \tag{5}$$

We then make the finite-temperature replacements (4) and obtain, after dropping the contact term arising from the constant ($\propto \zeta(-2) = 0$)

$$U = \frac{1}{a} \left(\frac{2\pi a}{\beta}\right)^4 \left[\sum_{n=0}^{\infty}{}' \frac{n^3}{e^{4\pi^2 an/\beta} - 1} + \frac{1}{240}\right], \tag{6}$$

which, since the summand vanishes at $n = 0$, gives only exponentially small corrections to Stefan's law, which is the result of Kutasov and Larsen [4]:

$$U \sim \frac{1}{a} \frac{(2\pi aT)^4}{240}, \quad (aT \gg 1). \tag{7}$$

2.2 Statistical-Mechanical Approach

We recall the usual statistical mechanical expression for the free energy,

$$F = -kT \ln Z, \quad \ln Z = -\sum_{n=0}^{\infty} (n+1)^{d-2} \ln\left(1 - e^{-\beta(n+1)/a}\right), \tag{8}$$

for conformally coupled scalars in S^{d-1}. Here the zero-point energy,

$$E_0 = \sum_{n=0}^{\infty} (n+1)^{d-2} \frac{n+1}{2a} = \frac{1}{2a}\zeta(1-d), \tag{9}$$

has been subtracted. The specific results for two and four dimensions are

$$E_0^{d=2} = -\frac{1}{2a}\frac{1}{2}B_2 = -\frac{1}{24a}, \quad E_0^{d=4} = -\frac{1}{2a}\frac{1}{4}B_4 = \frac{1}{240a}. \tag{10}$$

For the temperature dependence, we differentiate the partition function,

$$E = U - E_0 = -\frac{d}{d\beta}\ln Z = \frac{1}{a}\sum_{n=1}^{\infty} \frac{n^3}{e^{2\pi n\delta} - 1}, \quad \delta = \frac{\beta}{2\pi a}. \tag{11}$$

This is a very different representation from Eq. (6). Nevertheless, from it we may obtain the same result we found above if we use the Euler-Maclaurin sum formula.

2.3 Relation Between Representations

We have two representation for the Casimir energy, the one obtained from the Green's function, Eq. (6), and the one obtained from the partition function, Eq. (11). The relation between the two can be found from the Poisson sum formula. If the Fourier transform of a function $b(x)$ is defined by

$$c(\alpha) = \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{-i\alpha x} b(x), \quad \text{then} \quad \sum_{n=-\infty}^{\infty} b(n) = 2\pi \sum_{n=-\infty}^{\infty} c(2\pi n). \tag{12}$$

It is then easily seen that the energy (11) is

$$E = \frac{1}{a}\left(\frac{a}{\beta}\right)^4 \Gamma(4) \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(1 + k - ia2\pi n/\beta)^4}. \tag{13}$$

If we sum this on n first, we obtain the alternative expression (6)

$$E=\frac{1}{a}\left(\frac{2\pi a}{\beta}\right)^4\sum_{n=1}^\infty\frac{n^3}{e^{4\pi^2an/\beta}-1}+\frac{1}{240a}\left[(2\pi aT)^4-1\right]. \tag{14}$$

The two representations are best adapted for the low- and high-temperature limits, respectively:

$$U=\frac{1}{240a}+\frac{1}{a}\sum_{n=1}^\infty\frac{n^3}{e^{2\pi n\delta}-1}=\frac{(2\pi aT)^4}{240a}+\frac{(2\pi aT)^4}{a}\sum_{n=1}^\infty\frac{n^3}{e^{2\pi na/\delta}-1}. \tag{15}$$

3 $d=2$ Conformal Scalar

Because there is a subtle issue involving zero-modes here, it is useful to repeat the above calculation for $d=2$. From the partition function we immediately obtain the low-temperature representation,[1]

$$U=-\frac{1}{24a}+\frac{1}{a}\sum_{n=1}^\infty\frac{n}{e^{2\pi n\delta}-1}, \tag{16}$$

displaying an exponentially small correction to the zero-point energy if $\beta\gg1$. Again, by use of the Euler-Maclaurin sum formula we can obtain the high-temperature limit,

$$U\sim\frac{1}{24a}(2\pi aT)^2-\frac{1}{2}T,\quad (aT\gg1). \tag{17}$$

This again coincides with the result found in Kutasov and Larsen [4]. However, the linear term in T is omitted in the analysis of Klemm et al. [5] with an apparently erroneous remark that it only contributes when the saddle-point method breaks down, for a small central charge (the number of fields). So their derivation of the Cardy formula cannot be sustained.

It is easy to reproduce this result from the Green's function method. After the finite-temperature substitutions, the expression is

$$U=\frac{1}{24a\delta^2}-\frac{1}{2}T-\frac{1}{a\delta^2}\sum_{n=1}^\infty\frac{n}{e^{2\pi n/\delta}-1}, \tag{18}$$

[1] Dowker [8] has suggested that the zero-mode contribution be retained here. We see no reason to include the $n=0$ term, which would in any case lead to a violation of the Third Law of Thermodynamics [6, 9].

which gives the explicit exponential corrections to the high temperature limit (17). The low-temperature limit displayed in Eq. (16) may be easily obtained from this by using the Euler-Maclaurin sum formula. A proof of the equivalence of the two representations (18) and (16) can be carried out along the lines sketched above, with due care for the presence of the zero-mode at $n = 0$ [6].

4 Vector Field

The analysis proceeds similarly to that given above. For S^{d-1} the degeneracy and eigenvalues are

$$D_l = \frac{2l\left(l+\frac{d}{2}-1\right)(l+d-2)(l+d-4)!}{(d-3)(l+1)!}, \quad M_l^2 = l(l+d-2), \qquad (19)$$

so for $d = 4$ if we add the conformal coupling value 1 to M_l^2 we obtain for the Green's function mode sum

$$\sum_{l=0}^{\infty} \frac{2l(l+2)}{(l+1)^2/a^2-\omega^2} \to -\frac{1}{\omega^2} + i\pi a^2\left(\omega a - \frac{1}{\omega a}\right)\left(1+\frac{2}{e^{-2\pi i\omega a}-1}\right). \quad (20)$$

After making the finite temperature replacement, we carry out the sum on n, with the result

$$U = \frac{(2\pi aT)^4}{120a} - \frac{(2\pi aT)^2}{12a} + T + \frac{2(2\pi aT)^2}{a}\sum_{n=1}^{\infty}\frac{n+(2\pi aT)^2 n^3}{e^{4\pi^2 aTn}-1}, \qquad (21)$$

where the T term comes from the $n = 0$ term in the sum. Since the remaining sum is exponentially small in the large T limit, this form is well-adapted for high temperature. (The T^4 and T^2 terms are as given in Kutasov and Larsen [4].) However, it is exact, and by using the Euler-Maclaurin sum formula it yields the low temperature limit, $U \sim \frac{11}{120a}$, $aT \ll 1$, up to exponentially small corrections. The latter may be directly inferred from the partition function,

$$\ln Z = -\sum_{l=1}^{\infty} 2l(l+2)\ln\left(1-e^{-\beta(l+1)/a}\right). \qquad (22)$$

By taking the negative derivative of this with respect to β we obtain the alternative representation

$$U = \frac{11}{120a} + \frac{2}{a}\sum_{l=1}^{\infty}\frac{l(l^2-1)}{e^{\beta l/a}-1}. \qquad (23)$$

The exact equivalence of the two expressions (23) and (21) again is demonstrated either by use of the Euler-Maclaurin sum formula, or by the Poisson sum formula,.

5 Weyl Fermions

Here, the degeneracies and eigenvalues are $D_l = 2(l+2)(l+1)$, $M_l^2 = (l+3/2)^2$, so including the minus sign associated with a fermionic trace, and the antiperiodicity of the fermionic thermal Green's functions, we have the following expression for the energy,

$$U = \frac{1}{a}\biggl\{\frac{7}{960}\delta^{-4} - \frac{1}{96}\delta^{-2} - \frac{1}{4}\sum_{n=0}^{\infty}\left[(2n+1)^3\delta^{-4} + (2n+1)\delta^{-1}\right] \times \left(\frac{2}{e^{2\pi(2n+1)/\delta}-1} - \frac{1}{e^{\pi(2n+1)/\delta}-1}\right)\biggr\}. \tag{24}$$

The low-temperature limit (the zero-point energy) may be obtained from this by the Euler-Maclaurin formula, and the exponential corrections in that limit may be obtained directly from the partition function,

$$\ln Z = \sum_{n=1}^{\infty} 2n(n+1)\ln\left(1 + e^{-\beta(2n+1)/2a}\right). \tag{25}$$

That is

$$U = \frac{1}{a}\left[\frac{17}{960} + \sum_{n=1}^{\infty}\frac{n(n+1)(2n+1)}{e^{\beta(2n+1)/2a}+1}\right]. \tag{26}$$

The equivalence between Eqs. (24) and (26) may be demonstrated as above.

6 Entropy Bounds

From the above results, thermodynamic information may extracted in terms of the free energy, in terms of which the energy and the entropy may be extracted:

$$E \equiv U - E_0 = -\frac{\partial}{\partial\beta}\ln Z = \frac{\partial}{\partial\delta}\delta F, \quad S = 2\pi a\delta^2\frac{\partial}{\partial\delta}F = \beta(E-F). \tag{27}$$

6.1 Two-dimensional scalar

Klemm et al. [5] ignore the linear T term in the energy, and so have

$$E = \frac{1}{24a}(\delta^{-2} + 1), \quad F = -\frac{1}{24a}(\delta^{-2} - 1), \quad S = \frac{\pi}{6}\delta^{-1}. \tag{28}$$

These imply the Verlinde-Cardy formula, and the entropy bound,

$$S = 4\pi a\sqrt{E_0(E + E_0)}, \quad \frac{S}{2\pi aE} = 2\frac{\delta}{\delta^2 + 1} \leq 1. \tag{29}$$

However, this result is not meaningful as it stands. Even in the high-temperature limit we must add the term linear in temperature to the energy, which implies instead from Eq. (17), for $\delta \ll 1$, that

$$E = \frac{\delta^{-2} + 1}{24a} - \frac{1}{4\pi a\delta}, \; F = -\frac{\delta^{-2} - 1}{24a} - \frac{\ln\delta}{4\pi a\delta}, \; S = \frac{\pi}{6\delta} + \frac{(\ln\delta - 1)}{2}. \tag{30}$$

The ratio of S to E is then unbounded as $\delta \to \infty$. Yet this takes us to the low-temperature regime, where we must use the leading exponential corrections,

$$E \sim \frac{1}{a}e^{-\beta/a}, \quad F \sim -\frac{1}{\beta}e^{-\beta/a}, \quad S \sim \frac{\beta}{a}e^{-\beta/a}, \quad (\beta \gg 1) \tag{31}$$

so the entropy-energy ratio is

$$\frac{S}{2\pi aE} = \delta, \quad (\delta \gg 1). \tag{32}$$

It is apparent that this latter result is universal because the energy always dominates the free energy in the low temperature regime.

6.2 Entropy Bounds in Four Dimensions

In the following we will consider cases with N_s conformal scalars, N_v vectors, and N_f Weyl fermions. In the high-temperature regime we may write the free energy, energy, and entropy as[2]

$$F \sim -\frac{1}{a}[a_4\delta^{-4} + a_2\delta^{-2} + a_1\delta^{-1}\ln\delta + a_0], \tag{33}$$

$$E \sim \frac{1}{a}[3a_4\delta^{-4} + a_2\delta^{-2} - a_0 - a_1\delta^{-1}], \tag{34}$$

$$S \sim 2\pi[4a_4\delta^{-3} + 2a_2\delta^{-1} - a_1(1 - \ln\delta)]. \tag{35}$$

[2]What is called the Cardy formula is simply the observation that the leading behavior of S is the geometric mean of the leading and subleading terms in E. The term "Casimir energy" for the latter is misleading in other than $1 + 1$ dimensions. The entire energy U is due to quantum and thermal fluctuations, so it all should properly be reckoned as Casimir energy.

Here the coefficients were determined in the previous sections to be [see Eqs. (14), (21), and (24)]

$$a_4 = \frac{N_s}{720} + \frac{N_v}{360} + \frac{7N_f}{2880},\ a_2 = -\frac{N_v}{12} - \frac{N_f}{96},\ a_0 = 3a_4 - a_2,\ a_1 = -\frac{N_v}{2\pi}. \quad (36)$$

Even ignoring the a_1 term, Klemm et al. [5] note that no entropy bound is possible, unless special choices are made for the field multiplicities. For the $\mathcal{N} = 4$ case ($N_s = 6$, $N_v = 1$, $N_f = 4$) the entropy-energy ratio becomes

$$\frac{S}{2\pi a E} = \frac{1 - \ln\delta + \frac{\pi}{6}\delta^{-3}(1 - 3\delta^2)}{\delta^{-1} + \frac{\pi}{8}\delta^{-4}(1 + \delta^2)(1 - 3\delta^2)}. \quad (37)$$

If the a_1 terms here were omitted, the zero in both the energy and entropy at $\delta^2 = 1/3$ would cancel, and we would have the limit given by Klemm et al. [5]:

$$\frac{S}{2\pi a E} = \frac{4}{3}\frac{\delta}{1 + \delta^2} \leq \frac{4}{3} \quad (38)$$

in the high temperature regime. But $a_1 \neq 0$, and the ratio (37) diverges as $\delta \to \infty$. Of course that limit is the low-temperature one, but the argument given above then applies and shows that

$$\frac{S}{2\pi a E} \sim \delta, \quad (\delta \to \infty). \quad (39)$$

Although in this limit both the entropy and the subtracted energy are exponentially small, their ratio is unbounded.

It should noted that we are not in formal disagreement with previous studies. The interest there was restricted to high temperature, which is presumably all that is relevant to nearly the entire history of the universe.[3] In that case, only the leading terms in $1/\delta$ are relevant, and the ratio of entropy to energy is always of order $\delta \ll 1$. It is not surprising that such results as Eq. (38) are an unreliable guide to the moderate and low temperature regimes, which might be relevant in the very earliest (pre-inflationary) stages of the universe.

Another point, which is more closely connected with physics, is that it is permissible to make use of the thermodynamical formalism for fluctuating quasi-classical systems only when the temperature T is sufficiently high [10], that is, $\delta \ll 1$. It seems that we must be careful in not assigning too much physical significance to the subleading corrections.

[3] Before or after photon decoupling, but after inflation, the value of aT stays nearly constant, essentially reflecting entropy conservation. That value is far into the high temperature regime, the present value of δ being $\delta_0 \sim 10^{-30}$. Insofar as it is permissible to speak of temperature during inflation, aT is also constant then, but of a much smaller value, which value increases dramatically during reheating.

7 Conclusion

Our main qualitative result is that entropy/energy bounds should be relevant only in the ultra-high temperature limit, which applies to the universe after inflation. With the decrease of temperature the bound becomes much less reliable, until at low temperature ($aT \ll 1$), which might be the case in the very early universe, the entropy dominates the energy. This effect occurs already for conformal matter. The situation for non-conformal matter is much more complicated. Hence, it is unclear if entropy/energy bounds should exist at all even for high temperature. This question will be discussed elsewhere.

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[1] E. Verlinde, hep-th/00084140 v2.

[2] J. L. Cardy, Nucl. Phys. B **270**, 186 (1986).

[3] K. A. Milton, *The Casimir Effect: Physical Manifestations of Zero-Point Energy* (World Scientific, Singapore, 2001).

[4] D. Kutasov and F. Larsen, JHEP **0101**, 001 (2001).

[5] D. Klemm, A. C. Petkou, and G. Siopsis, Nucl. Phys. B **601**, 380 (2001).

[6] I. Brevik, K. A. Milton, and S. D. Odintsov, hep-th/0202048 v3, to appear in Ann. Phys. (N.Y.).

[7] R. Kantowski and K. A. Milton, Phys. Rev. D **35**, 549 (1987).

[8] J. S. Dowker, hep-th/0203026.

[9] E. Elizalde and A. C. Tort, Phys. Rev. D **66**, 045033 (2002).

[10] S. Das, P. Majumdar, and R. K. Bhaduri, Class. Quant. Grav. **19**, 2355 (2002).